МАТЕМАТИКА

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О КОНЕЧНЫХ ПОЛУ-π-СПЕЦИАЛЬНЫХ ГРУППАХ

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ON FINITE SEMI-π-SPECIAL GROUPS

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Конечная группа G называется π*-специальной*, если $G = O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G)$, где $\pi = \{p_1,..., p_n\}$. Мы говорим, что конечная группа G является *полу-*π*-специальной*, если нормализатор любой ненормальной π-специальной подгруппы группы G является π-специальной. Доказано, что если G не является π-специальной группой, но $N_G(A)$ является π-специальным для каждой подгруппы A в G такой, что A является либо π'-группой, либо p-группой для некоторой $p\in\pi$, тогда справедливы следующие утверждения: (i) $G/F(G)$ *является* π*-специальной группой. Следовательно, G имеет холлову* π' *-подгруппу H и разрешимую холлову* π *-подгруппу E.* (ii) *Если G не является p-замкнутой для каждого* $p\in\pi$, *то*: (1) *H нормальна в G и E нильпотентна.* (2) $O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times H$ *является максимальной* π *-специальной подгруппой в G и каждая минимальная нормальная подгруппа группы G содержится в* $F(G)$.

Ключевые слова*: конечная группа*, π*-специальная группа*, π*-разрешимая группа*, *силова подгруппа*, *холлова подгруппа*.

A finite group G is called π*-special* if $G = O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G)$, where $\pi = \{p_1,..., p_n\}$. We say that a finite group G is *semi-*π*-special* if the normalizer of every non-normal π-special subgroup of G is π-special. We prove that if G is not π-special but $N_G(A)$ is π-special for every subgroup A of G such that A is either a π'-group or a p-group for some $p\in\pi$, then the following statements hold: (i) $G/F(G)$ *is* π*-special. Hence G has a Hall* π' *-subgroup H and a soluble Hall* π *-subgroup E.* (ii) *If G is not p-closed for each* $p\in\pi$, *then*: (1) *H is normal in G and E is nilpotent.* (2) $O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times H$ *is a maximal* π *-special subgroup of G and every minimal normal subgroup of G is contained in* $F(G)$.

Keywords: *finite group*, π*-soluble group*, π*-special group*, *Sylow subgroup*, *Hall subgroup*.

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $p\in\pi\subseteq\mathbb{P}$ and $\pi' = \mathbb{P}\setminus\pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. Throughout this paper, all groups

A group G is called π*-special* [1]–[3] if

$$G = O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times O_{\pi'}(G),$$

where $\pi = \{p_1,..., p_n\}$.

Recall that the group G is called *semi-nilpotent* [4] if the normalizer of every non-normal nilpotent subgroup of G is nilpotent. We say, by analogy with it, that G is *semi-*π*-special* if the normalizer of every non-normal π-special subgroup of G is π-special.

Remark **0.1.** We show that G is semi-π-special if and only if the normalizer of every non-normal subgroup A of G which is either a p-group for some $p\in\pi$ or a π'-group is π-special. Since every such a subgroup is π-special, it is enough to show that if the normalizer of every non-normal subgroup A of G which is either a p-group for some $p\in\pi$ or a π'-group is π-special, then G is semi-π-special.

Let H be any non-normal π-special subgroup of G. Then

$$H = O_{p_1}(H)\times\cdots\times O_{p_n}(H)\times O_{\pi'}(H),$$

where $\pi = \{p_1,..., p_n\}$, and

$$N_G(H) = \\ = N_G(O_{\pi'}(H))\cap N_G(O_{p_1}(H)\cap\cdots\cap N_G(O_{p_n}(H)).$$

Moreover, since H is non-normal in G, at least one of the subgroups $O_{\pi'}(H), O_{p_1}(H),..., O_{p_n}(H)$ is not normal in G. But then at least one of the subgroups $N_G(O_{\pi'}(H)), N_G(O_{p_1}(H)),..., N_G(O_{p_n}(H))$ is π-special and so $N_G(H)$ is π-special.

The structure of semi-nilpotent groups is well-known (see [4] or [5, Chapter 4, Section 7]). In this paper we prove the following

***Theorem* 0.2.** *Suppose that G is not π-special but $N_G(A)$ is π-special for every subgroup A of G such that A is either a π'-group or a p-group for some $p \in \pi$. Then the following statements hold*:

(i) *$G/F(G)$ is π-special. Hence G has a Hall π'-subgroup H and a soluble Hall π-subgroup E.*

(ii) *If G is not p-closed for each $p \in \pi$, then*:

(1) *H is normal in G and E is nilpotent.*

(2) *$O_{p_1}(G)\times\cdots\times O_{p_n}(G)\times H$ is a maximal π-special subgroup of G and every minimal normal subgroup of G is contained in $F(G)$.*

In the case when $\pi = \mathbb{P}$ we get from Theorem 0.2 the following known result.

***Corollary* 1.3** (Sah [4]). *If G is semi-nilpotent, then $G/F(G)$ is nilpotent.*

In the case when $\pi = \{p\}$ we get from Theorem 1.2 the following known result.

***Corollary* 0.4** (Adarchenko, Blisnets, Rizhik [6]). *Suppose that $N_G(A)$ is p-decomposable for every subgroup A of G such that either A is either a p-group or a p'-group. If a Sylow p-subgroup P of G is not normal in G, then the following conditions hold*:

(i) *G is p-soluble and G has a normal Hall p'-subgroup H.*

(ii) *$G/F(G)$ is p-decomposable.*

1 Preliminaries

The first lemma can be proved by direct calculations.

***Lemma* 1.1.** *Let $\mathfrak{F}$ be the class of all π-special groups. Then*:

(1) *If $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$ for every normal subgroup N of G.*

(2) *If $G \in \mathfrak{F}$, then $E \in \mathfrak{F}$ for every subgroup E of G.*

(3) *If G/N, $G/L \in \mathfrak{F}$, then $G/N \cap L \in \mathfrak{F}$.*

(4) *If $G/\Phi \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

***Lemma* 1.2.** *Suppose that a group G is π-soluble and let P_i be a Sylow p_i-subgroup of G for all $p \in \pi = \{p_1,\ldots,p_n\}$ and C a Hall π'-subgroup of G. If $N_G(C)$ and $N_G(P_i)$ are π-special for all i, then G is π-special.*

Proof. Let R be a minimal normal subgroup of G. Then R is either a p-group for some prime $p \in \pi$ or a π'-group since G is π-soluble by hypothesis. Moreover, P_iR/R is a Sylow p_i-subgroup of G for all $p \in \pi$, CR/R is a Hall π'-subgroup of G/R and

$$N_G(P_iR/R) = N_G(_i)R/R \simeq N_G(P_i)/(N_G(P_i)\cap P_i)$$

and

$$N_G(CR/R) = N_G(C)R/R \simeq N_G(C)/(N_G(C)\cap C)$$

are π-special by Lemma 1.1 (1). Hence the hypothesis holds for G/R. Therefore

$$G/R = O_{\pi'}(G/R)\times O_{p_1}(G/R)\times\cdots\times O_{p_n}(G/R)$$

is π-special by induction.

Suppose that R is a p_i-group for some $p_i \in \pi$. Then

$$O_{p_i}(G/R) = O_{p_i}(G)/R = P_i/R$$

is normal in G/R, so P_i is normal in G. But then $G = N_G(P_i)$ is π-special by hypothesis. Similarly one can show that $G = N_G(C)$ is π-special in the case when R is a π'-group. □

***Lemma* 1.3** (See [8, Chapter V, Theorem 26.1]). *If G is a Schmidt group, then $G = P \rtimes Q$, where $P = G^{\mathfrak{N}} = G'$ is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G for some primes $p \neq q$.*

***Lemma* 2.4.** *If G is a minimal non-π-special group, then G is a Schmidt group.*

Proof. Assume that this is false and let G be a counterexample of minimal order. Then for some $p \in \pi$ we have $p \in \pi(G)$. Moreover, G is p-nilpotent for all $p \in \pi$. Indeed, if G is not p-nilpotent, then G is a minimal non-p-nilpotent group and so G is a Schmidt group by [9, IV, Satz 5.4], contrary to our assumption on G. Thus $G = V \rtimes H$, where V is a normal Hall π'-subgroup of G and H is a nilpotent Hall π-subgroup of G. Moreover, $|\pi(G)| > 2$ since otherwise, every proper subgroup of G is nilpotent and so G is a Schmidt group.

Now let $q \in \pi(V)$, $p \in \pi(H)$ and let Q be a Sylow q-subgroup of V and P the Sylow p-subgroup of H. Then $G = VN_G(Q)$ by the Frattini argument, so for some $x \in G$ we have $P \leq N_G(Q^x)$. But then $Q^x \rtimes P = Q^x \times P$ since $2 = |\pi(Q^x \rtimes P)| < |\pi(G)|$ and every proper subgroup of G is π-special. Therefore $|G : C_G(P)|$ is a q'-number for every $q \in \pi(V)$. Hence $G = V \times H$ is π-special, a contradiction. □

2 Proof of Theorem 0.2

Assume that this theorem is false and let G be a counter example of minimal order. Then G is not π-special. Hence $D := G^{\mathfrak{F}} \neq 1$, where $\mathfrak{F}$ is the class of all π-special groups.

(1) *Every proper subgroup E of G is semi-π-special. Hence Statement* (i) *holds for E.*

Let V be a non-normal π-special subgroup of E. Then V is not normal in G, so $N_G(V)$ is π-special by hypothesis. Hence $N_E(V) = N_G(V) \cap E$ is π-special by Lamma 1.1 (2). Hence E is semi-π-special. Hence we have (1) by the choice of G.

(2) *Every proper quotient* G/N *of G* (*that is*, $N \neq 1$) *is semi-*π*-special. Hence Statement* (*i*) *holds for* G/N.

In view of Remark 0.1 and the choice of *G*, it is enough to show that if U/N is any non-normal subgroup of G/N such that U/N is either a *p*-group for some prime $p \in \pi$ or a π'-group, then $N_{G/N}(U/N)$ is π-special. We can assume without loss of generality that *N* is a minimal normal subgroup of *G*.

Since U/N is not normal in

$$U/N < G/NU/N < G/N$$

and *U* is not normal in *G*. Hence *U* is a proper subgroup of *G*, which implies that *U* is π-soluble by Claim (1). Hence *N* is either a *p*-group for some prime $p \in \pi$ or a p'-group. First suppose that *N* is a π'-group.

If U/N is a π'–group, then *U* is a π'-group and so $N_G(U)$ is π-special by hypothesis. Hence

$$N_{G/N}(U/N) = N_G(U)/N \simeq N_G(U)/(N_G(U) \cap N)$$

is π-special by Lemma 1.1 (1). Now suppose that U/N is a *p*-group for some $p \in \pi$. Then *N* has a complement *V* in *U* and every two complements to *N* in *U* are conjugate in *U* since *U* is π-soluble. Therefore $N_G(U) = N_G(NV) = NN_G(V)$. Since $U = NV$ is not normal in *G*, *V* is not normal in *G* and so $N_G(V)$ is π-special. Hence $N_{G/N}(U/N) = = N_G(U)/N$ is π-special.

(3) *If A is a minimal non-*π*-special subgroup of G*, *then* $A = R \rtimes Q$, *where* $R = A^{\mathfrak{N}} = A'$ *is a Sylow r-subgroup of A and Q is a Sylow q-subgroup of A for some different primes r and q. Moreover*, *R is normal in G and so* $R \leq O_r(G)$.

The first assertion of the claim directly follows from Lemmas 1.3. Since *A* is not π-special, *R* is normal in *G* by hypothesis. Therefore $R \leq O_r(G)$.

(4) *G is* π*-soluble. Hence G has a Hall* π'*-subgroup H and a soluble Hall* π*-subgroup E.*

First we show that *G* is π-soluble. Suppose that this is false. Then *G* is a non-abelian simple group since every proper section of *G* is π-soluble by Claims (1) and (2). Moreover, *G* is not π-special and so it has a minimal non-π-special subgroup *A*. Claim (3) implies that for some prime *r* and for some Sylow *r*-subgroup *R* of *A* we have $1 < R \leq O_r(G) < G$. This contradiction shows that *G* is π-soluble. Hence *G* has a Hall π'-subgroup *H* and a soluble Hall π-subgroup *E*.

(5) *Statement* (*i*) *holds for G.*

In view of Lemma 1.1 (1), it is enough to show that $D = G^{\mathfrak{F}}$ is nilpotent. Assume that this is false. Then $D \neq 1$, and for any minimal normal subgroup *R* of *G* we have that

$$(G/R)^{F} = RD/R \simeq D/D \cap R$$

is nilpotent by Claim (2) and Lemma 1.1 (1). Moreover, *R* is the unique minimal normal subgroup of *G*, $R \leq D$ and $R \nleq \Phi(G)$ by Lemma 1.1 (3, 4).

Since *G* is not π-special, Claim (3) and [7, Ch. A, 15.6] imply that $R = C_G(R) = O_r(G) = F(G)$ for some prime *r*.

Then $R < D$ and $G = R \rtimes M$, where *M* is not π-special, so *M* has a minimal non-π-special subgroup *A*. Claim (3) implies that for some prime *q* dividing $|A|$ and for a Sylow *q*-subgroup *Q* of *A* we have $1 < Q \leq F(G) \cap M = R \cap M = 1$. This contradiction completes the proof of (5).

In what follows, we assume that *G* is not *p*-closed for each $p \in \pi$.

(6) *A Hall* π'*-subgroup H of G is normal in G and a Hall* π*-subgroup E of G is nilpotent. Hence* G/H *is nilpotent.*

Since P_i is not normal in *G* for all *i* by hypothesis, $N_G(P_i)$ is π-special for all *i*. Therefore, since *G* is not π-special, Lemma 1.2 implies that *H* is normal in *G*. By hypothesis, $N_E(P)$ is π-special for every Sylow *p*-subgroup of *E* and every $p \in \pi$ by our assumption of *G*. Therefore *E* is nilpotent. Hence we have (6).

(7) $H \times O_{p_1}(G) \times \cdots \times O_{p_n}(G) = H \times O_{p_1}(V) \times \cdots \times O_{p_n}()$ *for every subgroup V of G containing*

$$H \times O_{p_1}(G) \times \cdots \times O_{p_n}(G).$$

Hence $O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times H$ *is a maximal* π*-special subgroup of G.*

Indeed, since $H = O_{\pi'}(G)$ and $G/H \simeq E$ is nilpotent by Claim (6), every subgroup of *G* containing $H \times O_{p_1}(G) \times \cdots \times O_{p_n}(G)$ is subnormal in *G*. Therefore *V* is subnormal in *G*, so

$$O_{\pi'}(G) \times O_{p_1}(G) \times \cdots \times O_{p_n}(G) =$$
$$= H \times O_{p_1}(G) \times \cdots \times O_{p_n}(G) \leq$$
$$\leq O_{\pi'}(V) \times O_{p_1}(V) \times \cdots \times O_{p_n}(V) \leq$$
$$\leq O_{\pi'}(G) \times O_{p_1}(G) \times \cdots \times O_{p_n}(G).$$

Thus we have (7).

(8) *Every minimal normal subgroup of G is contained in* $F(G)$.

Let *R* be any minimal normal subgroup of *G*. Assume that $R \nleq F(G)$. From Claim (5) it follows that $D = G^{\mathfrak{F}} \leq F(G)$, so $R \nleq D$ and hence from the *G*-isomorphism $RD/D \simeq R$ it follows that *R* is a non-abelian π'-group. Let R_p be a Sylow *p*-subgroup of *R*, where $p \in \pi(R)$. Then R_p is not normal in *G* and so $G = RN_G(R_p)$ by the Frattini argument. But then

$$G / R \simeq N_G(R_p) / (N_G(R_p) \cap R)$$

is π-special and hence $R = D$. This contradiction completes the proof of Claim (8)

Final contradiction. From Claims (5)–(8) it follows that that the conclusion of the theorem holds for *G*, contrary to the choice of *G*. This final contradiction completes the proof of the theorem. □

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