Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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To cite this article: S I Fialka *et al* 2017 *J. Phys.: Conf. Ser.* **869** 012013

IOP Conf. Series: Journal of Physics: Conf. Series **869** (2017) 012013 doi:10.1088/1742-6596/869/1/012013

An approximate solution method for the quasipotential equations with local spin-orbital potentials

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Abstract. An exact solution of partial quasipotential equations for spin-0 and spin-½ particles with superposition of N δ-potentials is obtained. Based on this, a method for an approximate solution using a superposition of δ-potentials instead of an arbitrary smooth potential is presented. A smooth analytic potential allowing existence of resonant states is considered. Scattering cross section dependences on various potential parameters are investigated.

1. Introduction

Quasipotential approach had proved highly successful as a transparent and sequential method for describing relativistic two-particle systems [1]. But the construction of the interaction quasipotential within the framework of quantum field theory is a complicated problem. For this reason phenomenological potentials are often used. The explicit analogy between the nonrelativistic (the Schrödinger and the Lippmann-Schwinger equations) and the relativistic quasipotential equations, thus, turns out to be very useful. In spite of this there are only a few potentials which allow exact analytical solution.

We focus on the δ-potentials which have been well investigated in quantum mechanics [2-5], but have not been widely used in the quasipotential approach up to the present time.

In this paper we present the solution of quasipotential equations for scattering states with nonzero orbital momentum. A system of two relativistic equal-mass particles with spin 0 and 1/2 is considered. We use a local spin-orbital quasipotential, which reduces the three-dimensional integral equation to partial equations – one-dimensional integral equations for a certain orbital momentum.

2. The local spin-orbital quasipotential case

In the case of a spin-orbital quasipotential, which is local in the Lobachevski momentum space $V_{\sigma\mu}(E_q;\vec{p},\vec{k}) = V_{\sigma\rho}(E_q;(\vec{p}(-)\vec{k})^2)D^{1/2}_{\rho\mu}(\vec{n}_{kp})$, with D the Wigner rotation) [6], the quasipotential equation for scattering states of particles with spin 0 and 1/2 in the relativistic configuration representation takes the form

$$\psi_\sigma(\vec{q},\vec{r}) = \xi(\vec{q},\vec{r})\chi^{1/2}_\sigma + \int G_0(E_q;r,r')V_{\sigma\rho}(E_q;r')\psi_\rho(\vec{q},\vec{r}')d\vec{r}' . \tag{1}$$

Here $\psi_\sigma(\vec{q},\vec{r})$ is two-component wave function of the relative motion of two particles, $\vec{p}$ and $\vec{k}$ are the initial and final relative momenta of the particles in the center-of-inertia system, $2E_q = 2\sqrt{\vec{q}^2 + m^2}$ is the two-particle system's energy, $\chi^{1/2}_\sigma$ are the normalized eigenvectors of the spin. In this paper we limit ourselves to the Green's function of Logunov-Tavkhelidze equation, which in momentum representation is $G_0(E_q,E_p)=(E_q^2-E_p^2+i0)^{-1}$. Transformation between momentum representation and configuration representation is performed by functions $\xi(\vec{q},\vec{r})$, the "plane waves" in Lobachevsky space [7].

The local quasipotential allows to reduce three-dimensional integral equations to partial (one-dimensional) equations. Expansion of wave function, "plane wave", Green's function on the basis of the spherical spinors – eigenfunctions of $\hat{\vec{J}}^2$, $\hat{J}_Z$, $\hat{\vec{L}}^2$, $\hat{\vec{S}}^2$ operators [8, 9], yields:

$$\xi(\vec{q},\vec{r})\chi_\sigma^{1/2}=\frac{4\pi}{q\,r}\sum_{j\ell M} i^\ell s_\ell(w_q,r)\Omega_{jlM}^{1/2}(\vec{n}_r)\chi_\sigma^{1/2\dagger}\,\Omega_{jlM}^{1/2*}(\vec{n}_q)\,; \tag{2}$$

$$\psi_\sigma(\vec{q},\vec{r})=\frac{4\pi}{q\,r}\sum_{j\ell M} i^\ell \psi_\ell(w_q,r)\Omega_{jlM}^{1/2}(\vec{n}_r)\chi_\sigma^{1/2\dagger}\,\Omega_{jlM}^{1/2*}(\vec{n}_q)\,; \tag{3}$$

$$G_0(E_q;r,r')=\frac{1}{r\,r'}\sum_{j\ell M} G_0^{(\ell)}(E_q;r,r')\Omega_{jlM}^{1/2}(\vec{n}_r)\Omega_{jlM}^{1/2\dagger}(\vec{n}_{r'})\,. \tag{4}$$

Here we use the following parametrization: $E_q=m\cosh w_q$, $q=m\sinh w_q$; where w_q is the rapidity. Partial Green's functions of Logunov-Tavkhelidze equation [7] and their asymptotic behavior have the following form

$$\begin{aligned} G_0^{(\ell)}(E_q;r,r') &= \frac{1}{i\,m\sinh(2w_q)}\times \\ &\times\left[\frac{e_\ell^{(1)}(w_q,r)e_\ell^{(1)*}(w_q,r')}{1-e^{-\pi m(r-r')}}+\frac{e_\ell^{(2)}(w_q,r)e_\ell^{(2)*}(w_q,r')}{1-e^{\pi m(r-r')}}-\frac{e_\ell^{(1)}(w_q,r)e_\ell^{(2)*}(w_q,r')}{1-e^{-\pi m(r+r')}}-\frac{e_\ell^{(2)}(w_q,r)e_\ell^{(1)*}(w_q,r')}{1-e^{\pi m(r+r')}}\right]; \end{aligned} \tag{5}$$

$$G_0^{(\ell)}(E_q;r,r')\Big|_{r\to\infty}=-\frac{2s_\ell^*(w_q,r')}{m\sinh(2w_q)}e^{i(mr\,w_q-\pi/2\,\ell)}. \tag{6}$$

Explicit form of functions $s_\ell(w_q,r)$ and $e_\ell^{\pm}(w_q,r)$ are given by Kadyshevsky *et al.* [7] in terms of Legendre functions of complex order. We have established that the partial functions can be expressed simply in terms of Legendre functions of integer order

$$s_\ell(w,r)=i^{-(\ell+1)}\frac{e^{mr\pi}}{\Gamma(i\,m\,r)}Q_\ell^{i\,m\,r}(\coth w)\,;\quad e_\ell^{(1,2)}(w,r)=i^\ell\,\frac{\Gamma(i\,m\,r-\ell)\Gamma(\mp i\,m\,r+\ell+1)}{\Gamma(i\,m\,r)}P_\ell^{\pm i\,m\,r}(\coth w)\,.$$

We will consider a simple spin-orbital potential without E_q dependence (I – identity matrix)

$$V_{\sigma\rho}(E_q;r)=V_{\sigma\rho}(r)=\left(I_{\sigma\rho}+g^2\,2(\hat{\vec{\ell}}\hat{\vec{s}})_{\sigma\rho}\right)V(r)\,. \tag{7}$$

Substituting Eqs. (2), (3), (4) and (7) into Eq. (1), we obtain the partial equations

$$\psi_\ell(w_q,r)=s_\ell(w_q,r)+\int_0^\infty G_0^{(\ell)}(w_q;r,r')\left(1-g^2(1+\kappa)\right)V(r')\,\psi_\ell(w_q,r')\,dr'\,, \tag{8}$$

where the parameter κ depends on the spin orientation and takes on all integer values except zero: $-(\ell+1)$ if $j=\ell-1/2$ and ℓ if $j=\ell+1/2$.

3. Superposition of δ-potentials

Consider Eq. (8) with a finite, but varied, number of δ-potentials

$$V(r)=\sum_{n=1}^{N}A_n\,\delta(r-a_n)\,. \tag{9}$$

Substituting Eq. (9) into Eq. (8), we obtain the following algebraic system of equations

$$\psi_\ell(w_q,r) = s_\ell(w_q,r) + \sum_{n=1}^{N} G_0^{(\ell)}(w_q;r,a_n)\left(1-g^2(1+\kappa)\right)A_n\,\psi_\ell(w_q,a_n)\,. \tag{10}$$

Using a_ν $(\nu = 1,2,...,N)$ instead of r, we obtain a system of linear equations with respect to $\psi_\ell(w_q,a_n)$:

$$\sum_{n=1}^{N}\left[I_{\nu n} - G_0^{(\ell)}(w_q;a_\nu,a_n)\left(1-g^2(1+\kappa)\right)A_n\right]\psi_\ell(w_q,a_n) = s_\ell(w_q,a_\nu)\,.$$

To obtain the quantities which characterize scattering states, we have to consider the wave function at large distance. Taking into account asymptotic behavior of the partial Green's functions (6), equations (10) as $r \to \infty$ yields

$$\psi_\ell(w_q,r)\Big|_{r\to\infty} = s_\ell(w_q,r) + \left(\sum_{n=1}^{N}\frac{-2s_\ell^*(w_q,a_n)}{m\sinh(2w_q)}\left(1-g^2(1+\kappa)\right)A_n\,\psi_\ell(w_q,a_n)\right)e^{i(mrw_q-\pi/2\ell)}\,. \tag{11}$$

In quantum mechanics, the partial scattering amplitude is defined as the coefficient divided by momentum in front of the scattered wave. In our case, relativistic partial amplitudes have the form

$$f_{\ell,\kappa}(w_q) = \frac{1}{m\sinh w_q}\sum_{n=1}^{N}\frac{-2s_\ell^*(w_q,a_n)}{m\sinh(2w_q)}\left(1-g^2(1+\kappa)\right)A_n\,\psi_\ell(w_q,a_n)\,. \tag{12}$$

If the initial state is not polarized, the total scattering cross section can be represented as the sum of the partial cross sections and can be expressed via the relativistic scattering amplitude [10]

$$\sigma = \sum_{\ell=0}^{\infty}\sigma_\ell = 4\pi\sum_{\ell=0}^{\infty}\left[(\ell+1)\left|f_{\ell,-(\ell+1)}(w_q)\right|^2 + \ell\left|f_{\ell,\ell}(w_q)\right|^2\right]. \tag{13}$$

Substituting A_n and a_n parameters into expression for determining the scattering amplitude (12) and producing calculation of the partial cross sections (13), we obtain exact solution of the two-particle equation with superposition of δ-potentials. Based on these results, we can also find numerical approximation to the solution for a wide class of potential fields, with the condition that the potential function tends to zero fast enough as $r \to \infty$.

4. Numerical solution for smooth potential

As an example, consider the following analytic potential [11]

$$V(r) = U\,r^2\,\frac{\cosh((\pi-\beta)mr)}{\cosh(\pi mr)}\,. \tag{14}$$

Range of the variable r $(0 \le r \le \infty)$ should first be divided into two parts: semi-infinite $r \ge r_N$, where potential field is assumed to be zero, and finite $0 \le r < r_N$, where the superposition of δ-potentials (9) is used instead of the smooth potential (14). The last interval is divided into N sub-intervals of the form $[r_n;r_{n+1}]$ with δ-potential in the middle and the parameters take the following form

$$A_n = (r_n - r_{n-1})V(a_n); \quad a_n = (r_n + r_{n-1})/2.$$

Accuracy of the numerical method can be improved by increasing the density of δ-functions near sharp variations of the potential field. Fig. 1a shows the dependence of partial cross section σ_0 on energy E_q for different numbers of δ-functions N at $U = 30, \beta = \pi/3$. In the case of s-states ($\ell = 0$) the partial cross section is independent of g^2. Clearly, with increasing number of δ-functions the result converges quickly to the constant (exact) solution.

Figure 1b shows dependence of the total cross section σ on energy E_q for $g^2 = 0.015$ at $U = 4$ and $\beta = \pi/3$. In order to obtain exact dependence, 7 first partial cross sections were summed. The partial cross sections with $\ell \ge 7$ give no contribution to the total cross section in shown energy range for considered parameters. The superposition of $N = 100$ δ-functions was used.

In contrast to the case of two spinless particles ($g^2 = 0$), resonance energy can be divided depending on strength of the spin-orbital interaction, as shown in Fig. 1b. Just as in the case of two spinless particles, increased height of the barrier (increase parameter U and decrease β) leads to an increased number of resonant states. Resonance energies can be calculated using the complex scaling method [11].

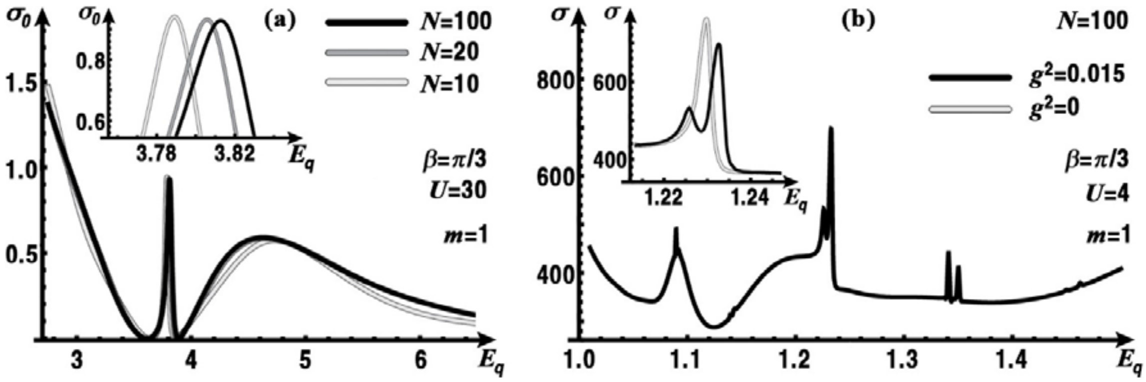

Figure 1. (a) The dependence of partial cross section σ_0 on energy E_q for different number of δ-functions. (b) The dependence of total cross section σ on two-particle system's energy E_q .

It should be mentioned that consideration of scattering states with $\ell > 0$ is important, as is clearly shown in Fig. 1b. In this case, when $\ell = 0, 1, 2,$ there is no resonance, with a small imaginary part and partial cross sections show no narrow peaks, in contrast to the states with $\ell > 2$.

5. Conclusion

Exact solution of partial quasipotential equations for spin-0 and spin-½ particles with superposition of N δ-potentials was presented. The compact general expression for determining the partial scattering amplitudes was obtained. By changing coe□cients next to δ-functions, a numerical solution for a wide class of potential fields can be found out and more realistic systems can be considered.

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