

On permutizers of subgroups of finite groups

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Dedicated to Professor Victor D. Mazurov
in honor of his 70th birthday

Abstract

Let H be a subgroup of a group G . The permutizer of H in G is the subgroup $P_G(H) = \langle x \in G \mid \langle x \rangle H = H \langle x \rangle \rangle$. A subgroup H of a group G is called permuteral in G , if $P_G(H) = G$, and strongly permuteral in G , if $P_U(H) = U$ whenever $H \leq U \leq G$. Finite groups with given systems of permuteral and strongly permuteral subgroups are studied. New characterizations of w-supersoluble and supersoluble groups are received.

Keywords: permutizer of a subgroup, supersoluble group, w-supersoluble group, \mathbb{P} -subnormal subgroup, permuteral subgroup, strongly permuteral subgroup, product of subgroups

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Introduction

All groups considered in this paper are finite. In the theory of groups the normalizer of a subgroup is a classical concept, about which there are many well-known results. For example,

The following statements about a group G are equivalent:

- (1) G is nilpotent;
- (2) $H < P_G(H)$ for every $H < G$ (the normalizer condition);
- (3) $N_G(M) = G$ for every maximal subgroup M of G (the maximal normalizer condition);
- (4) $N_G(P) = G$ for every Sylow subgroup P of G ;
- (5) $N_G(S) = G$ for every Hall subgroup S of G ;
- (6) $G = AB$ where A and B are nilpotent subgroups of G and $N_G(A) = N_G(B) = G$.

A natural generalization of subgroup's normalizer is the concept of the permutizer of a subgroup introduced in [1].

Definition 1 [1, p. 27]. Let $H \leq G$. The permutizer of H in G is the subgroup $P_G(H) = \langle x \in G \mid \langle x \rangle H = H \langle x \rangle \rangle$.

Replacing in (2)–(6) the normalizer of a subgroup by its permutizer, we obtain the following interesting problems.

Problem 1. Describe all groups G , satisfying the permutizer condition, i.e. $G : H < P_G(H)$ for every $H < G$.

This problem was investigated by W.E. Deskins and P. Venzke [1, pp. 27–29], J. Zhang [2], J.C. Beidleman and D.J.S. Robinson [3], A. Ballester-Bolinches and R. Esteban-Romero [4] and others.

Problem 2. *Describe all groups G , satisfying the maximal permutizer condition, i.e. $G : N_G(M) = G$ for every maximal subgroup M of G .*

This problem was considered by W.E. Deskins and P. Venzke [1, pp. 27–29], X. Liu and Ya. Wang [5], Sh. Qiao, G. Qian and Ya. Wang [6] and others.

In order to briefly formulate assertions (4)–(6) in terms of the permutizers of subgroups, we need to introduce the following definition.

Definition 2. *Let H be a subgroup of a group G . We say that*

- (1) *H is permuteral in G , if $P_G(H) = G$;*
- (2) *H is strongly permuteral in G , if $H \leq U \leq G$ then $P_U(H) = U$.*

There exists groups which have permuteral but not strongly permuteral subgroups. For example, it's easy to check that in the group $G = PSL(2, 7)$ a Sylow 3-subgroup Z_3 is permuteral in G . Since $Z_3 \leq U \leq G$, where U is isomorphic to the alternating group A_4 of degree 4 and $P_U(Z_3) = Z_3$, then Z_3 is not strongly permuteral in G .

We note the following problems.

Problem 3. *Describe all groups G such that:*

- a) *every Sylow (Hall) subgroup of G is permuteral in G ;*
- b) *every Sylow (Hall) subgroup of G is strongly permuteral in G .*

Problem 4. *Describe all groups $G = AB$ where A and B are permuteral (strongly permuteral) nilpotent subgroups of G .*

This paper is devoted the solution of the problems 3 and 4.

1. Preliminary results

We use the notation and terminology from [7, 8]. We recall some concepts significant in the paper.

Let G be a group. If H is a subgroup of G , we write $H \leq G$ and if $H \neq G$, we write $H < G$. We denote by $|G|$ the order of G ; by $\pi(G)$ the set of all distinct prime divisors of the order of G ; by $\text{Syl}_p(G)$ the set of all Sylow p -subgroups of G ; by $\text{Syl}(G)$ the set of all Sylow subgroups of G ; by $\text{Core}_G(M)$ the core of subgroup M in G , i.e. intersection of all subgroups conjugated with M in G ; by $F(G)$ the Fitting subgroup of G ; by $F_p(G)$ the p -nilpotent radical of G , i.e. the product of all normal p -nilpotent subgroups of G . Z_n denotes cyclic group of order n ; \mathbb{P} denotes the set of all primes; π denotes a set of primes; $\pi' = \mathbb{P} \setminus \pi$; \mathfrak{S} denotes the class of all soluble groups; \mathfrak{U} denotes the class of all supersoluble groups; \mathfrak{N} denotes the class of all nilpotent groups; $\mathfrak{A}(p-1)$ denotes the class of all abelian groups of exponent dividing $p-1$.

A group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ (where p_i is a prime, $i = 1, 2, \dots, n$) is called *Ore dispersive* [7, p. 251], whenever $p_1 > p_2 > \cdots > p_n$ and G has a normal subgroup of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$ for every $i = 1, 2, \dots, n$. A *Carter subgroup* of a group G is called a nilpotent subgroup H that $N_G(H) = H$. A group G is *p -closed* if G has a normal Sylow p -subgroup.

A class of groups \mathfrak{F} is called a *formation* if the following conditions hold: (a) every quotient group of a group lying in \mathfrak{F} also lies in \mathfrak{F} ; (b) if $H/A \in \mathfrak{F}$ and $H/B \in \mathfrak{F}$ then $H/A \cap B \in \mathfrak{F}$. A formation \mathfrak{F} is called *hereditary* whenever \mathfrak{F} together with every group contains all its subgroups, and *saturated*, if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. Denote by $G^\mathfrak{F}$ the \mathfrak{F} -residual of a group G , i.e. the smallest normal subgroup of G with $G/G^\mathfrak{F} \in \mathfrak{F}$.

A function $f : \mathbb{P} \rightarrow \{\text{formations}\}$ is called a *local function*. A formation \mathfrak{F} is called *local*, if there exists a local function f such that \mathfrak{F} coincides with the class of groups $(G|G/F_p(G) \in f(p) \text{ for every } p \in \pi(G))$.

Lemma 1.1 [7, Lemma 3.9]. *If H/K is a chief factor of a group G and $p \in \pi(H/K)$ then $G/C_G(H/K)$ doesn't contain nonidentity normal p -subgroup and besides $F_p(G) \leq C_G(H/K)$.*

Lemma 1.2 [7, Lemma 1.2]. *Let \mathfrak{F} be a nonempty formation, K be a normal subgroup of a group G . Then $(G/K)^\mathfrak{F} = G^\mathfrak{F}K/K$.*

Theorem 1.3 [8, A, theorem 2.7 (ii)]. *Let G be a soluble group. Then $F(G)/\Phi(G) = C_{G/\Phi(G)}(F(G)/\Phi(G)) = \text{Soc}(G/\Phi(G))$.*

Definition 1.4 [9]. *A subgroup H of a group G is called \mathbb{P} -subnormal in G (denoted by $H \mathbb{P}\text{-sn } G$), if either $H = G$, or there exists a chain of subgroups $H = H_0 < H_1 < \dots < H_{n-1} < H_n = G$ such that $|H_{i+1} : H_i|$ is a prime for every $i = 0, 1, \dots, n-1$.*

Lemma 1.5 [10, Lemmas 3.1 and 3.4]. *Let H be a subgroup of a group G . Then:*

- (1) *If $N \trianglelefteq G$ and $H \mathbb{P}\text{-sn } G$ then $(H \cap N) \mathbb{P}\text{-sn } N$ and $HN/N \mathbb{P}\text{-sn } G/N$.*
- (2) *If $N \trianglelefteq G$, $N \leq H$ and $H/N \mathbb{P}\text{-sn } G/N$ then $H \mathbb{P}\text{-sn } G$.*
- (3) *If $HN_i \mathbb{P}\text{-sn } G$, $N_i \trianglelefteq G$, $i = 1, 2$ then $(HN_1 \cap HN_2) \mathbb{P}\text{-sn } G$.*
- (4) *If $H \mathbb{P}\text{-sn } K$ and $K \mathbb{P}\text{-sn } G$ then $H \mathbb{P}\text{-sn } G$.*
- (5) *If $H \mathbb{P}\text{-sn } G$ then $H^x \mathbb{P}\text{-sn } G$ for every $x \in G$.*
- (6) *If $G^\mathfrak{U} \leq H$ then $H \mathbb{P}\text{-sn } G$.*
- (7) *If G is soluble, $H \mathbb{P}\text{-sn } G$ and K is a subgroup of G then $(H \cap K) \mathbb{P}\text{-sn } K$.*
- (8) *If G is soluble, $H_i \mathbb{P}\text{-sn } G$, $i = 1, 2$ then $(H_1 \cap H_2) \mathbb{P}\text{-sn } G$.*

A group G is called *w-supersoluble* [9], if every Sylow subgroup of G is \mathbb{P} -subnormal in G . Denote by $w\mathfrak{U}$ the class of all w-supersoluble groups. Observe that $\mathfrak{U} \subseteq w\mathfrak{U}$.

The following example [9] shows that $\mathfrak{U} \neq w\mathfrak{U}$. Take the symmetric group S of degree 3. According to [8, Chapter B, Theorem 10.6], there exists a faithful irreducible S -module U over the field F_7 with 7 elements. Consider the semidirect product $G = [U]S$. Since S is a nonabelian group, G is not supersoluble. The supersolubility of G/U implies that $H_1 = UG_2, H_2 = UG_3$, and $H_3 = UG_7 = G_7$ are \mathbb{P} -subnormal subgroups of G , where G_p is a Sylow p -subgroup of G for $p \in \{2, 3, 7\}$. Observe that H_i supersoluble subgroup of G for $i = 1, 2, 3$. Consequently, $G_2 \mathbb{P}\text{-sn } H_1$ and $G_3 \mathbb{P}\text{-sn } H_2$. This implies that $G_p \mathbb{P}\text{-sn } G$ for $p \in \{2, 3, 7\}$, and so $G \in w\mathfrak{U}$.

We present some properties of w-supersoluble groups.

Proposition 1.6 [9, Proposition 2.8]. *Every w-supersoluble group is Ore dispersive.*

Theorem 1.7 [9, Theorems 2.7, 2.10]. *The class $w\mathfrak{U}$ is a hereditary saturated formation and it has a local function f such that $f(p) = (G \in \mathfrak{G} | \text{Syl}(G) \subseteq \mathfrak{A}(p-1))$ for every prime p .*

Theorem 1.8 [9, Theorem 2.13]. *Every biprimary subgroup of a w-supersoluble group is supersoluble.*

Theorem 1.9 [1, I, Theorem 1.4]. *Let H/K be a chief p -factor of a group G . $|H/K| = p$ if and only if $\text{Aut}_G(H/K)$ is abelian group of exponent dividing $p - 1$.*

A subgroup H of a group G is called: 1) *pronormal in G* , if the subgroups H and H^x are conjugated in their join $\langle H, H^x \rangle$ for all $x \in G$; 2) *abnormal in G* , if $x \in \langle H, H^x \rangle$ for all $x \in G$.

Lemma 1.10 [7, Lemma 17.1]. *If a subgroup H is pronormal in G then $N_G(H)$ is abnormal in G .*

Lemma 1.11 [7, Lemma 17.2]. *Let H be a subgroup of a group G . Then the following conditions are equivalent:*

- (1) H is abnormal in G ;
- (2) If $H \leq U \leq G$ and $H \leq U \cap U^x$ implies that $x \in U$;
- (3) H is pronormal in G and $U = N_G(U)$ for $H \leq U \cap U^x$;
- (4) H is pronormal in G and $H = N_G(H)$.

Lemma 1.12 [7, Lemma 17.5]. *Let H be a subgroup of a group G . Then:*

- (1) *If H is pronormal in G and $H \leq U \leq G$, then H is pronormal in U .*
- (2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is pronormal in G if and only if H/N is pronormal in G/N .*
- (3) *If $N \trianglelefteq G$ and H is pronormal in G then HN/N is pronormal in G/N .*

Lemma 1.13 [13, Lemma 2]. *Assume that G is the product of two nilpotent subgroup A u B , and that G possesses in addition a minimal normal subgroup N such that $N = C_G(N) \neq G$. Then*

- (1) $A \cap B = 1$.
- (2) $N \leq A \cup B$.
- (3) *If $N \leq A$ then A is a p -group for some prime p , and B is a p' -group.*

The group G satisfies E_π if G has at least one Hall π -subgroup; it satisfies D_π if G there exists precisely one conjugacy class of Hall π -subgroups and if every π -subgroup of G is contained in a Hall π -subgroup of G .

Lemma 1.14 [14]. *Let $G = AB$ be a group satisfying D_π and suppose both A and B satisfy E_π . Then there exists Hall π -subgroups A_π and B_π of A and B respectively such that $A_\pi B_\pi = B_\pi A_\pi$ is a Hall π -subgroup of G .*

2. Properties of permutizers of subgroups

Lemma 2.1. *Let H be a subgroup of a group G . Then:*

- (1) $P_U(H) \leq P_G(H)$ for every subgroup U of G such that $H \leq U$.
- (2) $P_G(H)^g = P_G(H^g)$ for every $g \in G$.
- (3) $N_G(H) \leq P_G(H)$.
- (4) *If $N \trianglelefteq G$ then $P_G(H)N/N \leq P_{G/N}(HN/N)$.*
- (5) *If $N \trianglelefteq G$ and $N \leq H$ then $P_{G/N}(H/N) = P_G(H)/N$.*

Proof. (1) follows from the definition of $P_G(H)$.

(2) Let $g \in G$. Suppose that $P_G(H) = \langle L \rangle$, where $L = \{x \in G \mid \langle x \rangle H = H \langle x \rangle\}$, and $P_G(H^g) = \langle K \rangle$, where $K = \{y \in G \mid \langle y \rangle H^g = H^g \langle y \rangle\}$. Clearly, $P_G(H)^g = \langle L^g \rangle$.

Let's take any $z \in L^g$. Then $z = x^g$ for some $x \in L$. From $\langle x^g \rangle H^g = \langle x \rangle^g H^g = (\langle x \rangle H)^g = (H \langle x \rangle)^g = H^g \langle x^g \rangle$ we obtain that $z \in K$. Hence $L^g \subseteq K$.

We consider any $y \in K$. From $y^{g^{-1}} \in K^{g^{-1}}$ we have $\langle y^{g^{-1}} \rangle H = \langle y \rangle^{g^{-1}} (H^g)^{g^{-1}} = (\langle y \rangle H^g)^{g^{-1}} = (H^g \langle y \rangle)^{g^{-1}} = H \langle y^{g^{-1}} \rangle$. Hence $y^{g^{-1}} \in L$ and $K \subseteq L^g$.

Thus $P_G(H)^g = \langle L^g \rangle = \langle K \rangle = P_G(H^g)$.

Statements (3)–(5) follows from Lemma 2.4 [6]. \square

It is easy to verify the following result.

Lemma 2.2. *Let H be a subgroup of a group G and $N \trianglelefteq G$. Then:*

- (1) *If H is permuteral in G then HN/N is permuteral in G/N .*
- (2) *If H is permuteral in G then HN is permuteral in G .*
- (3) *If $N \leq H$ then H is permuteral in G if and only if H/N is permuteral in G/N .*
- (4) *If H is strongly permuteral in G then HN/N is strongly permuteral in G/N .*

Lemma 2.3. *Let $G = HQ$ be a group, where $H \in \text{Syl}_p(G)$, p is the largest prime divisor of $|G|$, Q is a cyclic subgroup of G . Then G is p -closed.*

Proof. Let G be a group of minimal order for which the lemma is false. Since G is a product of nilpotent subgroups, by the Theorem of Kegel-Wilandt [11], [12] G is soluble. Let N be a minimal normal subgroup of G . Then G/N p -closed. Since the class of all p -closed groups is a saturated formation, N is an unique minimal normal subgroup of G , $\Phi(G) = 1$. Then there exists a maximal subgroup M of G such that $G = NM$, where $M \cap N = 1$, $\text{Core}_G(M) = 1$ and $N = C_G(N)$. If N is a p -group then $HN/N = H/N \in \text{Syl}_p(G/N)$. Hence $H \trianglelefteq G$, a contradiction. Let N be a q -group, $q \neq p$. In view of Sylow Theorem $H^g \leq M$ for some $g \in G$ and $N \leq Q$. Then $|N| = q$. Hence $M \simeq G/C_G(N)$ can be embedded in $\text{Aut}(Z_q) \simeq Z_{q-1}$, a contradiction with $p > q$. \square

Lemma 2.4. *Let $H \in \text{Syl}_p(G)$ and p be the largest prime divisor of $|G|$. If H is permuteral in G then G p -closed.*

Proof. Let x be an arbitrary element of a group G such that $\langle x \rangle H = H \langle x \rangle$. Then $\langle x \rangle H$ is a subgroup of G . By Lemma 2.3 $H \trianglelefteq \langle x \rangle H$. So $\langle x \rangle \leq N_G(H)$ and $G = P_G(H) \leq N_G(H)$. \square

Lemma 2.5. *If every Sylow subgroup of a group G is permuteral in G then G is Ore dispersive.*

Proof. We prove the lemma by using induction on $|G|$. We may assume that $|\pi(G)| > 1$. Let $|G| = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $p_1 > p_2 > \cdots > p_k$, p_i are primes, $i = 1, 2, \dots, k$. For $P_1 \in \text{Syl}_{p_1}(G)$ by Lemma 2.4 $P_1 \trianglelefteq G$. Every Sylow p_i -subgroup of a quotient group G/P_1 has the form $P_i P_1 / P_1$, where $P_i \in \text{Syl}_{p_i}(G)$, $i = 2, \dots, k$. In view of Lemma 2.2(1) $P_i P_1 / P_1$ is permuteral in G/P_1 . By induction G/P_1 is Ore dispersive. Hence G is Ore dispersive. \square

Lemma 2.6. *If G is a supersoluble group then every pronormal subgroup of G is strongly permuteral in G .*

Proof. In view of the heredity of \mathfrak{U} and Lemma 1.12(1) it suffices to prove that every pronormal subgroup of $G \in \mathfrak{U}$ is permuteral in G .

Let G be a supersoluble group of minimal order such that $P_G(H) \neq G$ for some pronormal subgroup H of G .

We suppose that $\Phi = \Phi(G) \neq 1$. Then $G/\Phi \in \mathfrak{U}$, by Lemma 1.12(3) $H\Phi/\Phi$ is pronormal in G/Φ . By the choice of G it follows $P_{G/\Phi}(H\Phi/\Phi) = G/\Phi$. In view of Lemma 2.1(5) we conclude that $P_G(H\Phi) = G$. Since $P_G(H) \neq G$, there exists an element $x \in G$ such that $x \notin P_G(H)$ and $\langle x \rangle H\Phi = H\Phi \langle x \rangle$. Then $R = \langle x \rangle H\Phi$ is a subgroup of G . If $R \neq G$ then the choice of G implies that $P_R(H) = R$. So $x \in R = P_R(H) \leq P_G(H)$, a contradiction. So $R = \langle x \rangle H\Phi = G = \langle x \rangle H$. Hence $x \in P_G(H)$, a contradiction.

Thus $\Phi(G) = 1$. Group $G \in \mathfrak{U}$, so its commutator subgroup $G' \in \mathfrak{N}$. The choice of G implies that $N_G(H) \neq G$. The abnormality of $N_G(H)$ in G implies $G = G'N_G(H) = F(G)N_G(H)$. By Theorem 1.3 $F(G) = N_1 \cdots N_k$, where N_i is a minimal normal subgroup of G and $i = 1, \dots, k$. The supersolvability of G implies N_i is a cyclic subgroup of a prime order. From $N_i H = H N_i$ we get $N_i \leq P_G(H)$ for any $i = 1, \dots, k$. So $F(G) \leq P_G(H)$. But then $G = F(G)N_G(H) \leq P_G(H)$, a contradiction. \square

Corollary 2.6.1. *If G is a supersoluble group then every Sylow subgroup of G is strongly permuteral in G .*

Corollary 2.6.2. *If G is a supersoluble group then every Carter subgroup of G is strongly permuteral in G .*

Corollary 2.6.3. *If G is a supersoluble group then every Hall subgroup of G is strongly permuteral in G .*

The following example shows that there exists supersoluble groups with nonpermuteral subgroups.

Example 2.7. Let $G = (Z_4 \times Z_2)Z_2 = \langle a, b \mid a^4 = b^4 = (ab)^2 = (a^{-1}b)^2 = 1 \rangle$ be a group of order 16. Then the subgroup $H = \langle ba \rangle$ is not permuteral in G , since $P_G(H)$ is an elementary abelian 2-group of order 8.

Lemma 2.8. *Let G be a soluble group. If H is a \mathbb{P} -subnormal Hall subgroup of G then H is strongly permuteral in G .*

Proof. In view of the heredity of \mathfrak{S} and Lemma 1.5(7) it suffices to prove that every \mathbb{P} -subnormal Hall subgroup of $G \in \mathfrak{S}$ is permuteral in G .

Let G be a soluble group of minimal order which has a Hall π -subgroup H , H \mathbb{P} -sn G and $P_G(H) \neq G$. Let N be a minimal normal subgroup of G . Then HN/N is a Hall π -subgroup of G/N . By Lemma 1.5(1) HN/N \mathbb{P} -sn G/N . By the choice of G , the Hall π -subgroup HN/N is permuteral in G/N . By Lemma 2.2(3) HN is permuteral in G . Therefore N is a q -group for some prime $q \notin \pi$. Since H \mathbb{P} -sn G , it follows that there exists a maximal subgroup M in G such that $H \leq M$ and $|G : M|$ is a prime. By Lemma 1.5(7) H \mathbb{P} -sn M . The choice of G implies that $M = P_M(H) \leq P_G(H) \neq G$. Therefore $M = P_G(H)$. Since $G = P_G(HN)$, there exists $x \in G$ such that $\langle x \rangle HN = \langle x \rangle HN$ and $x \notin M$. Hence $P_G(H) = M$ implies that $G = \langle x \rangle HN$. If $N \leq \Phi(G)$ then $G = \langle x \rangle H$. Hence $x \in P_G(H) = M$ that contradicts $x \notin M$.

So $N \not\leq \Phi(G)$. Then there exists a maximal subgroup W in G such that $N \not\leq W$ and $G = NW$. Hence $|G : W|$ is a q -number and $H \leq W^g$ for some $g \in G$. Then $W^g = P_{W^g}(H) \leq P_G(H) = M$ and $G = NM$.

Assume that $HN \neq G$. Then by the choice of G we conclude that $HN = P_{HN}(H) \leq P_G(H) = M$. We get the contradiction $G = NM \leq M \neq G$.

Hence $HN = G$. Since $N \cap M = 1$, it follows that $H = M$. Then $|N| = q$. In view of $HN = NH$ we get $N \leq P_G(H) = M$. Hence $G \leq M \neq G$, a contradiction. \square

Corollary 2.8.1. *If G is a w-supersoluble group then every Sylow subgroup of G is strongly permuteral in G .*

3. Characterizations of w-supersoluble and supersoluble groups

Theorem 3.1. *A group G is w-supersoluble if and only if every Sylow subgroup of G is strongly permuteral in G .*

Proof. Let $\mathfrak{F} = (G \mid \text{any Sylow subgroup of the group } G \text{ is strongly permuteral in } G)$. Corollary 2.8.1 implies that $w\mathfrak{U} \subseteq \mathfrak{F}$.

Let G be a group of minimal order in $\mathfrak{F} \setminus w\mathfrak{U}$. Since $P_G(H) = G$ for every Sylow subgroup H of G , by Lemma 2.5 G is Ore dispersive. Let N be a minimal normal subgroup of G . Take an arbitrary Sylow p -subgroup R/N of G/N . There exists $P \in \text{Syl}_p(G)$ such that $R/N = PN/N$. Since $G \in \mathfrak{F}$, by Lemma 2.2(4) it follows $G/N \in \mathfrak{F}$. By the choice of G the quotient group $G/N \in w\mathfrak{U}$. Since $w\mathfrak{U}$ is a saturated formation by Theorem 1.7, so N is an unique minimal normal subgroup of G and $\Phi(G) = 1$. Let $|G| = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i are primes, $i = 1, 2, \dots, k$ and $p_1 > p_2 > \cdots > p_k$. Denote by P_i Sylow p_i -subgroup of G , $i = 1, 2, \dots, k$. Then $P_1 \trianglelefteq G$ and $N \leq P_1$. Note that $P_1 \mathbb{P}\text{-sn } G$.

Let's denote $H_i = P_i P_1$ for $i \in \{2, \dots, k\}$.

If $H_i \neq G$ for every $i \in \{2, \dots, k\}$, then $H_i \in \mathfrak{F}$. By the choice of G , H_i is w-supersoluble. The heredity of $w\mathfrak{U}$ implies that $P_i N \in w\mathfrak{U}$. So $P_i \mathbb{P}\text{-sn } P_i N$. Since $G/N \in w\mathfrak{U}$, $P_i N \mathbb{P}\text{-sn } G$ by Lemma 1.5(2). By Lemma 1.5(4) we get $P_i \mathbb{P}\text{-sn } G$. Hence $G \in w\mathfrak{U}$. It contradicts the choice of G .

Hence $H_i = G$ for some $i \in \{2, \dots, k\}$. Since $\Phi(G) = 1$, so $G = MN$, where M is some maximal subgroup of G , $M \cap N = 1$, $N = C_G(N)$. Since $G/N \simeq M$, by Lemma 1.1 it implies $P_1 \cap M = 1$. Therefore $N = P_1$ and $M = P_i$. In view of $P_G(P_i) = G$, there exists an element y of G such that $y \notin P_i$, $\langle y \rangle P_i = P_i \langle y \rangle$. Then $G = \langle y \rangle P_i$. Hence $|N| = p_1$. So G is supersoluble, a contradiction. \square

Recall [8, p. 519] that the nilpotent length of a soluble group G , is the smallest integer l such that $F_l(G) = G$, where the subgroups $F_i(G)$ are determined recursively as $F_0(G) = 1$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$ for all $i \geq 1$. The proposition 2.5 [9] shows that the nilpotent length of a w-supersoluble group cannot be bounded by a fixed integer n . Since the commutator subgroup of every supersoluble group is nilpotent, the nilpotent length of a supersoluble group is at most 2, i.e. every supersoluble group is metanilpotent.

Theorem 3.2. *Let G be a metanilpotent group. Then the following statements are equivalent:*

- (1) G is supersoluble;
- (2) Every Sylow subgroup of G is strongly permuteral in G ;
- (3) Every Sylow subgroup of G is permuteral in G .

Proof. (1) \Rightarrow (2) by Corollary 2.6.1.

(2) \Rightarrow (3) obviously.

(3) \Rightarrow (1). Let $\mathfrak{F} = (G \mid \text{the group } G \text{ is metanilpotent and every Sylow subgroup of } G \text{ is permuteral in } G)$.

Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{U}$. The nilpotency of \mathfrak{N} -residual $G^{\mathfrak{N}}$ and the quotient group $G/G^{\mathfrak{N}}$ implies the solvability of G . Let N be a minimal normal subgroup

of G . By Lemma 1.2 and the nilpotency of $G^{\mathfrak{N}}$ implies that $(G/N)^{\mathfrak{N}} = G^{\mathfrak{N}}N/N \simeq G^{\mathfrak{N}}/G^{\mathfrak{N}} \cap N$ is nilpotent. For $R/N \in \text{Syl}_p(G/N)$ (p is any prime of $\pi(G)$) there exists $P \in \text{Syl}_p(G)$ such that $R/N = PN/N$. In view of Lemma 2.2(1), R/N is permuteral in G/N . By the choice of G , we get G/N is supersoluble. From the saturation of the formation \mathfrak{A} of all supersoluble groups, we conclude that N is an unique minimal normal subgroup of G , $\Phi(G) = 1$. Then $G = NM$, where M is a maximal subgroup of G , $N \cap M = 1$, $\text{Core}_G(M) = 1$. Since N is an elementary abelian p -group for some $p \in \pi(G)$ and $N = C_G(N)$, by using of Lemma 1.1, we get $N = G^{\mathfrak{N}} = F(G)$ and $G/N \simeq M$ is nilpotent p' -group. By Lemma 2.5, G is Ore dispersive. Then for $|G| = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ (p_i are primes, $i = 1, 2, \dots, k$ and $p_1 > p_2 > \cdots > p_k$), Sylow p_1 -subgroup $P_1 \trianglelefteq G$. Hence $p = p_1$ and $N = P_1$.

We will fix $i \in \{2, \dots, k\}$. Let $P_i \in \text{Syl}_{p_i}(G)$ and $P_i \leq M$. Since $P_G(P_i) = G$, there exists $x \in G$ such that $\langle x \rangle P_i = P_i \langle x \rangle$ and $x \notin M$.

If $\langle x \rangle$ is a p_1' -group then $\langle x \rangle P_i$ is also a p_1' -group. The solvability of G implies that $\langle x \rangle P_i \leq M^g$ for some $g \in G$. Then $\langle x \rangle^{g^{-1}} P_i^{g^{-1}} \leq M$. Since $P_i^{g^{-1}}$ is a Sylow p_i -subgroup of the nilpotent group M , we get $P_i^{g^{-1}} = P_i$. Hence $g^{-1} \in N_G(P_i) = M$. Therefore $g \in M$. Hence $x \in \langle x \rangle P_i \leq M^g = M$. We get a contradiction with $x \notin M$.

Thus $\langle x \rangle$ is not a p_1' -group. Let $\langle z \rangle \in \text{Syl}_{p_1}(\langle x \rangle)$. Clearly, $\langle z \rangle \in \text{Syl}_{p_1}(\langle x \rangle P_i)$ and $\langle z \rangle = P_1 \cap \langle x \rangle P_i \trianglelefteq \langle x \rangle P_i$. So $\langle z \rangle P_i$ is a subgroup of $\langle x \rangle P_i$. Since $\langle z \rangle \leq P_1$ and P_1 is an elementary abelian p_1 -group, we conclude $|\langle z \rangle| = p_1$.

Let's denote $R_i = \langle z \rangle P_i$. The subgroup P_i is maximal in R_i . Since $N_G(P_i) = M$, it follows $\langle z \rangle \not\leq N_{R_i}(P_i)$. Therefore $N_{R_i}(P_i) = P_i$. We will show $C_i = C_{R_i}(\langle z \rangle) \cap P_i = 1$. Assume that $C_i \neq 1$. Then $\langle z \rangle \leq N_G(C_i)$ and $P_i \leq N_{R_i}(C_i) \leq N_G(C_i)$. Since M is nilpotent, we get $P_j \leq N_G(C_i)$ for any $j \in \{2, \dots, k\}, j \neq i$. So $M \leq N_G(C_i)$ and $M \neq N_G(C_i)$. Hence $C_i \trianglelefteq G$. Therefore $1 \neq C_i \leq \text{Core}_G(M) = 1$, a contradiction. So $C_i = C_{R_i}(\langle z \rangle) \cap P_i = 1$. Then $P_i \simeq R_i / \langle z \rangle = R_i / C_{R_i}(\langle z \rangle)$ can be embedded in $\text{Aut}(Z_{p_1}) \simeq Z_{p_1-1}$.

Thus $P_i \in \mathfrak{A}(p_1 - 1)$ for all $i \in \{2, \dots, k\}$. Hence the nilpotency of M implies that $M \in \mathfrak{A}(p_1 - 1)$. Since $M \simeq G/N = G/C_G(N)$, $|N| = p_1$ by Theorem 1.9. So G is supersoluble, a contradiction. \square

Theorem 3.3. *Let G be a group. Then the following statements are equivalent:*

- (1) G is supersoluble;
- (2) Every pronormal subgroup of G is strongly permuteral in G ;
- (3) Every pronormal subgroup of G is permuteral in G ;
- (4) Every Hall subgroup of G is strongly permuteral in G ;
- (5) Every Hall subgroup of G is permuteral in G .

Proof. (1) \Rightarrow (2) by Lemma 2.6.

(2) \Rightarrow (3) obviously.

(2) \Rightarrow (4). Since every Sylow subgroup of G is pronormal in G , by (2) and Lemma 2.5 G is solvable. Then every Hall subgroup of G is pronormal in G and by (2) is strongly permuteral in G .

(4) \Rightarrow (5) obviously.

(3) \Rightarrow (5). By (3) and Lemma 2.5 G is soluble. Then every Hall subgroup of G is pronormal in G and by (3) is permuteral in G .

(5) \Rightarrow (1). Let $\mathfrak{F} = (G \mid \text{every Hall subgroup of } G \text{ is permuteral in } G)$.

Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{U}$. Then every Sylow subgroup of G is permuteral in G . By Lemma 2.5, G is Ore dispersive. Let N be a minimal normal subgroup of G . For a set of primes π take any Hall subgroup π -subgroup K/N of G/N . The solvability of G implies that $K/N = SN/N$ for some Hall π -subgroup S of G . Since $G \in \mathfrak{F}$, by Lemma 2.2(1), we conclude $K/N = SN/N$ is permuteral in G/N . By the choice of G , it follows that $G/N \in \mathfrak{U}$. Since \mathfrak{U} is a saturated formation, N is an unique minimal normal subgroup of G , $\Phi(G) = 1$. Then there exists a maximal subgroup M in G such that $G = MN$, $M \cap N = 1$, $N = C_G(N)$. In view of $P \trianglelefteq G$ for $P \in \text{Syl}_p(G)$ (p is the largest prime divisor of $|G|$) we get $N \leq P$. Since $P \cap M \trianglelefteq M$, by Lemma 1.1 it implies that $P \cap M = 1$. Therefore $N = P$ and M is Hall ω -subgroup for $\omega = \pi(G) \setminus \{p\}$. Then M is permuteral in G . So there exists $x \in G$, $x \notin M$ such that $\langle x \rangle M = G$. Then a Sylow p -subgroup of $\langle x \rangle$ is a Sylow p -subgroup of G . Therefore $|N| = p$. Hence $M \simeq G/C_G(N)$ is isomorphic embedded in $\text{Aut}Z_p \simeq Z_{p-1}$. So $G \in \mathfrak{U}$, a contradiction. \square

Corollary 3.3.1 [15]. *If every Hall subgroup of a group G is \mathbb{P} -subnormal in G then G is supersoluble.*

Proof. Since every Sylow subgroup of a group G is \mathbb{P} -subnormal in G , G is soluble by Proposition 1.6. By Lemma 2.8 and Theorem 3.2 we conclude that G is supersoluble.

Theorem 3.4. *Let G be a group. Then the following statements are equivalent:*

- (1) G is supersoluble;
- (2) $G = AB$ is the product of strongly permuteral nilpotent subgroups A and B of G ;
- (3) $G = AB$ is the product of permuteral nilpotent subgroups A and B of G .

Proof. (1) \Rightarrow (2). If G is supersoluble, then $G = F(G)H$, where H is a Carter subgroup of G . Subgroups $F(G)$ and H are nilpotent. In view of $F(G) \trianglelefteq G$ and Corollary 2.6.2 we conclude that $F(G)$ and H are strongly permuteral in G .

(2) \Rightarrow (3) obviously.

(3) \Rightarrow (1). Let $\mathfrak{F} = (G \mid \text{the group } G = AB \text{ is the product of permuteral nilpotent subgroups } A \text{ and } B \text{ of } G)$.

Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{U}$. By the Theorem of Kegel-Wilandt [11], [12] G is soluble. Let N be a minimal normal subgroup of G . Since $AN/N \simeq A/A \cap N \in \mathfrak{N}$, $BN/N \simeq B/B \cap N \in \mathfrak{N}$, by Lemma 2.2(1) and by the choice of G it follows that $G/N \in \mathfrak{U}$. Therefore N is an unique minimal normal subgroup of G and $\Phi(G) = 1$. There exists a maximal subgroup M in G such that $G = NM$, $N \cap M = 1$, $\text{Core}_G(M) = 1$. $N = C_G(N)$ and N is an elementary abelian p -group for some $p \in \pi(G)$. By Lemma 1.13, if $N \leq A$ then A is a Sylow p -group and B is a Hall p' -group of G .

Consider any $x \in G$ for which $\langle x \rangle B = B \langle x \rangle$ and $x \notin B$. Let $R = \langle x \rangle B$ and $\langle x \rangle = \langle z \rangle \langle y \rangle$, where $\langle z \rangle \in \text{Syl}_p(\langle x \rangle)$ and $\langle y \rangle$ is a Hall p' -subgroup of $\langle x \rangle$. Clearly, $\langle x \rangle$ is not a p' -group. By Lemma 1.14 $\langle y \rangle B$ is a Hall p' -subgroup of R . So $\langle y \rangle \leq B$ and $R = \langle z \rangle B$.

Let $B = P_1 \cdots P_k$, where $P_i \in \text{Syl}_{p_i}(B)$, $i = 1, \dots, k$. Take any $i \in \{1, \dots, k\}$. By Lemma 1.14 for $\pi_i = \{p, p_i\}$ there exists a Hall π_i -subgroup $\langle z \rangle P_i = P_i \langle z \rangle$ in R . Then $\langle x \rangle P_i = \langle z \rangle \langle y \rangle P_i = \langle z \rangle P_i \langle y \rangle = P_i \langle z \rangle \langle y \rangle = P_i \langle x \rangle$. Hence $x \in P_G(P_i)$. Since $B \leq N_G(P_i) \leq P_G(P_i)$, we have $G = P_G(B) \leq P_G(P_i)$, i.e. $G = P_G(P_i)$. Since $A \in \text{Syl}_p(G)$, $P_i \in \text{Syl}_{p_i}(G)$, in view of Lemma 2.1(2) we conclude that $G = P_G(H)$ for every Sylow subgroup H of G .

By Lemma 2.5 G is Ore dispersive. Since N is an unique minimal normal subgroup of G and N is a p -group, we obtain p is the largest prime divisor $|G|$. Then since $N \leq A \in \text{Syl}_p(G)$, it follows that $A \trianglelefteq G$. By Lemma 1.1 $G/C_G(N) = G/N \simeq M$ has not

nonidentity normal p -subgroups. Therefore $A \cap M = 1$ and $N = A$. Since $G = AB = NB$, it follows that B is a maximal subgroup of G . In view of $P_G(B) = G$ and $B \neq G$, there exists $g \in G$ such that $\langle g \rangle B = B \langle g \rangle$ and $g \notin B$. Then $G = \langle g \rangle B$. So A is cyclic and $|A| = p$. Now then $G/C_G(A) = G/A$ can be embedded in $\text{Aut}(Z_p) \simeq Z_{p-1}$. Hence G is supersoluble, a contradiction. \square

Corollary 3.4.1. *Let G be a group, and let $G = AB$ be a product of its Sylow subgroups A and B . Then G is supersolvable if and only if A and B is permuteral in G .*

Corollary 3.4.2. *Let G be a group. Then G is supersoluble if and only if $G = F(G)H$, where H is a permuteral Carter subgroup of G .*

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