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Abstract

The covariant single-time equations of the quantum field theory are formulated in the relativistic configurational representation. The explicit formulas for the Green functions corresponding to the scattering states are calculated in this representation. Using the derived nonhomogeneous equations the scattering problem is solved exactly for certain potentials (combinations of zero-range potentials).

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In the momentum representation the quasipotential equations [1, 2] are analogous to the nonrelativistic Lippman-Schwinger and Schrödinger integral equation (IE). As known in the nonrelativistic theory the solution of the Schrödinger equation can easily be obtained in the coordinate representation, not in the momentum one. In the coordinate representation this equation is a differential one. However, for the relativistic equations the Fourier analysis is not too useful, since the equations transform into differential equations [3, 4].

The construction of the relativistic configurational representation (RCR) [5, 6], which is introduced using the harmonic analysis for the Lorentz group, makes it possible to obtain finite-difference equations for the wave function. However, in the general case, solutions of the finite-difference equations contain arbitrary "i - periodical multipliers". The attractive RCR construction gives a physically evident description of the local potentials $V(r)$ and wave functions $\Psi_{\vec{q}}(\vec{r})$. At the same time RCR have no ambiguity for IE [7].

In the present paper, with the help of the IE approach, we obtain the wave functions corresponding to the scattering states and calculate the corresponding reflection and penetration coefficients.

The nonhomogeneous integral equation in the RCR for the scattering state is

$$\Psi_j(r) = e^{iqr} - \int g_j(q;r,r')\,V(r')\,\Psi_j(r')\,dr', \qquad j=1,..4, \tag{1}$$

where $E = \cosh q > 1$ corresponds to the continuous spectrum. The Green functions for the quasipotential equations are

$$g(q;r,r') = \int e^{iq(r-r')} G(E_q,E)dq; \qquad E_q = \cosh q, \tag{2}$$

where G is the Green function in the momentum representation. In the present paper we are going to consider four different relativistic equations: Logunov-Tavkhelidze, Kadyshevsky, modified Logunov-Tavkhelidze and modified Kadyshevsky. The corresponding Green functions in the momentum representations are given by:

$$G_1(E_q,E) = \frac{1}{E_q^2 - E^2}; \qquad G_2(E_q,E) = \frac{1}{E_q(E_q - E)}; \tag{3}$$

$$G_3(E_q,E) = \frac{E_q}{E_q^2 - E^2}; \qquad G_4(E_q,E) = \frac{1}{E_q - E} \tag{4}$$

respectively. The Green functions (2) can be calculated explicitly

$$\begin{aligned}
g_1(q;r,r') &= \int \frac{\exp(i\chi(r-r'))}{(\cosh\chi - i0)^2 - \cosh^2 q} d\chi = \frac{2\pi i}{\sinh 2q} \frac{\sinh\left[(r-r')\left(\frac{\pi}{2} + iq\right)\right]}{\sinh\left(\frac{\pi}{2}r - \frac{\pi}{2}r'\right)}, \\
g_2(q;r,r') &= \int \frac{\exp(i\chi(r-r'))}{\cosh\chi\,(\cosh\chi - \cosh q - i0)} d\chi \\
&= \frac{2\pi}{\sinh 2q}\left[\frac{2i\sinh\left[(r-r')(\pi + iq)\right]}{\sinh(\pi r - \pi r')} - \frac{\sinh q}{\cosh\left(\frac{\pi}{2}r - \frac{\pi}{2}r'\right)}\right], \\
g_3(q;r,r') &= \int \frac{\exp(i\chi(r-r'))\cosh\chi}{(\cosh\chi - i0)^2 - \cosh^2 q} d\chi = \frac{i\pi}{\sinh q} \frac{\cosh\left[(r-r')\left(\frac{\pi}{2} + iq\right)\right]}{\cosh\left(\frac{\pi}{2}r - \frac{\pi}{2}r'\right)}, \\
g_4(q;r,r') &= \int \frac{\exp(i\chi(r-r'))}{\cosh\chi - \cosh q - i0} d\chi = \frac{2\pi i}{\sinh q} \frac{\sinh\left[(r-r')(\pi + iq)\right]}{\sinh(\pi r - \pi r')}.
\end{aligned} \tag{5}$$

Our programme for the future includes the investigation of equation (1) for quasipotentials derived on the basis of field theory. For example, we are planning to consider the potential

$$V(r)=\frac{1}{r}\tanh\left(\frac{\pi}{2}r\right) \tag{6}$$

(some superposition of potentials of one-boson exchange) for which the homogeneous integral equation for bound state has been solved exactly. However, in the current paper, we decided to confine our consideration to the "toy-model" potential

$$V(r)=V_1\delta(r-a)+V_2\delta(r+a), \tag{7}$$

which can be considered as an approximation of the potential (6).

The construction of IE in the RCR makes it possible to use zero radius potentials, which are widely used in the nonrelativistic theory [8-10]. The one-dimensional differential Schrödinger equation with δ- potentials [9, 10, 14] has attracted a lot of attention. In this connection it is interesting to take a further look at the relativistic equations with such potentials. The Dirac equation with point potentials has already been studied [11, 12, 13]. Therefore we are going to study the Logunov-Tavkhelidze and Kadyshevsky equations only. It is obvious that the investigation of the difference equations with the δ-potentials is difficult, since it is impossible to formulate boundary conditions for the difference equations similar to quantum-mechanical boundary conditions for differential equations. And with it IE in the RCR, on the contrary, are well suited for the study of δ-potentials.

This model is characterized by three parameters V_1, V_2 and $2a$, the parameter $2a$ is the width of hole (barrier). Substituting potential (7) into equation (1) we get the following expression for the wave function

$$\Psi_j(r)=e^{iqr}-V_1g_j(q;r,a)\Psi_j(a)-V_2g_j(q;r,-a)\Psi_j(-a), \tag{8}$$

where the constants $\Psi_j(a)$ and $\Psi_j(-a)$ should satisfy the algebraic system

$$\begin{bmatrix} 1+V_1g_j(q;a,a) & V_2g_j(q;a,-a) \\ V_1g_j(q;-a,a) & 1+V_2g_j(q;-a,-a)\end{bmatrix}\begin{bmatrix}\Psi_j(a)\\ \Psi_j(-a)\end{bmatrix}=\begin{bmatrix}\exp(iqa)\\ \exp(-iqa)\end{bmatrix}. \tag{9}$$

The constants are given by

$$\Psi_j(a)=\frac{\Delta_{1j}}{\Delta_j};\quad \Psi_j(-a)=\frac{\Delta_{2j}}{\Delta_j}, \tag{10}$$

where

$$\begin{aligned}\Delta_j &= (1+V_1g_j(q,a,a))(1+V_2g_j(q,-a,-a))-V_1V_2g_j(q,a,-a)g_j(q,-a,a);\\ \Delta_{1j} &= e^{iqa}(1+V_2g_j(q,a,a))-e^{-iqa}V_2g_j(q,a,-a);\\ \Delta_{2j} &= (1+V_1g_j(q,a,a))e^{-iqa}-V_1g_j(q,-a,a)e^{iqa},\ j=1,2,3,4.\end{aligned} \tag{11}$$

Consider now the asymptotic behaviour of the wave function for $r\to\pm\infty$. We start from the first equation. Using the following representation for the Green function

$$g_1(q;r,r')=\frac{2\pi i}{\sinh 2q}\{\cos[q(r-r')]+i\coth[\pi/2(r-r')]\sin[q(r-r')]\} \tag{12}$$

we derive

$$\Psi_1(r)=\exp(iqr)-\frac{2\pi i}{\sinh 2q}V_1\Psi_1(a)\{\cos[q(r-a)]+i\coth[\pi/2(r-a)]\sin[q(r-a)]\}-$$

$$-\frac{2\pi i}{\sinh 2q} V_2 \Psi_1(-a) \left\{\cos\left[q(r+a)\right] + i \coth\left[\pi/2(r+a)\right] \sin\left[q(r+a)\right]\right\}. \tag{13}$$

By analogy we obtain explicit formulas for $\Psi_j(r)$, $j = 2, 3, 4$ and in all four cases the following asymptotic formulas are valid

$$\Psi_j(r)\,|_{r\to-\infty} = \exp(iqr) + B_j \cdot \exp(-iqr); \quad \Psi_j(r)\,|_{r\to\infty} = A_j \cdot \exp(iqr). \tag{14}$$

The constants B_j and A_j, amplitudes of the reflected and transmitted waves, are given by

$$\begin{aligned} B_j &= K_j \left\{V_1 \exp(iqa) \Psi_j(a) + V_2 \exp(-iqa) \Psi_j(-a)\right\} \\ &= \frac{K_j}{\Delta_j} \left\{V_1 \exp(2iqa) + V_2 \exp(-2iqa) + 2V_1V_2 \left[g_j(q) \cos 2qa - g_j(q; a, -a)\right]\right\}, \\ A_j &= 1 + K_j \left\{V_1 \exp(-iqa) \Psi_j(a) + V_2 \exp(iqa) \Psi_j(-a)\right\} \\ &= 1 + \frac{K_j}{\Delta_j} \left\{V_1 + V_2 + 2V_1V_2 \left[g_j(q) - g_j(q; a, -a) \cos 2qa\right]\right\}, \end{aligned} \tag{15}$$

where

$$K_1 = -\frac{2\pi i}{\sinh 2q}; \quad K_2 = -\frac{4\pi i}{\sinh 2q}; \quad K_3 = -\frac{\pi i}{\sinh q}; \quad K_4 = -\frac{2\pi i}{\sinh q}. \tag{16}$$

Using the explicit formulas for the Green function $g_j(q; a, a)$, $g_j(q; a, -a)$, we calculate the amplitude coefficients A_j and B_j. For the Logunov-Tavkhelidze and Kadyshevsky equations we obtain

$$\begin{aligned} B_1 &= -\frac{2\pi i}{D_1} \left\{(V_1 \exp(2iqa) + V_2 \exp(-2iqa)) \sinh 2q + 4V_1V_2 (\pi \sin 2qa \coth a\pi - 2q \cos 2qa)\right\}, \\ A_1 &= 1 - \frac{2\pi i}{D_1} \left\{(V_1 + V_2) \sinh 2q + 2V_1V_2 (\pi \sin 4qa \coth a\pi - 4q) + 4\pi i V_1V_2 \sin^2 2qa\right\}, \end{aligned} \tag{17}$$

where

$$D_1 = \left[\sinh 2q + (2\pi i - 4q) V_1\right] \left[\sinh 2q + (2\pi i - 4q) V_2\right] + 4\pi^2 V_1V_2 \left[\frac{\sinh(a\pi + 2iqa)}{\sinh(a\pi)}\right]^2, \tag{18}$$

and

$$\begin{aligned} B_2 &= -\frac{4\pi i}{D_2} \Big\{(V_1 \exp(2iqa) + V_2 \exp(-2iqa)) \sinh 2q + \\ &\quad +4 \left[2\pi \sin 2qa \coth 2a\pi - (2q + \pi \sinh q) \cos 2qa + \frac{\pi \sinh q}{\cosh a\pi}\right] V_1V_2\Big\}, \\ A_2 &= 1 - \frac{4\pi i}{D_2} \Big\{4 \left[\pi \sin 4qa \coth 2a\pi + \pi \sinh q \left(\frac{\cos 2qa}{\cosh a\pi} - 1\right) - 2q\right] V_1V_2 + \\ &\quad (V_1 + V_2) \sinh 2q + 8\pi i V_1V_2 \sin^2 2qa\Big\}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} D_2 &= \left[\sinh 2q + 2(2\pi i - 2q - \pi \sinh q) V_1\right] \left[\sinh 2q + 2(2\pi i - 2q - \pi \sinh q) V_2\right] - \\ &\quad -4\pi^2 V_1V_2 \left[\frac{2i \sinh(2a\pi + 2iqa)}{\sinh(2a\pi)} - \frac{\sinh q}{\cosh(a\pi)}\right]^2. \end{aligned} \tag{20}$$

For the modified Logunov-Tavkhelidze and modified Kadyshevsky equations the amplitudes are given by

$$\begin{aligned} B_3 &= -\frac{\pi i}{D_3} \left\{(V_1 \exp(2iqa) + V_2 \exp(-2iqa)) \sinh q + 2\pi V_1V_2 \sin 2qa \tanh a\pi\right\}, \\ A_3 &= 1 - \frac{\pi i}{D_3} \left\{(V_1 + V_2) \sinh q + \pi \sin 4qa \tanh a\pi V_1V_2 + 2\pi i V_1V_2 \sin^2 2qa\right\}, \end{aligned} \tag{21}$$

where

$$D_3 = [\sinh q + \pi i V_1] [\sinh q + \pi i V_2] + \pi^2 V_1 V_2 \left[\frac{\cosh(a\pi + 2iqa)}{\cosh(a\pi)}\right]^2 . \quad (22)$$

and

$$B_4 = -\frac{2\pi i}{D_4} \{(V_1 \exp(2iqa) + V_2 \exp(-2iqa)) \sinh q + 4V_1 V_2 (\pi \sin 2qa \coth 2a\pi - q \cos 2qa)\},$$

$$A_4 = 1 - \frac{2\pi i}{D_4} \left\{(V_1 + V_2) \sinh q + 2(\pi \sin 4qa \coth 2a\pi - 2q) V_1 V_2 + 4\pi i V_1 V_2 \sin^2 2qa\right\}, \quad (23)$$

where

$$D_4 = [\sinh q + (2\pi i - 2q) V_1] [\sinh q + (2\pi i - 2q) V_2] + 4\pi^2 V_1 V_2 \left[\frac{\sinh(2a\pi + 2iqa)}{\sinh(2a\pi)}\right]^2 . \quad (24)$$

From the asymptotic expressions for $\Psi_j(r)$ (14) we define the reflection and transition probabilities

$$R_j = |B_j|^2 ; \quad P_j = |A_j|^2 . \quad (25)$$

The unitarity of the scattering matrix implies that

$$R_j + P_j = 1.$$

We have checked that the calculated reflection and penetration coefficients satisfy the latter equality. (The calculations have been carried out using the REDUCE programme.)

Let us consider now the results of the numerical calculations. In Fig.1 and Fig.2 the reflectivity R is given as a function of the rapidity q for fixed parameters V_1 ,V_2 , a. The reflectivity R_1 vanishes (total penetration) when $V_1 = V_2$ provided that

$$\tan 2qa = \left[\frac{2q}{\pi} - \frac{(V_1 + V_2 \sinh 2q)}{4\pi V_1 V_2}\right] \tanh a\pi. \quad (26)$$

As we can see this transcendental equation has infinite set of solutions.

A similar equation for R_3 is given by

$$\tan 2qa = -\frac{V_1 + V_2}{2\pi V_1 V_2} \coth a\pi \sinh q. \quad (27)$$

In just the same way it is possible to analyze the reflectivity as a function of the width a for fixed V_1, V_2, q. Since the curves have a similar form for all four equations we plot in Fig.3 the curves for R_3. Solving equation (27) with respect to a we obtain again an infinite set of solutions.

Let us consider now in more detail the cases where $V_1 = V_2 > 0$ (two barriers) and $V_1 = -V_2$ (barrier-hole). The corresponding curves of the reflectivity R as a function of the rapidity are given in Fig.4 and Fig.5.

A typical feature of these curves is the existence of points where the reflectivity is equal to unit (total reflection) with the exception for R_3 in the case of two barriers. Numerical analysis does not, in principle, determine whether the unit is approached exactly. Naturally it is necessary to analyze the reflectivity behaviour analytically and locate the points where the penetration vanishes, that is, the potential becomes nonpenetrable. This analysis has been made. At first sight it seems impossible that the penetration is equal to zero, since the amplitude A for all cases is complex-valued and vanishes only if both the imaginary and real parts are equal to zero, i.e. the following equalities hold

$$\left.\begin{array}{l} ReA = 0 \\ ImA = 0 \end{array}\right\}.$$

But for all cases the imaginary part of the numerator of A is equal to zero identically. Therefore it is necessary to consider the condition only for the real part of the numerator. The explicit formulas for the conditions when the potential is nonpenetrable for all cases are as follows

$$\begin{aligned}
&1) \quad [\sinh 2q - 4qV_1]\,[\sinh 2q - 4qV_2] - 4\pi^2 V_1 V_2 \frac{\sin^2 2qa}{\sinh^2 a\pi} = 0,\\
&2) \quad [\sinh 2q - 2\,(2q + \pi \sinh q)\, V_1]\,[\sinh 2q - 2\,(2q + \pi \sinh q)\, V_2] + \\
&\quad +16\pi^2 V_1 V_2 \sin^2 2qa - 4\pi^2 V_1 V_2 \left[2 \coth 2a\pi \sin 2qa + \frac{\sinh q}{\cosh a\pi}\right]^2 = 0,\\
&3) \quad \sinh^2 q + \pi^2 V_1 V_2 \frac{\sin^2 2qa}{\cosh^2 a\pi} = 0,\\
&4) \quad [\sinh q - 2qV_1]\,[\sinh q - 2qV_2] - 4\pi^2 V_1 V_2 \frac{\sin^2 2qa}{\sinh^2 a\pi} = 0.
\end{aligned} \tag{28}$$

By the way, considering the "toy-model" potential we discovered the phenomenon of total reflection which is missing in the nonrelativistic model with the potential

$$V(x) = V_1 \delta(x-a) + V_2 \delta(x+a).$$

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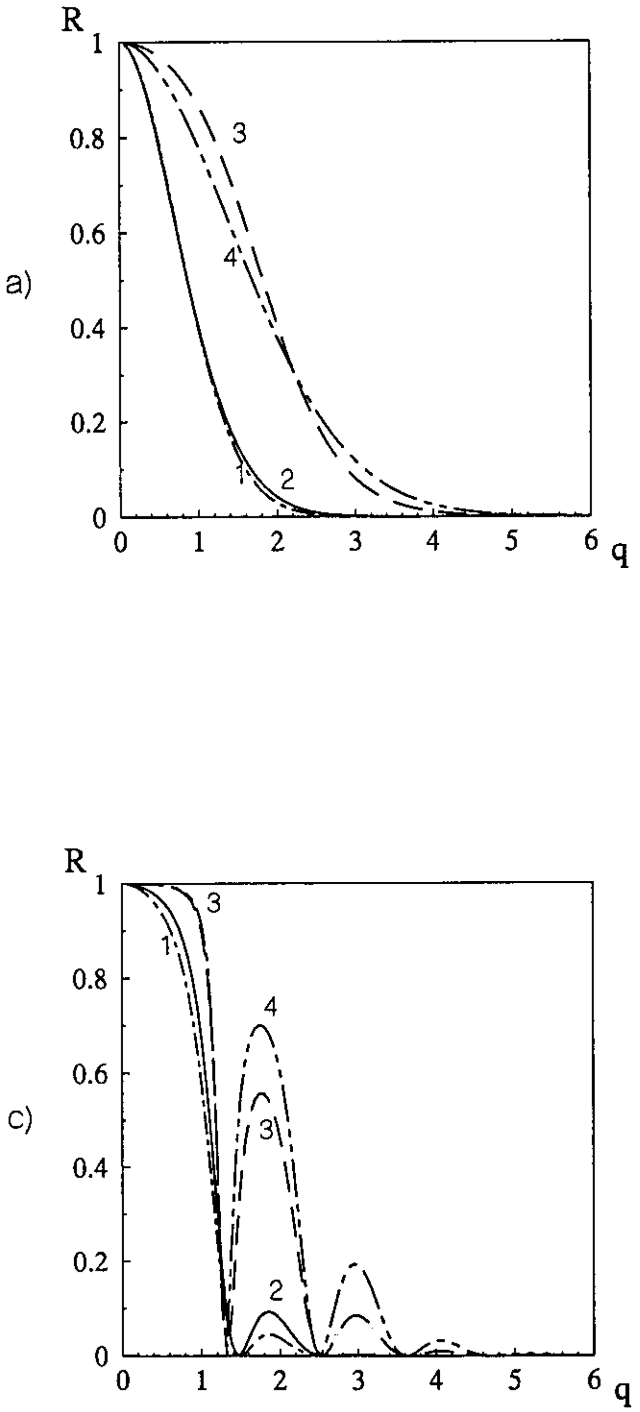


Figure 1: Reflectivity as a function of rapidity q at $V_1 = V_2 = -0.5$: a) $a = 0.1$; c) $a = 1.5$.

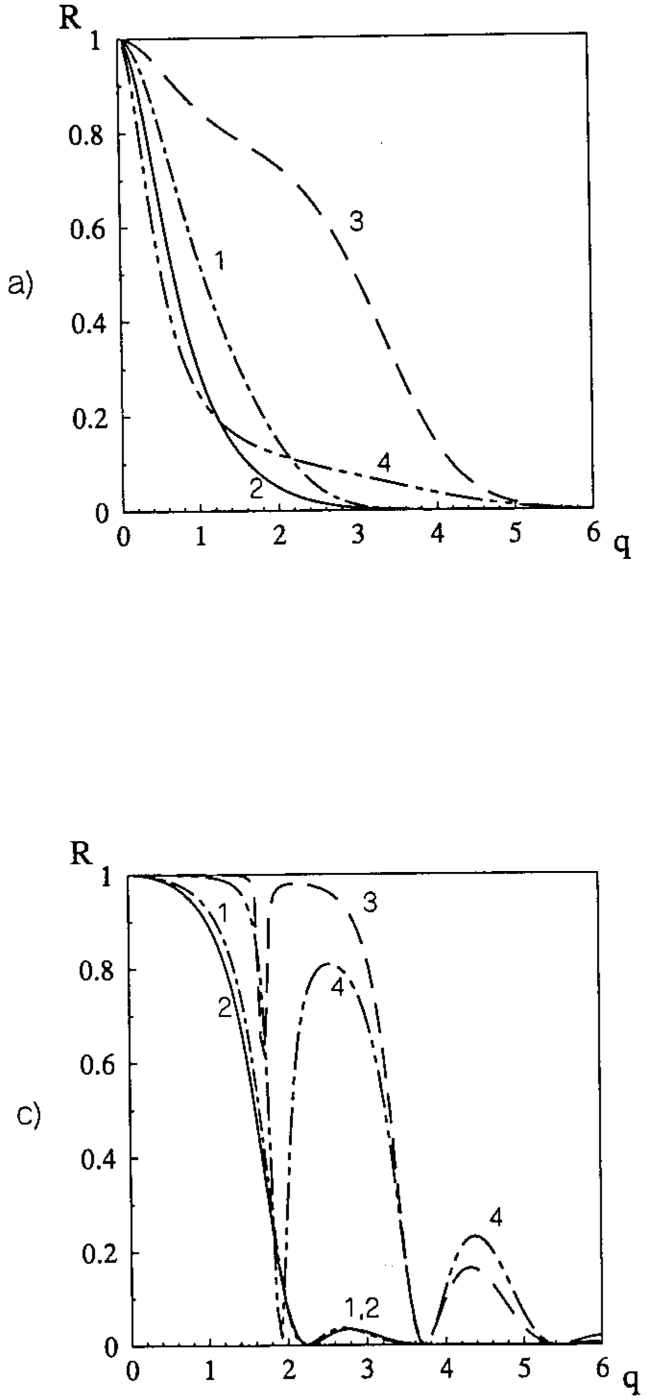


Figure 2: Reflectivity as a function of rapidity q at $V_1 = V_2 = -3$: a) $a = 0.1$; c) $a = 1$.

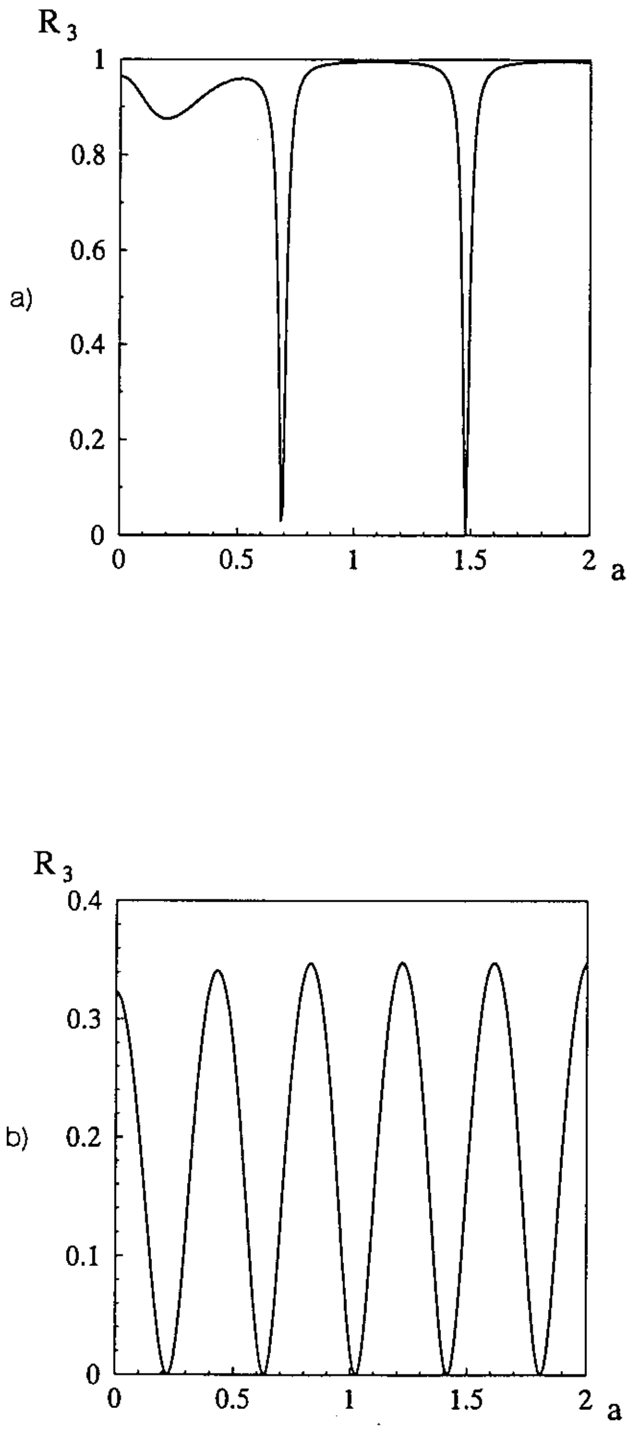
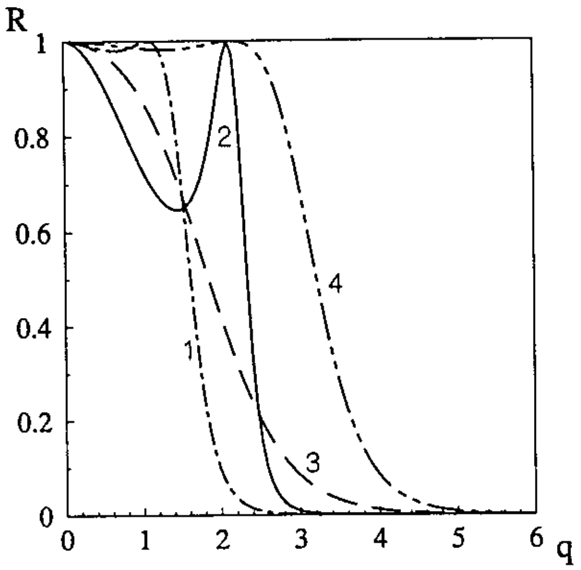


Figure 3: Reflectivity as a function of width a at $V_1 = V_2 = 3$: a) $q = 2$; b) $q = 4$.

a)



c)

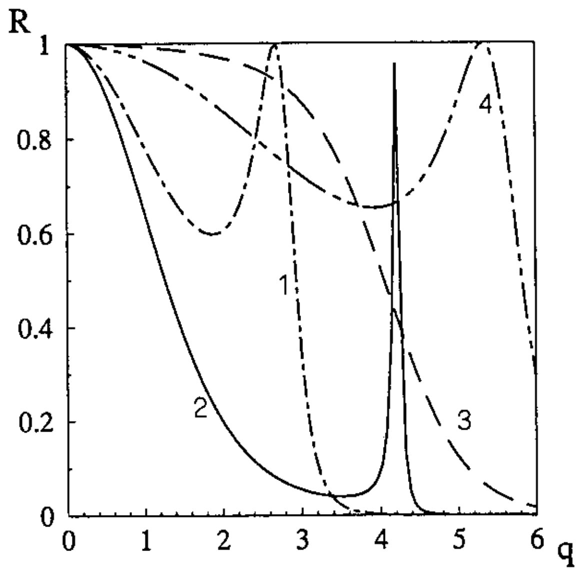
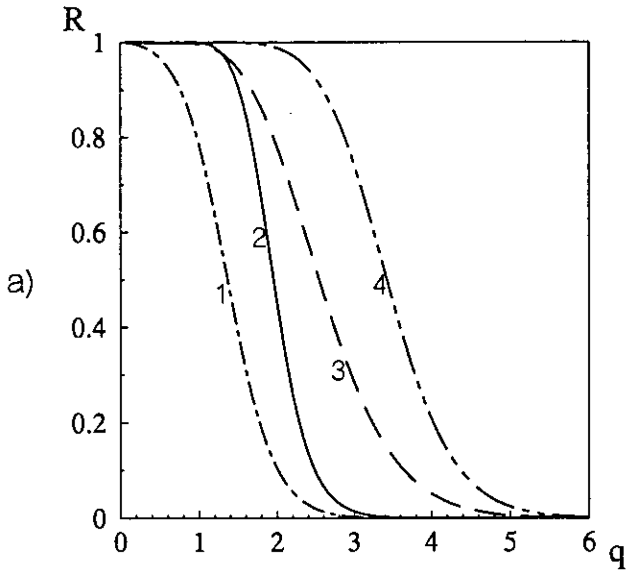


Figure 4: Reflectivity as a function of particle momentum at $a = 0.05$, $V_1 = V_2 = V_0$: a) $V_0 = 0.5$; c) $V_0 = 5$.



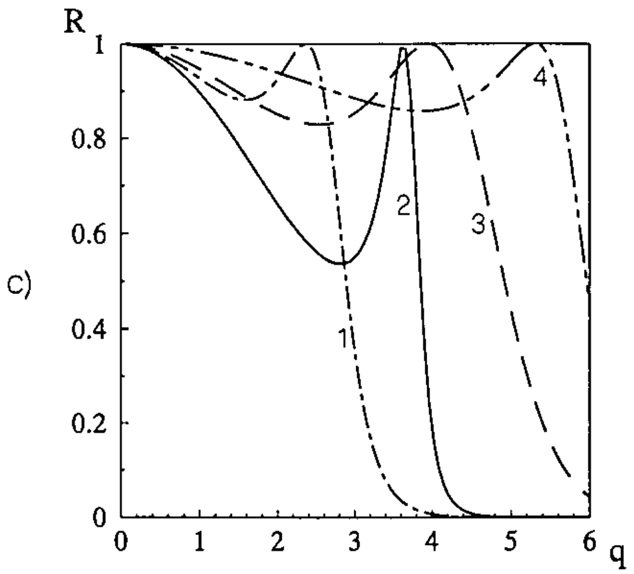


Figure 5: Reflectivity as a function of particle momentum at $a = 0.2$, $V_1 = -V_2 = V_0$: a) $V_0 = 1$; c) $V_0 = 10$.