

RELATIVISTIC TWO-PARTICLE EQUATIONS WITH SUPERPOSITION OF DELTA-SHELL POTENTIALS: SCATTERING AND BOUND STATES

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Exact solutions of two-particle relativistic equations of quantum field theory describing the scattering s -states and the bound s -states are found in the cases of delta-shell potential and superposition of delta-shell potentials. Some properties of obtained relativistic wave functions, scattering amplitudes and quantization conditions are investigated. The resonance character of scattering processes is demonstrated by the behavior of amplitudes. It is shown that the non-relativistic limits of these relativistic values coincide with respective non-relativistic ones obtained from the Schrödinger equation.

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1. Introduction

In this paper we consider three-dimensional relativistic equations of quasipotential type, which describe systems of two particles with equal masses. Originally these two-particle equations were obtained in the momentum representation, in which they are analogous to the Schrödinger and the Lippmann-Schwinger equations [1, 2]. In the center of mass system, the equations for the scattering state wave function of relative movement $\psi_{(j)}(\mathbf{q}, \mathbf{p})$ have the form

$$\psi_{(j)}(\mathbf{q}, \mathbf{p}) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) + G_{(j)}(E_q, p) \int V(E_q, \mathbf{p}, \mathbf{k}) \psi_{(j)}(\mathbf{q}, \mathbf{k}) \frac{d\mathbf{k}}{(2\pi)^3}. \quad (1)$$

In the most general case, the quasipotential $V(E_q, \mathbf{p}, \mathbf{k})$ depends on the two-particle energy $2E_q$ in the center of mass system and on the spin variables.

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In this paper we only discuss the case of energy and spin independent potentials. Energy for the scattering states can be parameterized as follows: $2E_q = 2m \cosh \chi_q$, where m - is the mass of each particle, and χ_q - is the rapidity. The Green functions (GFs) $G_{(j)}(E_q, p)$, in the momentum representation for the Logunov-Tavkhelidze equation ($j = 1$) and for the Kadyshevsky equation ($j = 2$), have the following form ($E_p = \sqrt{\mathbf{p}^2 + m^2} = m \cosh \chi_p$):

$$G_{(1)}(E_q, p) = \frac{1}{E_q^2 - E_p^2 + i0} \frac{m}{E_p}; \quad G_{(2)}(E_q, p) = \frac{1}{2E_q - 2E_p + i0} \frac{m}{E_p}. \quad (2)$$

We also consider the modified Logunov-Tavkhelidze ($j = 3$) and the modified Kadyshevsky ($j = 4$) equations, for which the GFs are

$$G_{(3)}(E_q, p) = \frac{1}{E_q^2 - E_p^2 + i0}; \quad G_{(4)}(E_q, p) = \frac{1}{2E_q - 2E_p + i0} \frac{1}{E_p}. \quad (3)$$

All four relativistic Green functions in the non-relativistic limit ($m \rightarrow \infty$; $\chi_q, \chi_p \rightarrow 0$; $m\chi_q = q$; $m\chi_p = p$) turn into the GF of the Schrödinger equation:

$$\lim_{\substack{\chi_q, \chi_p \rightarrow 0 \\ m \rightarrow \infty}} G_{(j)}(E_q, p) = G_{q(0)}(p) = \frac{1}{q^2 - p^2 + i0}. \quad (4)$$

The so called relativistic configurational representation (RCR) is used for two-particle equations along with the momentum representation [3, 4]. The RCR was introduced by means of expansion of all the values in equations over the principal series of matrix elements of irreducible unitary representations the Lorentz group, which are realized by functions ($\mathbf{r} = r\mathbf{n}$)

$$\xi(\mathbf{p}, \mathbf{r}) = \left(\frac{E_p - \mathbf{p}\mathbf{n}}{m} \right)^{-1-imr}. \quad (5)$$

The group parameter r is treated as a relative relativistic coordinate in the center of mass system [3, 4]. Functions $\xi(\mathbf{p}, \mathbf{r})$ play the role of plane waves in the RCR; in the nonrelativistic limit they turn into functions $\exp(i\mathbf{p}\mathbf{r})$, where \mathbf{r} - is the radius-vector in the ordinary coordinate space. For instance, transformation of the wave function to the RCR and its inverse transformation have the form [3, 4]

$$\psi_{(j)}(\mathbf{q}, \mathbf{r}) = \frac{1}{(2\pi)^3} \int \xi(\mathbf{p}, \mathbf{r}) \psi_{(j)}(\mathbf{q}, \mathbf{p}) \frac{m}{E_p} d\mathbf{p}, \quad (6)$$

$$\psi_{(j)}(\mathbf{q}, \mathbf{p}) = \int \xi^*(\mathbf{p}, \mathbf{r}) \psi_{(j)}(\mathbf{q}, \mathbf{r}) d\mathbf{r}, \quad (7)$$

respectively. Properties of functions $\xi(\mathbf{p}, \mathbf{r})$ and transformations (6), (7) as well as analogous transformations for quasipotentials $V(E_q, \mathbf{p}, \mathbf{k})$ and GFs $G_{(j)}(E_q, p)$ are discussed in detail in [3, 4].

After the transformation into the RCR, the equations for relativistic wave functions have the following form [3, 4] (in this paper we discuss only the case of the RCR-local, spherically symmetric, spin independent potentials $V(\mathbf{r}) \equiv V(r)$):

$$\psi_{(j)}(\mathbf{q}, \mathbf{r}) = \xi(\mathbf{q}, \mathbf{r}) + \int G_{(j)}(E_q, \mathbf{r}, \mathbf{r}') V(r') \psi_{(j)}(\mathbf{q}, \mathbf{r}') d\mathbf{r}'. \quad (8)$$

The expansion of functions $\psi_{(j)}(\mathbf{q}, \mathbf{r})$, $\xi(\mathbf{q}, \mathbf{r})$ into series in the Legendre polynomials $P_l(\mathbf{q}\mathbf{r}/qr)$, and of functions $G_{(j)}(E_q, \mathbf{r}, \mathbf{r}')$ into series in the polynomials $P_l(\mathbf{r}\mathbf{r}'/rr')$, results in equations for the partial wave functions, which in the s -wave case have the form

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q m r) + \int_0^\infty G_{(j)}(\chi_q, r, r') V(r') \psi_{(j)}(\chi_q, r') dr'. \quad (9)$$

Partial Green functions in the RCR $G_{(j)}(\chi_q, r, r')$ are connected with the GFs in the momentum representation as follows:

$$G_{(j)}(\chi_q, r, r') = \frac{2}{\pi} \int_0^\infty \sin(\chi_k m r) G_{(j)}(m \cosh \chi_q, k) E_k \sin(\chi_k m r') d\chi_k. \quad (10)$$

The direct calculation of GFs (10) for specific j leads to the following expressions:

$$G_{(j)}(\chi_q, r, r') = G_{(j)}(\chi_q, r - r') - G_{(j)}(\chi_q, r + r'), \quad (11)$$

where [5]

$$G_{(1)}(\chi_q, r) = \frac{-i \sinh[(\pi/2 + i\chi_q)m r]}{K_q^{(1)} \sinh[\pi m r/2]}, \quad (12)$$

$$G_{(2)}(\chi_q, r) = \frac{(4m \cosh \chi_q)^{-1}}{\cosh[\pi m r/2]} - \frac{i \sinh[(\pi + i\chi_q)m r]}{K_q^{(2)} \sinh[\pi m r]},$$

$$G_{(3)}(\chi_q, r) = \frac{-i \cosh[(\pi/2 + i\chi_q)m r]}{K_q^{(3)} \cosh[\pi m r/2]}; \quad G_{(4)}(\chi_q, r) = \frac{-i \sinh[(\pi + i\chi_q)m r]}{K_q^{(4)} \sinh[\pi m r]}.$$

In formulae (12) we used the notations

$$K_q^{(1)} = K_q^{(2)} = m \sinh 2\chi_q; \quad K_q^{(3)} = K_q^{(4)} = 2m \sinh \chi_q. \quad (13)$$

In what follows, we will need the asymptotics of GFs (11) at $r \rightarrow \infty$

$$G_{(j)}(\chi_q, r, r') \Big|_{r \rightarrow \infty} \cong -\frac{2}{K_q^{(j)}} \sin(\chi_q m r') \exp(i\chi_q m r). \quad (14)$$

In the bound state case, equations (9) (as well as (8)) are modified to the homogeneous form, and the rapidity χ_q becomes imaginary ($\chi_q = iw_q$; $0 \leq w_q < \pi/2$; $2E_q = 2m \cos w_q$)[6]:

$$\psi_{(j)}(iw_q, r) = \int_0^\infty G_{(j)}(iw_q, r, r') V(r') \psi_{(j)}(iw_q, r') dr'. \quad (15)$$

It is not difficult to see that in the nonrelativistic limit all GFs (11) transform into the GF of the three-dimensional Schrödinger equation for s -waves in the coordinate representation [7, 8]:

$$\lim_{\substack{\chi_q \rightarrow 0 \\ m \rightarrow \infty}} G_{(j)}(\chi_q, r, r') = G_{(0)}(q, r, r') = \frac{-1}{q} \sin(qr_<) \exp(iqr_>). \quad (16)$$

The nonrelativistic limit of equations (9), (15) gives the Schrödinger equation for scattering

$$\psi_{(0)}(q, r) = \sin(qr) + \int_0^\infty G_{(0)}(q, r, r') V(r') \psi_{(0)}(q, r') dr' \quad (17)$$

and bound states

$$\psi_{(0)}(i\kappa, r) = \int_0^\infty G_{(0)}(i\kappa, r, r') V(r') \psi_{(0)}(i\kappa, r') dr', \quad (18)$$

respectively. The quantity $i\kappa$ in equation (18) is defined as $i\kappa = \lim_{\substack{w_q \rightarrow 0 \\ m \rightarrow \infty}} imw_q$.

In general, potentials V in the RCR depend on the system energy $2E_q$. Since the problem of finding relativistic potentials in the frame of quantum field theory is quite complicated, various phenomenological potentials are commonly used analogous to the non-relativistic Schrödinger equation. In many cases such phenomenological potentials are chosen to be independent of $2E_q$. Moreover, the coordinate dependence of relativistic potentials is often the same as the coordinate dependence of the non-relativistic potentials. Even so, the number of cases with exact solutions to relativistic equations is much smaller than in the non-relativistic theory. Solutions found for such potentials and the values obtained based on these solutions allow us to make

definite important conclusions about the general properties of quasipotential type two-particle equations in the case of more complicated potentials which do not allow for exact solutions. The potentials which allow exact solutions and detailed analysis are, therefore, of special interest. To study the general properties based only on numerical solutions is quite difficult.

Delta-function potentials attract much attention in the non-relativistic theory [9, 10, 11, 12, 13, 14]. As a physical model, the delta-function potential is used to represent a potential whose spatial extension is very small compared to all other length scales. Superposition and array of delta-function potentials have been used in solid state physics and optics. The solution of one-dimensional relativistic two-particle equations with the delta-function potential and superposition of delta-function potentials was considered in [5, 6]. These equations with potential "delta-function derivative of n -th order" $\delta^{(n)}$ at $n = 1, 2, 3$ were solved in article [15]. The relativistic one-dimensional problem for nonlinear delta-function potential was considered in [16].

In this paper we consider the solution of relativistic partial two-particle equations for s -waves (9), (15) with the potential $V_0\delta(r - a)$, which is localized on the sphere of finite radius $a > 0$ (in the three-dimensional case such potential is often called "delta-shell potential"), and with a superposition of two such potentials. The article is organized as follows. In the second paragraph the wave functions for the scattering states are found. The scattering amplitudes, S -matrices and phase shifts are obtained using the wave functions. The analysis of obtained results is carried out. In the third paragraph the wave functions and the energy quantization conditions for them are found for bound states. In the fourth paragraph the results are studied in the non-relativistic limit.

2. The scattering states

Let us consider briefly the solutions of relativistic equations for the scattering states (9) in the case of the delta-shell potential

$$V(r) = V_0\delta(r - a), \quad (19)$$

where V_0 and $a > 0$ - are real constants. The wave functions for potential (19) are

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q mr) + V_0 A_{(j)}^{-1}(\chi_q) \sin(\chi_q ma) G_{(j)}(\chi_q, r, a); \quad (20)$$

$$A_{(j)}(\chi_q) = 1 - V_0 G_{(j)}(\chi_q, a, a).$$

Taking into account (14) one can represent the asymptotics of wave functions (20) at $r \rightarrow \infty$ in the form ($q = m \sinh \chi_q$ - is the relativistic

momentum)

$$\psi_{(j)}(\chi_q, r) \Big|_{r \rightarrow \infty} \cong \sin(\chi_q mr) + q f_{(j)}(\chi_q) \exp(i\chi_q mr), \quad (21)$$

$$f_{(j)}(\chi_q) = \frac{-2V_0 \sin^2(\chi_q ma)}{q K_q^{(j)} A_{(j)}(\chi_q)}, \quad (22)$$

where the relativistic scattering amplitude $f_{(j)}(\chi_q)$ is determined, in the same way as the non-relativistic one, as the coefficient at the scattered wave divided by momentum [7, 8] (the relativistic scattered s -wave has the form $\exp(i\chi_q mr)$ [3, 4]). Expressions for the scattering amplitudes corresponding to the four GF variants can be represented as

$$f_{(1)}(\chi_q) = \frac{-2V_0 q^{-1} s^2(a)}{K_q^{(1)} + V_0 \left[\frac{s(2a)}{\tilde{t}(a)} - \frac{2\chi_q}{\pi} + 2is^2(a) \right]}; \quad (23)$$

$$f_{(2)}(\chi_q) = \frac{-2V_0 q^{-1} s^2(a)}{K_q^{(2)} + V_0 \left[\frac{s(2a)}{\tilde{t}(2a)} - \frac{\chi_q}{\pi} + 2is^2(a) - \frac{\tilde{t}^2(a/2)}{1 + \tilde{t}^2(a/2)} \sinh(\chi_q) \right]};$$

$$f_{(3)}(\chi_q) = \frac{-2V_0 q^{-1} s^2(a)}{K_q^{(3)} + V_0 [\tilde{t}(a)s(2a) + 2is^2(a)]};$$

$$f_{(4)}(\chi_q) = \frac{-2V_0 q^{-1} s^2(a)}{K_q^{(4)} + V_0 \left[\frac{s(2a)}{\tilde{t}(2a)} - \frac{\chi_q}{\pi} + 2is^2(a) \right]}.$$

where we used notations

$$s(a) = \sin(\chi_q ma), \quad \tilde{t}(a) = \tanh(\pi ma). \quad (24)$$

Some numerical calculations results of scattering amplitudes (22) for $j = 1, 3$ will be shown later, namely after considering another important value — the phase shift.

Now let us find the solutions of equations (9) and the expressions for the scattering amplitudes in the case of superposition of two delta-shell potentials:

$$V(r) = V_1 \delta(r - a_1) + V_2 \delta(r - a_2), \quad (25)$$

where $V_{1,2}$, $a_{1,2}$ - are real constants and $a_2 > a_1 > 0$. Substitution of (25) into equations (9) gives the wave function expressions in the following

general form:

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q m r) + \sum_{k=1}^2 V_k G_{(j)}(\chi_q, r, a_k) \psi_{(j)}(\chi_q, a_k), \quad (26)$$

where the values $\psi_{(j)}(\chi_q, a_{1,2})$ are to be found. To determine these values one should take expressions (26) at the points $r = a_1$ and $r = a_2$. As a result, one obtains a system of two linear algebraic equations for $\psi_{(j)}(\chi_q, a_1)$, $\psi_{(j)}(\chi_q, a_2)$. Solving this system and substituting the solutions into (26) one obtains the following expressions for the wave functions:

$$\psi_{(j)}(\chi_q, r) = \sin(\chi_q m r) + \sum_{k=1}^2 V_k G_{(j)}(\chi_q, r, a_k) \frac{\Delta_{(j)k}(\chi_q)}{\Delta_{(j)}(\chi_q)}, \quad (27)$$

where notations

$$\Delta_{(j)}(\chi_q) = \prod_{k=1}^2 [1 - V_k G_{(j)}(\chi_q, a_k, a_k)] - V_1 V_2 G_{(j)}^2(\chi_q, a_1, a_2); \quad (28)$$

$$\Delta_{(j)1}(\chi_q) = s(a_1)[1 - V_2 G_{(j)}(\chi_q, a_2, a_2)] + V_2 s(a_2) G_{(j)}(\chi_q, a_1, a_2);$$

$$\Delta_{(j)2}(\chi_q) = s(a_2)[1 - V_1 G_{(j)}(\chi_q, a_1, a_1)] + V_1 s(a_1) G_{(j)}(\chi_q, a_1, a_2)$$

are introduced. Asymptotic behaviour at $r \rightarrow \infty$ of wave functions (27) yields the following formulae for the scattering amplitudes:

$$f_{(j)}(\chi_q) = \frac{-2}{q K_q^{(j)} \Delta_{(j)}(\chi_q)} \sum_{k=1}^2 V_k \Delta_{(j)k}(\chi_q) s(a_k). \quad (29)$$

The explicit form of expressions (27)-(29) for concrete j in the case of superposition of delta-shell potentials is quite cumbersome. For example, the scattering amplitude at $j = 3$ is

$$f_{(3)}(\chi_q) = \frac{F_1 + F_2}{q K_q^{(3)} (F_3 + i F_4)}, \quad (30)$$

where

$$F_1 = V_1 s^2(a_1) \left[1 + \frac{V_2}{K_q^{(3)}} \tilde{t}(a_2) s(2a_2) \right] + V_2 s^2(a_2) \left[1 + \frac{V_1}{K_q^{(3)}} \tilde{t}(a_1) s(2a_1) \right]; \quad (31)$$

$$F_2 = \frac{2V_1 V_2}{K_q^{(3)}} s(a_1) s(a_2) \left[\tilde{t}\left(\frac{a_2 - a_1}{2}\right) s(a_2 - a_1) - \tilde{t}\left(\frac{a_2 + a_1}{2}\right) s(a_2 + a_1) \right];$$

$$\begin{aligned}
F_3 &= \prod_{k=1}^2 \left[1 + \frac{V_k}{K_q^{(3)}} \tilde{t}(a_k) s(2a_k) \right] \\
&\quad - \frac{V_1 V_2}{(K_q^{(3)})^2} \left[\tilde{t}\left(\frac{a_2 - a_1}{2}\right) s(a_2 - a_1) - \tilde{t}\left(\frac{a_2 + a_1}{2}\right) s(a_2 + a_1) \right]^2; \\
F_4 &= \frac{2V_1}{K_q^{(3)}} s^2(a_1) \left[1 + \frac{V_2}{K_q^{(3)}} \tilde{t}(a_2) s(2a_2) \right] + \frac{2V_2}{K_q^{(3)}} s^2(a_2) \left[1 + \frac{V_1}{K_q^{(3)}} \tilde{t}(a_1) s(2a_1) \right] \\
&\quad + \frac{4V_1 V_2}{(K_q^{(3)})^2} s(a_1) s(a_2) \left[\tilde{t}\left(\frac{a_2 - a_1}{2}\right) s(a_2 - a_1) - \tilde{t}\left(\frac{a_2 + a_1}{2}\right) s(a_2 + a_1) \right].
\end{aligned}$$

Results of numerical calculations of scattering amplitudes (29) for $j = 1, 2, 3, 4$ will be presented after the phase shift consideration.

The scattering amplitude $f_{(j)}(\chi_q)$ provides information about particle scattering. For instance, the partial s -wave cross section $\sigma_{0(j)}(\chi_q)$ and the partial-wave S -matrix $S_{(j)}(\chi_q)$ are expressed through the scattering amplitude $f_{(j)}(\chi_q)$ by relations

$$\sigma_{0(j)}(\chi_q) = 4\pi |f_{(j)}(\chi_q)|^2; \quad S_{(j)}(\chi_q) = 1 + 2iq f_{(j)}(\chi_q). \quad (32)$$

It is not difficult to see that expressions for scattering amplitudes (22), (29) satisfy the unitarity condition [3, 4], which has the form similar to the non-relativistic one [7, 8]:

$$\text{Im} f_{(j)}(\chi_q) = q |f_{(j)}(\chi_q)|^2. \quad (33)$$

It follows from expressions (22), (29) that this unitarity condition is equivalent to the following property of GFs (11):

$$\text{Im} G_{(j)}(\chi_q, a, b) = -\frac{2s(a)s(b)}{K_q^{(j)}}. \quad (34)$$

In the case of delta-shell potential (19) the partial wave S -matrix can be represented in the form

$$S_{(j)}(\chi_q) = 1 - \frac{4iV_0 s^2(a)}{K_q^{(j)} A_{(j)}(\chi_q)} = \frac{(A_{(j)}(\chi_q))^*}{A_{(j)}(\chi_q)}, \quad (35)$$

from which its unitarity is obvious. For the superposition of delta-shell potentials (25) $S_{(j)}(\chi_q)$ can also be represented in analogous form to (35)

$$S_{(j)}(\chi_q) = 1 - \frac{4i \left(V_1 s(a_1) \Delta_{(j)1} + V_2 s(a_2) \Delta_{(j)2} \right)}{K_q^{(j)} \Delta_{(j)}} = \frac{(\Delta_{(j)})^*}{\Delta_{(j)}}. \quad (36)$$

The unitarity of the S -matrix is reflected in the following representation

$$S_{(j)}(\chi_q) = \exp(2i\phi_{(j)}(\chi_q)), \quad (37)$$

which defines the phase shift $\phi_{(j)}(\chi_q)$. The phase shifts can be found from expressions:

a) in the case of delta-shell potential

$$\tan(2\phi_{(j)}) = \frac{-4V_0K_q^{(j)}[1 - V_0\text{Re}G_{(j)}(\chi_q, a, a)]s^2(a)}{[K_q^{(j)}(1 - V_0\text{Re}G_{(j)}(\chi_q, a, a))]^2 - [2V_0s^2(a)]^2}; \quad (38)$$

b) in the case of delta-shell potential superposition

$$\tan(2\phi_{(j)}) = \frac{-2\text{Re}\Delta_{(j)}(\chi_q)\text{Im}\Delta_{(j)}(\chi_q)}{(\text{Re}\Delta_{(j)}(\chi_q))^2 - (\text{Im}\Delta_{(j)}(\chi_q))^2}, \quad (39)$$

where

$$\begin{aligned} \text{Re}\Delta_{(j)}(\chi_q) &= \prod_{k=1}^2 [1 - V_k\text{Re}G_{(j)}(\chi_q, a_k, a_k)] - V_1V_2[\text{Re}G_{(j)}(\chi_q, a_1, a_2)]^2; \\ \text{Im}\Delta_{(j)}(\chi_q) &= \frac{2V_1s^2(a_1)}{K_q^{(j)}}[1 - V_2\text{Re}G_{(j)}(\chi_q, a_2, a_2)] \\ &+ \frac{2V_2s^2(a_2)}{K_q^{(j)}}[1 - V_1\text{Re}G_{(j)}(\chi_q, a_1, a_1)] + \frac{4V_1V_2s(a_2)s(a_1)}{K_q^{(j)}}\text{Re}G_{(j)}(\chi_q, a_1, a_2). \end{aligned} \quad (40)$$

In figures 1 and 2 we present the results of numerical calculations of the cross sections and phase shifts (the curve number corresponds to the index of GF j). In the figures (and from expressions (22), (29)) it is seen that $\sigma_{0(j)}(\chi_q) \rightarrow 0$ at $\chi_q \rightarrow \infty$, which is natural. The figures also show that the amplitudes can be equal to zero at some finite values of rapidity χ_q . The analogous non-relativistic effect of the partial-wave cross section (scattering amplitude) vanishing at finite values of momentum is called the Ramsauer-Taunsend effect [7, 8, 17]. Scattering amplitudes of the delta-shell potential are equal to zero if the following condition (the same for all j) holds:

$$\chi_q m a = \pi n, \quad n = 1, 2, 3, \dots \quad (41)$$

From this condition it follows that the scattering amplitudes have an infinite number of zeros. For the superposition of two delta-shell potentials the vanishing condition of the cross section is

$$V_2s^2(a_2) + V_1s^2(a_1) + V_1V_2[2s(a_2)s(a_1)G_{(j)}(\chi_q, a_1, a_2)] \quad (42)$$

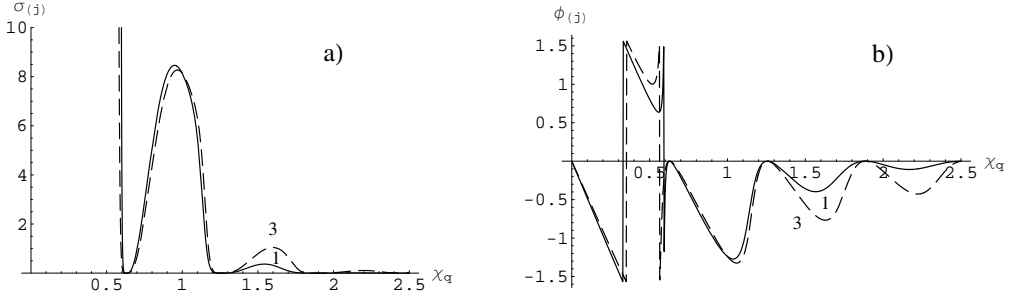


Figure 1. The cross sections (a) and the phase shifts (b) for the delta-shell potential at $m = 1$, $a = 5$, $V_0 = 2$

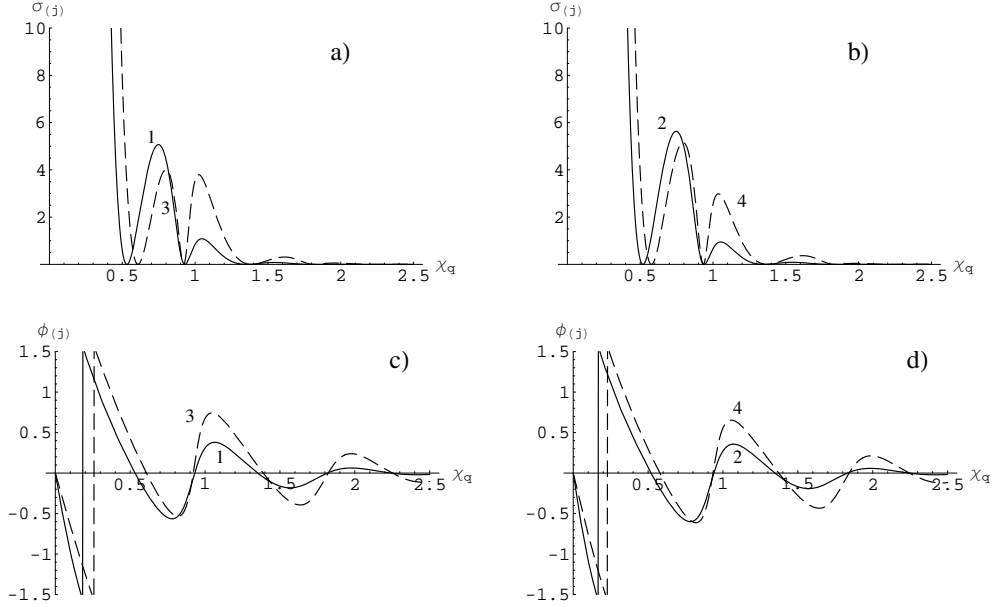


Figure 2. The cross sections (a, b) and the phase shifts (c, d) for the superposition of delta-shell potentials at $m = 1$, $a_1 = 3$, $a_2 = 4$, $V_1 = 1$, $V_2 = -1$

$$-s^2(a_1)G_{(j)}(\chi_q, a_2, a_2) - s^2(a_2)G_{(j)}(\chi_q, a_1, a_1) = 0.$$

For example, for $j = 3$ expression (42) has the form ($c(a) = \cos(\chi_q ma)$)

$$V_2 s^2(a_2) + V_1 s^2(a_1) + \frac{2V_1 V_2}{K_q^{(3)}} s(a_2) s(a_1) \left[\tilde{t} \left(\frac{a_2 - a_1}{2} \right) s(a_2 - a_1) \right] \quad (43)$$

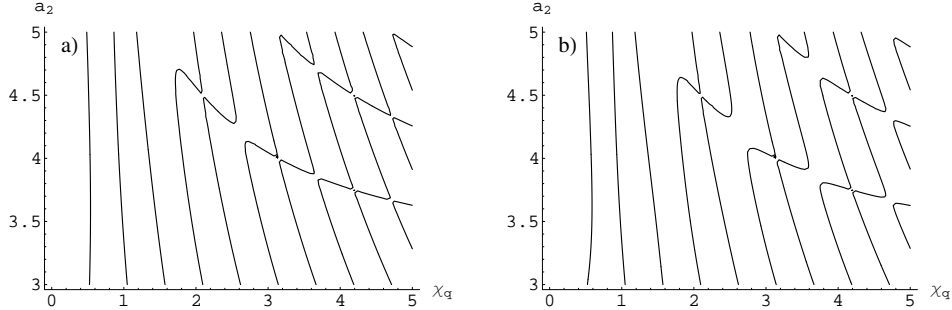


Figure 3. The scattering amplitudes vanishing conditions at $m = 1$, $a_1 = 3$, $V_1 = 1$, $V_2 = -1$: a) $j = 1$, b) $j = 4$

$$\left[-\tilde{t}\left(\frac{a_2 + a_1}{2}\right)s(a_2 + a_1) + \tilde{t}(a_1)s(a_2)c(a_1) + \tilde{t}(a_2)s(a_1)c(a_2) \right] = 0.$$

Condition (42) depends on five variables (V_1 , V_2 , a_1 , a_2 , χ_q). When three of the variables are fixed, a condition for the other two is obtained. In this 2-variable case the scattering amplitude zeros can be represented on a plane. Some results of numerical calculations (for expressions (42)) of this sort are represented in figure 3. At $\chi_q \rightarrow \infty$ expression (42) turns into equality

$$V_2 \sin^2(\chi_q m a_2) + V_1 \sin^2(\chi_q m a_1) = 0. \quad (44)$$

From (44) it follows:

- 1) at $V_1 V_2 < 0$ there is an infinite number of zeros of the scattering amplitude $f_{(j)}(\chi_q)$;
- 2) at $V_1 V_2 > 0$ condition (44) only has solutions if the quotient a_2/a_1 is a rational number, in such case there is an infinite number of scattering amplitude zeros, otherwise the scattering amplitude $f_{(j)}(\chi_q)$ has a finite number of zeros.

3. The bound states

Let us consider solutions of equations for the bound states (15) (for all j) with potentials (19) and (25). The wave functions of bound states for delta-shell potential (19) have the form

$$\psi_{(j)}(i w_q, r) = V_0 G_{(j)}(i w_q, r, a) \psi_{(j)}(i w_q, a). \quad (45)$$

Nontrivial $\psi_{(j)}(iw_q, a)$ (not equal to zero) exist only if the following conditions hold:

$$V_{0(j)} = [G_{(j)}(iw_q, a, a)]^{-1} = [G_{(j)}(iw_q, 0) - G_{(j)}(iw_q, 2a)]^{-1}, \quad (46)$$

these are the quantization conditions for the quantity w_q ($2E_q = 2m \cos w_q$). In conditions (46) the potential parameter V_0 is represented as a function of the energy parameter w_q . Of course, it would be more convenient to have w_q (or E_q) as a function of V_0 , but this is impossible in the case considered here. For $j = 1, 2, 3, 4$ conditions (46) have the following form:

$$V_{0(1)} = \frac{\pi m \sin 2w_q}{2w_q + \pi \left[2\tilde{s}^2(a) - \frac{\tilde{s}(2a)}{\tilde{t}(a)} \right]}; \quad (47)$$

$$V_{0(2)} = \frac{\pi m \sin 2w_q}{w_q + \pi \left[\frac{\tilde{t}^2(a/2) \sin w_q}{\tilde{t}^2(a/2) + 1} + 2\tilde{s}^2(a) - \frac{\tilde{s}(2a)}{\tilde{t}(2a)} \right]};$$

$$V_{0(3)} = \frac{2m \sin w_q}{2\tilde{s}^2(a) - \tilde{t}(a)\tilde{s}(2a)}; \quad V_{0(4)} = \frac{2\pi m \sin w_q}{w_q + \pi \left[2\tilde{s}^2(a) - \frac{\tilde{s}(2a)}{\tilde{t}(2a)} \right]},$$

where we used notations

$$\tilde{s}(a) = \sinh(w_q m a); \quad \tilde{t}(a) = \tanh(\pi m a). \quad (48)$$

From expressions (47) it follows that bound states can exist only if $V_{0(j)} < 0$. In figure 4 we present numerical calculation results of V_0 , w_q -dependence (for all j) given by (47). As seen in the figure, only one energy level can

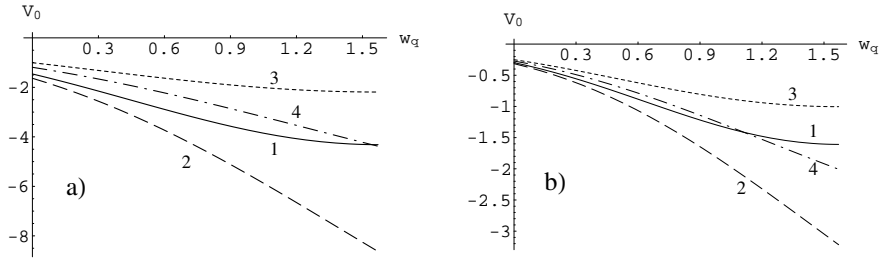


Figure 4. The quantization condition for the delta-shell potential at: a) $m = 1$, $a = 1$; b) $m = 0.5$, $a = 2$

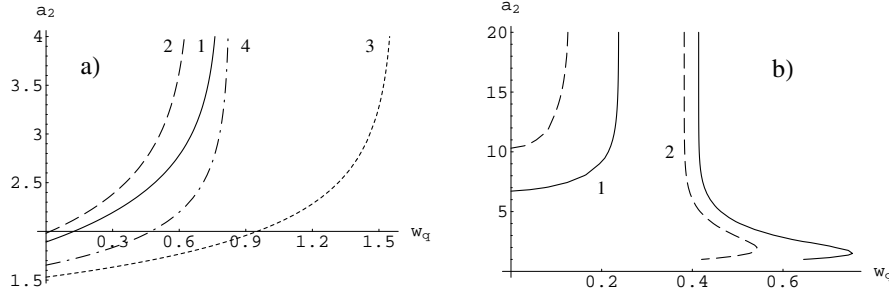


Figure 5. The quantization condition for the superposition of two delta-shell potentials at $m = 1$, $a_1 = 1$.; a) $V_1 = 7$, $V_2 = -2$; b) $V_1 = -2$, $V_2 = -1$

exist for the delta-shell potential (one value w_q with all the other parameters fixed). Unknown values $\psi_{(j)}(iw_q, a)$ in wave functions (45) can be determined from the normalization condition which has the same form in the RCR for all j :

$$\int_0^\infty |\psi_{(j)}(iw_q, r)|^2 dr = 1. \quad (49)$$

Solutions of equations (15) for the superposition of the delta-shell potentials can be found by analogy with the scattering state case. The quantization conditions for the superposition of delta-shell potentials (25) are

$$\prod_{k=1}^2 [1 - V_k G_{(j)}(iw_q, a_k, a_k)] - V_1 V_2 G_{(j)}^2(iw_q, a_1, a_2) = 0. \quad (50)$$

It is not difficult to see that at $a_1 \rightarrow \infty$ (or $a_2 \rightarrow \infty$) expressions (50) are transformed into quantization conditions similar to (46) for the single delta-shell. In figure 5 we present some numerical results for (50). The figure shows that one or two energy levels can exist for the superposition of two delta-shell potentials (one or two values w_q for fixed a_2). Expressing V_2 via all other parameters of the problem under consideration (V_1 , a_1 , a_2 , m , w_q) one obtains

$$V_{2(j)} = \frac{1 - V_1 G_{(j)}(iw_q, a_1, a_1)}{G_{(j)}(iw_q, a_2, a_2) + V_1 F_{(j)}(iw_q, a_1, a_2)}, \quad (51)$$

$$F_{(j)}(iw_q, a_1, a_2) = G_{(j)}^2(iw_q, a_1, a_2) - \prod_{s=1}^2 G_{(j)}(iw_q, a_s, a_s).$$

In figure 6 we show some numerical results of expressions (51) at fixed parameters V_1 , a_1 , a_2 , m . The figure illustrates that expressions (51) can have singularities at some parameters V_1 , a_1 , a_2 , m , w_q .

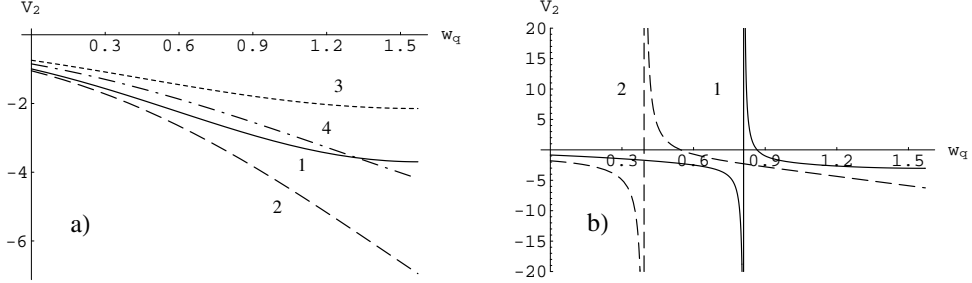


Figure 6. The quantization condition for the superposition of two delta-shell potentials at $m = 1$, $a_1 = 1$: a) $a_2 = 2$, $V_1 = 1.5$; b) $a_2 = 4$, $V_1 = -3.5$

For a more detailed analysis of expressions (50) let us define $V_2 = \alpha V_1$, where α is a dimensionless parameter ($\alpha \neq 0$). Equalities (50) then take the form of quadratic equations for V_1 , solving them one can obtain two following expressions for each j :

$$V_{1(j)}^{\pm} = \frac{\alpha G_{(j)}(iw_q, a_2, a_2) + G_{(j)}(iw_q, a_1, a_1) \pm \sqrt{D_{(j)}}}{2\alpha[G_{(j)}(iw_q, a_1, a_1)G_{(j)}(iw_q, a_2, a_2) - G_{(j)}^2(iw_q, a_1, a_2)]}, \quad (52)$$

$$D_{(j)} = [G_{(j)}(iw_q, a_1, a_1) - \alpha G_{(j)}(iw_q, a_2, a_2)]^2 + 4\alpha G_{(j)}^2(iw_q, a_1, a_2).$$

Numerical results for expressions (52) at $V_1 = V_2$ ($\alpha = 1$) and at $V_1 = -V_2$ ($\alpha = -1$) are shown in figures 7 and 8 respectively. From the figures and expressions (52) it follows that two values V_1 exist for each value w_q at fixed parameters m , a_1 , a_2 , α . Herewith, bound states can exist if one of the following conditions hold: 1) $V_1 < 0$, $V_2 < 0$; 2) $V_1 < 0$, $V_2 > 0$; 3) $V_1 > 0$, $V_2 < 0$. At $V_1 > 0$, $V_2 > 0$ bound states do not exist.

It has to be noted that quantization conditions (46) and (50) can be obtained by equating the expressions for $A_{(j)}(\chi_q)$, $\Delta_{(j)}(\chi_q)$ (expressions (20) and (28)) to zero at $\chi_q = iw_q$, $0 \leq w_q < \pi/2$.

4. The non-relativistic limit

Now let us consider the non-relativistic limit of the results obtained. In this limit the wave functions, the scattering amplitudes and the quantization conditions obtained give the same result for all equations under consideration. In the case of single delta-potential the limits are (q - is the non-relativistic momentum, and $\kappa = \lim_{\substack{w_q \rightarrow 0 \\ m \rightarrow \infty}} mw_q$.)

$$\lim_{\substack{\chi_q \rightarrow 0 \\ m \rightarrow \infty}} \psi_{(j)}(\chi_q, r) = \psi_{(0)}(q, r) = \sin(qr) + \frac{V_0 q \sin(qa) G_{(0)}(q, r, a)}{q + V_0 \sin(qa) \exp(iqa)}, \quad (53)$$

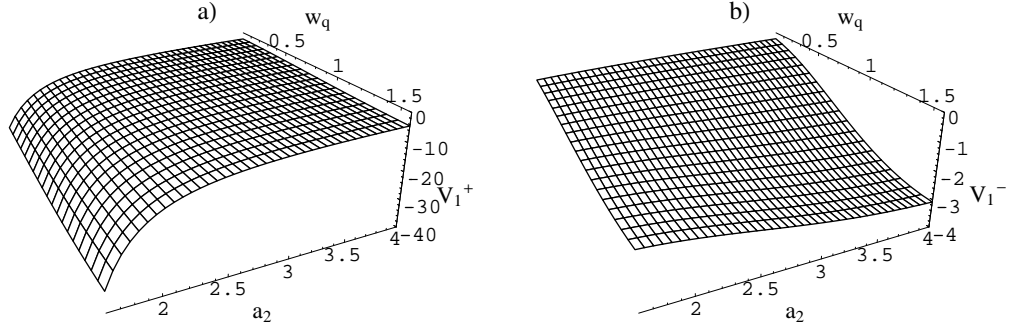


Figure 7. The quantization condition for the superposition of delta-shell potentials for $j = 1$ at $\alpha = 1$, $m = 1$, $a_1 = 1$

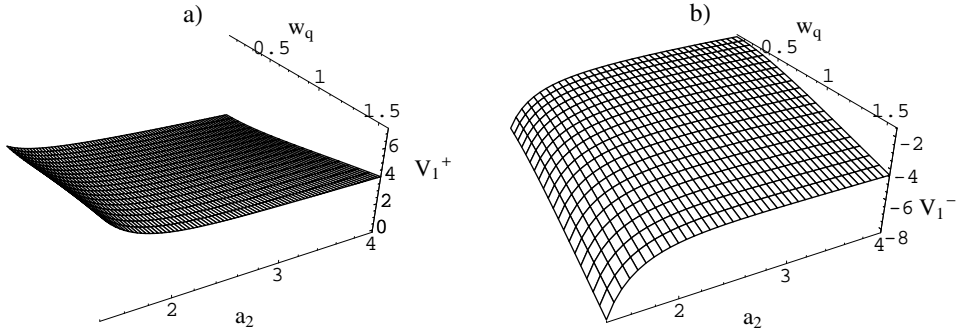


Figure 8. The quantization condition for the superposition of delta-shell potentials for $j = 4$ at $\alpha = -1$, $m = 1$, $a_1 = 1$

$$\lim_{\substack{\chi_q \rightarrow 0 \\ m \rightarrow \infty}} f_{(j)}(\chi_q) = f_{(0)}(q) = \frac{-V_0 \sin^2(qa)}{q[q + V_0 \sin(qa) \exp(ika)]}, \quad (54)$$

$$\lim_{\substack{w_q \rightarrow 0 \\ m \rightarrow \infty}} V_{0(j)}(w_q) = V_{0(0)}(\kappa) = \frac{-2\kappa}{1 - \exp(-2\kappa a)}. \quad (55)$$

Expressions (53)-(55) coincide with the corresponding results obtained based on the Schrödinger equation solution [18, 19, 20]. In the case of superposition of two delta-shell potentials the non-relativistic limits of the relativistic wave functions and scattering amplitudes for all j can be written as

$$\begin{aligned} \psi_{(0)}(q, r) = \sin(qr) + \frac{1}{\Delta_{(0)}(q)} \left[V_1 \sin(qa_2) G_{(0)}(q, r, a_1) \right. \\ \left. + V_2 \left(\sin(qa_2) + \frac{V_1}{q} \sin(qa_1) \sin[q(a_2 - a_1)] \right) G_{(0)}(q, r, a_2) \right], \end{aligned} \quad (56)$$

$$f_{(0)}(q) = \frac{-\sin(qa_2)}{q^2\Delta_{(0)}(q)} \left[\sum_{s=1}^2 V_s \sin(qa_s) + \frac{V_1 V_2}{q} \sin(qa_1) \sin[q(a_2 - a_1)] \right], \quad (57)$$

where

$$\begin{aligned} \Delta_{(0)}(q) &= 1 + \frac{1}{q} \sum_{s=1}^2 V_s \exp(iqa_s) \sin(qa_s) \\ &+ \frac{V_1 V_2}{q^2} \exp(iqa_2) \sin(qa_1) \sin[q(a_2 - a_1)]. \end{aligned} \quad (58)$$

The non-relativistic limit of the quantization conditions (50) takes the form

$$\Delta_{(0)}(i\kappa) = 1 + \frac{1}{2\kappa} \sum_{s=1}^2 V_s [1 - \exp(-2\kappa a_s)] \quad (59)$$

$$+ \frac{V_1 V_2}{\kappa^2} \exp(-\kappa a_2) \sinh(\kappa a_1) \sinh(\kappa(a_2 - a_1)) = 0.$$

All these expressions (56)-(59) coincide with the expressions obtained by solving the Schrödinger equation with the superposition of two delta-shell potentials.

5. Conclusion

In this paper we solved exactly relativistic two-particle integral equations describing the scattering and bound s -states in cases of the delta-shell potential and the superposition of two delta-shell potentials. These solutions yield the expressions for the partial scattering amplitudes, partial cross sections, partial S -matrices and phase shifts. The cross section behaviour demonstrates the resonant character of the scattering process for the potentials considered. Properties of results obtained for the scattering and bound states are investigated analytically and numerically. Namely: unitarity condition of the scattering amplitude is proved and the connection between the unitarity of scattering amplitudes and the imaginary part of the Green functions was found, the vanishing conditions of the scattering amplitudes at some finite quantities of the rapidity is observed, the analysis of the conditions for bound state existence is performed. The non-relativistic limits of all obtained results are found.

The properties of solutions studied in this paper, and the physical values which were obtained based on these solutions have a general character for all four equations. In the future, we plan to consider a numerical solutions of the two-particle integral equations for potentials which have more complicated form in the relativistic configurational representation.

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