A generalization of Hall's theorem on hypercenter

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Abstract

Let σ be a partition of the set of all primes and $\mathfrak{F}$ be a hereditary formation. We described all formations $\mathfrak{F}$ for which the $\mathfrak{F}$-hypercenter and the intersection of weak K-$\mathfrak{F}$-subnormalizers of all Sylow subgroups coincide in every group. In particular the formation of all σ-nilpotent groups has this property. With the help of our results we solve a particular case of L.A. Shemetkov's problem about the intersection of $\mathfrak{F}$-maximal subgroups and the $\mathfrak{F}$-hypercenter. As corollaries we obtained P. Hall's and R. Baer's classical results about the hypercenter. We proved that the non-σ-nilpotent graph of a group is connected and its diameter is at most 3.

KeywordsFinite group; σ-nilpotent group; hereditary formation; K-$\mathfrak{F}$-subnormal subgroup; $\mathfrak{F}$-hypercenter; non-$\mathfrak{F}$-graph of a group.

MSC(2010): Primary 20D25; Secondary 20F17; 20F19.

1 Introduction

Throughout this paper, all groups are finite; G and p always denote a finite group and a prime respectively. The notion of the hypercenter of a group naturally appears with the definition of nilpotency of a group through upper central series. R. Baer [4] introduced and studied the analogue of hypercenter for the class of all supersoluble groups. B. Huppert [18] considered the $\mathfrak{F}$-hypercenter where $\mathfrak{F}$ is a hereditary saturated formation. L.A. Shemetkov [28] extended the notion of $\mathfrak{F}$-hypercenter for graduated formations. The $\mathfrak{F}$-hypercenter for formations of algebraic systems (including finite groups) was suggested in [31].

Recall that a chief factor H/K of G is called $\mathfrak{X}$-*central* (see [31, p. 127–128]) in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{X}$. A normal subgroup N of G is said to be $\mathfrak{X}$-*hypercentral* in G if $N = 1$ or $N \neq 1$ and every chief factor of G below N is $\mathfrak{X}$-central. The symbol $\mathrm{Z}_{\mathfrak{X}}(G)$ denotes the $\mathfrak{X}$-*hypercenter* of G, that is, the product of all normal $\mathfrak{X}$-hypercentral in G subgroups. According to [31, Lemma 14.1] $\mathrm{Z}_{\mathfrak{X}}(G)$ is the largest normal $\mathfrak{X}$-hypercentral subgroup of G. If $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, then $\mathrm{Z}_{\mathfrak{N}}(G)$ is the hypercenter $\mathrm{Z}_{\infty}(G)$ of G.

One of the first characterizations of the hypercenter was obtained by P. Hall [16]. He proved that the hypercenter of a group coincides with the intersection of normalizers of all its Sylow subgroups. P. Schmid [27] proved the analogue of Hall's result in profinite groups. There were generalizations of P. Hall's theorem in terms of intersections of normalizers of π_i-maximal subgroups [23] or Hall π_i-subgroups [17] where π_i belongs to some partition σ of $\mathbb{P}$ (see Corollaries 1.2 and 1.3).

These results are the part of research project in which the $\mathfrak{F}$-hypercenter and its generalizations are used as descriptors for characterising some structural properties of the group. A useful tool that provides a suitable language in this direction is the theory of formations. Nowadays this project is actively developing by many researchers (for example [2, 5, 17] and [15, Chapter 1]). As part of the above mentioned project, the aim of our paper is to describe all hereditary (not necessary saturated) formations $\mathfrak{F}$ for which the analogue of Hall's result holds for the

[1]This document is the results of the research project funded by Belarusian Republican Foundation for Fundamental Research, project No. Φ20P-291.

$\mathfrak{F}$-hypercenter and find the applications of this result. To formulate our results we need the following definitions.

Let $\mathfrak{F}$ be a formation. O.H. Kegel [20] introduced the formation generalization of subnormality. Recall [8, Definition 6.1.4] that a subgroup H of G is called K-$\mathfrak{F}$-*subnormal* in G if there is a chain of subgroups $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ with $H_{i-1} \trianglelefteq H_i$ or $H_i/\mathrm{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, \ldots, n$. Denoted by $H\,K$-$\mathfrak{F}$-$sn\,G$. If $\mathfrak{F} = \mathfrak{N}$, then the notions of K-$\mathfrak{F}$-subnormal and subnormal subgroups coincide.

R.W. Carter [11] and C.J. Graddon [13] studied subnormalizers and $\mathfrak{F}$-subnormalizers respectively. Note that the subnormalizer of a Sylow subgroup always exists and coincides with its normalizer. For an arbitrary subgroup a subnormalizer or $\mathfrak{F}$-subnormalizer may not exist. A. Mann [22] suggested the concept of a weak subnormalizer that always exists but may be not unique. A subgroup T of G is called *a weak subnormalizer* of H in G if H is subnormal in T and if H is subnormal in $M \leq G$ and $T \leq M$, then $T = M$. Here we introduced its generalization.

Definition 1. Let $\mathfrak{F}$ be a formation. We shall call a subgroup T of G *a weak K-$\mathfrak{F}$-subnormalizer* of H in G if H is K-$\mathfrak{F}$-subnormal in T and if H is K-$\mathfrak{F}$-subnormal in $M \leq G$ and $T \leq M$, then $T = M$.

It is clear that a weak K-$\mathfrak{F}$-subnormalizer always exists. Note that the notions of weak subnormalizer and K-$\mathfrak{N}$-subnormalizer coincide. See [12, A, Example 14.12] for an example of a group that has a subgroup without an unique weak subnormalizer.

Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes. According to A.N. Skiba [33], a group G is called *σ-nilpotent* if G has a normal Hall π_i-subgroup for every $i \in I$ with $\pi(G) \cap \pi_i \neq \emptyset$. The class of all σ-nilpotent groups is denoted by $\mathfrak{N}_\sigma$. This class is a very interesting generalization of the class of nilpotent groups and widely studied, applied and are part of the actively developing nowadays σ-method, i.e. the studying the properties of a group that depends on the given partition σ (for example, see [6, 10, 17, 19, 29, 33]). The class $\mathfrak{N}$ of all nilpotent groups coincides with the class $\mathfrak{N}_\sigma$ for $\sigma = \{\{p\} \mid p \in \mathbb{P}\}$.

Recall [8, Example 2.2.12] that $\underset{i\in I}{\times}\mathfrak{F}_{\pi_i} = (G = \underset{i\in I, \pi_i\cap\pi(G)\neq\emptyset}{\times} \mathrm{O}_{\pi_i}(G) \mid \mathrm{O}_{\pi_i}(G) \in \mathfrak{F}_{\pi_i})$ is a hereditary formation where $\mathfrak{F}_{\pi_i}$ is a hereditary formation with $\pi(\mathfrak{F}_{\pi_i}) = \pi_i$ for all $i \in I$. The main result of this paper is

Theorem 1. *Let $\mathfrak{F}$ be a hereditary formation. The following statements are equivalent:*

(1) *The intersection of all weak K-$\mathfrak{F}$-subnormalizers of all cyclic primary subgroups coincides with the $\mathfrak{F}$-hypercenter in every group.*

(2) *The intersection of all weak K-$\mathfrak{F}$-subnormalizers of all Sylow subgroups coincides with the $\mathfrak{F}$-hypercenter in every group.*

(3) *There is a partition $\sigma = \{\pi_i \mid i \in I\}$ of $\mathbb{P}$ such that the $\mathfrak{F}$-hypercenter coincides with the σ-nilpotent-hypercenter in every group.*

(4) *There is a partition $\sigma = \{\pi_i \mid i \in I\}$ of $\mathbb{P}$ such that $\mathfrak{F} = \times_{i\in I}\mathfrak{F}_{\pi_i}$ where $\mathfrak{F}_{\pi_i}$ is a hereditary formation with $\pi(\mathfrak{F}_{\pi_i}) = \pi_i$ and $\mathfrak{F}_{\pi_i}$ coincides with the class of all π_i-groups for all $i \in I$ with $|\pi_i| \geq 2$.*

Remark 1. As follows from [36, Theorem] formations from (4) of Theorem 1 are lattice formations.

Corollary 1.1. *Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition $\mathbb{P}$, G be a group and $\mathcal{M}$ be a set of maximal π_i-subgroups of G, $\pi_i \in \sigma$, such that* *(a) if $H \in \mathcal{M}$, then $H^x \in \mathcal{M}$ for every $x \in G$;*

(b) for every Sylow subgroup P of G there is $H \in \mathcal{M}$ with $P \leq H$.

Then the intersection of normalizers in G of all subgroups from $\mathcal{M}$ is $\mathrm{Z}_{\mathfrak{N}_\sigma}(G)$.

Corollary 1.2 ([23, Corollary 3.7]). *Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition $\mathbb{P}$. The intersection of normalizers of all π_i-maximal subgroups of G, $\pi_i \in \sigma$, is $\mathrm{Z}_{\mathfrak{N}_\sigma}(G)$.*

Corollary 1.3 ([17, Theorem B(ii)]). *Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition $\mathbb{P}$. Assume that a group G posses a set $\mathcal{H}$ of Hall subgroups such that $\mathcal{H}$ contains exactly one Hall π_i-subgroup of G with $\pi_i \cap \pi(G) \neq \emptyset$. Then*

$$\bigcap_{x\in G}\bigcap_{H\in\mathcal{H}} N_G(H^x) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G).$$

Corollary 1.4 (P. Hall [16]). *The intersection of all normalizers of Sylow subgroups is the hypercenter in every group.*

Corollary 1.5. *The intersection of all weak subnormalizers of cyclic primary subgroups is the hypercenter in every group.*

Corollary 1.6 ([23, Theorem 3.1(2)]). *Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition $\mathbb{P}$. A π_i-element belongs to $\mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ iff its permutes with all π_i'-elements of a group G.*

Corollary 1.7 (R. Baer [3, 5, Theorem 1(ii)]). *Let p be a prime. A p-element belongs to $\mathrm{Z}_\infty(G)$ iff its permutes with all p'-elements of a group G.*

2 Preliminaries

The notation and terminology agree with [8] and [12]. We refer the reader to these books for the results about formations.

Recall that a *formation* is a class of groups which is closed under taking epimorphic images and subdirect products. A formation $\mathfrak{F}$ is called *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$; *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(N) \in \mathfrak{F}$ for some normal subgroup N of G.

Lemma 1 ([33, Lemma 2.5]). *The class of all σ-nilpotent groups is a hereditary saturated formation.*

The following two lemmas follow from [8, Lemmas 6.1.6 and 6.1.7].

Lemma 2. *Let $\mathfrak{F}$ be a formation, H and R be subgroups of G and $N \trianglelefteq G$.*

(1) *If H K-$\mathfrak{F}$-sn G, then HN/N K-$\mathfrak{F}$-sn G/N.*

(2) *If H/N K-$\mathfrak{F}$-sn G/N, then H K-$\mathfrak{F}$-sn G.*

(3) *If H K-$\mathfrak{F}$-sn R and R K-$\mathfrak{F}$-sn G, then H K-$\mathfrak{F}$-sn G.*

Lemma 3. *Let $\mathfrak{F}$ be a hereditary formation, H and R be subgroups of G.*

(1) *If H K-$\mathfrak{F}$-sn G, then $H \cap R$ K-$\mathfrak{F}$-sn R.*

(2) *If H K-$\mathfrak{F}$-sn G and R K-$\mathfrak{F}$-sn G, then $H \cap R$ K-$\mathfrak{F}$-sn G.*

The following lemma directly follows from Lemma 2.

Lemma 4. *Let $\mathfrak{F}$ be a formation, H and R be subgroups of G and $N \trianglelefteq G$. If H K-$\mathfrak{F}$-sn R, then HN K-$\mathfrak{F}$-sn RN.*

Recall that $\mathbb{F}_p$ denotes a field with p elements. The following result directly follows from [12, B, Theorem 10.3].

Lemma 5. *If $\mathrm{O}_p(G) = 1$ and G has a unique minimal normal subgroup, then G has a faithful irreducible module over $\mathbb{F}_p$.*

In [30] L.A. Shemetkov possed the problem to describe the set of formations $\mathfrak{F}$ having the following property

$$\mathfrak{F} = (G \mid \text{every chief factor of } G \text{ is } \mathfrak{F}\text{-central}) = (G \mid G = \mathrm{Z}_{\mathfrak{F}}(G)).$$

This class of formations contains saturated (local) and solubly saturated (composition or Baer-local) formations and other. Shortly we shall call formations from this class *Z-saturated.* In [7] A. Ballester-Bolinches and M. Pérez-Ramos showed that for a formation $\mathfrak{F}$ the class

$$Z\mathfrak{F} = (G \mid G = \mathrm{Z}_{\mathfrak{F}}(G))$$

is a formation and $\mathfrak{F} \subseteq Z\mathfrak{F} \subseteq \mathbf{E}_{\Phi}\mathfrak{F}$.

Let $\mathfrak{F}$ be a hereditary formation. In [24] and [34] the classes $\overline{w}\mathfrak{F}$ and $v^*\mathfrak{F}$ of all groups all whose Sylow and cyclic primary subgroups respectively are K-$\mathfrak{F}$-subnormal were studied. From the results of these papers follows

Proposition 1. *If $\mathfrak{F}$ is a hereditary formation, then $\overline{w}\mathfrak{F}$ and $v^*\mathfrak{F}$ are hereditary formations and $\mathfrak{N} \cup \mathfrak{F} \subseteq \overline{w}\mathfrak{F} \subseteq v^*\mathfrak{F}$.*

Recall that a Schmidt group G is a non-nilpotent group all whose proper subgroups are nilpotent. It is well known that $\pi(G) = \{p, q\}$ and G has a unique normal Sylow subgroup. Recall [35] that a Schmidt (p, q)-group is a Schmidt group with a normal Sylow p-subgroup. An *N-critical graph* $\Gamma_{Nc}(G)$ of a group G [35, Definition 1.3] is a directed graph on the vertex set $\pi(G)$ of all prime divisors of $|G|$ and (p, q) is an edge of $\Gamma_{Nc}(G)$ iff G has a Schmidt (p, q)-subgroup. An *N-critical graph* $\Gamma_{Nc}(\mathfrak{X})$ of a class of groups $\mathfrak{X}$ [35, Definition 3.1] is a directed graph on the vertex set $\pi(\mathfrak{X}) = \cup_{G\in\mathfrak{X}}\pi(G)$ such that $\Gamma_{Nc}(\mathfrak{X}) = \cup_{G\in\mathfrak{X}}\Gamma_{Nc}(G)$.

Proposition 2 ([35, Theorem 5.4]). *Let $\sigma = \{\pi_i \mid i \in I\}$ be a partition of the vertex set $V(\Gamma_{Nc}(\mathfrak{X}))$ such that for $i \neq j$ there are no edges between π_i and π_j. Then every $\mathfrak{X}$-group is the direct product of its Hall π_k-subgroups, where $k \in \{i \in I \mid \pi(G) \cap \pi_k \neq \emptyset\}$.*

3 The proof of Theorem 1 and its corollaries

The proof of Theorem 1 is rather complicated and require various preliminary results and definitions. A subgroup U of G is called *$\mathfrak{X}$-maximal* in G provided that (a) $U \in \mathfrak{X}$, and (b) if $U \leq V \leq G$ and $V \in \mathfrak{X}$, then $U = V$ [12, III, Definition 3.1]. The symbol $\mathrm{Int}_{\mathfrak{X}}(G)$ denotes the intersection of all $\mathfrak{X}$-maximal subgroups of G [32].

Proposition 3. *Let $\mathfrak{F}$ be a hereditary formation. Then*

(1) *[2, Lemma 2.4] $\mathrm{Z}_{\mathfrak{F}}(G) \cap H \leq \mathrm{Z}_{\mathfrak{F}}(H)$ for every subgroup H of a group G.*

(2) *$\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(\mathrm{Z}_{\mathfrak{F}}(G))$ for every group G.*

(3) *Assume that H is an $\mathfrak{F}$-subgroup of a group G. If $\mathfrak{F}$ is Z-saturated, then $H\mathrm{Z}_{\mathfrak{F}}(G) \in \mathfrak{F}$. In particular $\mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{Int}_{\mathfrak{F}}(G)$ for every group G.*

Proof. (2) From (1) it follows that $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G) \cap \mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{Z}_{\mathfrak{F}}(\mathrm{Z}_{\mathfrak{F}}(G)) \leq \mathrm{Z}_{\mathfrak{F}}(G)$. Thus $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(\mathrm{Z}_{\mathfrak{F}}(G))$.

(3) From (1) it follows that $\mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{Z}_{\mathfrak{F}}(G) \cap H\mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{Z}_{\mathfrak{F}}(H\mathrm{Z}_{\mathfrak{F}}(G))$. Since the group $H\mathrm{Z}_{\mathfrak{F}}(G)/\mathrm{Z}_{\mathfrak{F}}(G) \in \mathfrak{F}$, we see that $H\mathrm{Z}_{\mathfrak{F}}(G)/\mathrm{Z}_{\mathfrak{F}}(H\mathrm{Z}_{\mathfrak{F}}(G)) \in \mathfrak{F}$. Hence $H\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(H\mathrm{Z}_{\mathfrak{F}}(G)) \in Z\mathfrak{F} = \mathfrak{F}$.

Let M be an $\mathfrak{F}$-maximal subgroup of G. Then $M\mathrm{Z}_{\mathfrak{F}}(G) \in \mathfrak{F}$. It means that $M\mathrm{Z}_{\mathfrak{F}}(G) = M$. Thus $\mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{Int}_{\mathfrak{F}}(G)$. □

The following result plays the key role in the proof of Theorem 1.

Proposition 4. *Let $\mathfrak{F}$ be a formation.*

(1) $\mathrm{Z}_{Z\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ *holds for every group G.*

(2) *Assume that $\mathfrak{F}$ is hereditary. A subgroup H is K-$\mathfrak{F}$-subnormal in a group G iff it is K-$Z\mathfrak{F}$-subnormal in G.*

Proof. (1) Let H/K be a chief factor of a group G. Now $(H/K) \rtimes G/C_G(H/K)$ is a primitive group. It means that the $\mathfrak{F}$-hypercenter is defined by the set of all primitive $\mathfrak{F}$-groups. According to [7] $\mathfrak{F} \subseteq Z\mathfrak{F} \subseteq \mathbf{E}_\Phi\mathfrak{F}$. It means that every $Z\mathfrak{F}$-group G with $\Phi(G) = 1$ belongs $\mathfrak{F}$. Thus the sets of all primitive $\mathfrak{F}$-groups and $Z\mathfrak{F}$-groups coincide. Hence $\mathrm{Z}_{Z\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$.

(2) Note that $Z\mathfrak{F}$ is a hereditary formation by Statement (1) of Proposition 3. Since $\mathfrak{F}$ is a hereditary formation, we see that H is a K-$\mathfrak{F}$-subnormal subgroup of a group G if and only if there is a chain of subgroups $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ with $H_{i-1} \trianglelefteq H_i$ or $H_i/\mathrm{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ and H_{i-1} is a maximal subgroup of H_i for all $i = 1, \dots, n$. It means that K-$\mathfrak{F}$-subnormality is defined by the set of all primitive $\mathfrak{F}$-groups for a hereditary formation $\mathfrak{F}$. As we have already mentioned the sets of all primitive $\mathfrak{F}$-groups and $Z\mathfrak{F}$-groups coincide. Thus a subgroup is K-$\mathfrak{F}$-subnormal in a group G iff it is K-$Z\mathfrak{F}$-subnormal in G. □

The next step in the proof of Theorem 1 is to characterize the intersections $S_{\mathfrak{F}}(G)$ and $C_{\mathfrak{F}}(G)$ of all weak K-$\mathfrak{F}$-subnormalizers of all Sylow and all cyclic primary subgroups of G respectively.

Proposition 5. *Let $\mathfrak{F}$ be a hereditary formation.*

(1) *$S_{\mathfrak{F}}(G)$ is the largest subgroup among normal subgroups N of G with P K-$\mathfrak{F}$-sn PN for every Sylow subgroup P of G.*

(2) *$C_{\mathfrak{F}}(G)$ is the largest subgroup among normal subgroups N of G with C K-$\mathfrak{F}$-sn CN for every cyclic primary subgroup C of G.*

Proof. (1) Let $N \trianglelefteq G$ with P K-$\mathfrak{F}$-*sn* PN for every Sylow subgroup P of G. If S is a weak K-$\mathfrak{F}$-subnormalizer of P in G, then PN K-$\mathfrak{F}$-*sn* SN by Lemma 4. Hence P K-$\mathfrak{F}$-*sn* SN by (3) of Lemma 2. Now $SN = S$ by the definition of a weak K-$\mathfrak{F}$-subnormalizer. Thus $N \leq S_{\mathfrak{F}}(G)$.

From the other hand, since $\mathfrak{F}$ is a hereditary formation and $PS_{\mathfrak{F}}(G)$ lies in every weak K-$\mathfrak{F}$-subnormalizer of every Sylow subgroup P of G, we see that P K-$\mathfrak{F}$-*sn* $PS_{\mathfrak{F}}(G)$ for every Sylow subgroup P of G by Lemma 3. Thus $S_{\mathfrak{F}}(G)$ is the largest normal subgroup N of G with P K-$\mathfrak{F}$-*sn* PN for every Sylow subgroup P of G.

The proof of (2) is the same. □

The connections between the previous steps are shown in the following proposition:

Proposition 6. *Let $\mathfrak{F}$ be a hereditary formation. Then $\overline{w}\mathfrak{F}$ and $v^*\mathfrak{F}$ are hereditary Z-saturated formations and* $\mathrm{Int}_{\overline{w}\mathfrak{F}}(G) = S_{\mathfrak{F}}(G) \leq C_{\mathfrak{F}}(G) = \mathrm{Int}_{v^*\mathfrak{F}}(G)$ *holds for every group G.*

Proof. Note that $v^*\mathfrak{F}$ and $\overline{w}\mathfrak{F}$ are hereditary formations by Proposition 1. Assume that $\overline{w}\mathfrak{F}$ is not a Z-saturated formation. Let chose a minimal order group G from $Z(\overline{w}\mathfrak{F}) \setminus \overline{w}\mathfrak{F}$. From Proposition 3 it follows that $Z\overline{w}\mathfrak{F}$ is a hereditary formation. So G is $\overline{w}\mathfrak{F}$-critical. Now $|\pi(G)| > 1$ by Proposition 1. From $\overline{w}\mathfrak{F} \subset Z\overline{w}\mathfrak{F} \subseteq \mathbf{E}_\Phi\overline{w}\mathfrak{F}$ it follows that $\Phi(G) \neq 1$ and $G/\Phi(G) \in \overline{w}\mathfrak{F}$. Let P be a Sylow subgroup of G. Then $P\Phi(G) < G$ and $P\Phi(G) \in \overline{w}\mathfrak{F}$. Hence P K-$\mathfrak{F}$-$sn\, P\Phi(G)$. From $G/\Phi(G) \in \overline{w}\mathfrak{F}$ it follows that $P\Phi(G)/\Phi(G)$ K-$\mathfrak{F}$-$sn\, G/\Phi(G)$. Therefore $P\Phi(G)$ K-$\mathfrak{F}$-$sn\, G$. Thus P K-$\mathfrak{F}$-$sn\, G$. It means that $G \in \overline{w}\mathfrak{F}$, a contradiction. Thus $\overline{w}\mathfrak{F}$ is a Z-saturated formation. The proof for $v^*\mathfrak{F}$ is the same.

Note that $\mathfrak{N} \subseteq v^*\mathfrak{F}$ by Proposition 1. Hence $C\mathrm{Int}_{v^*\mathfrak{F}}(G) \in v^*\mathfrak{F}$ for every cyclic primary subgroup C of G. Therefore C K-$\mathfrak{F}$-$sn\, C\mathrm{Int}_{v^*\mathfrak{F}}(G)$ for every cyclic primary subgroup C of G. Thus $\mathrm{Int}_{v^*\mathfrak{F}}(G) \leq C_\mathfrak{F}(G)$ by (2) of Proposition 5.

From the other hand let M be a $v^*\mathfrak{F}$-maximal subgroup of G and C be a cyclic primary subgroup of $MC_\mathfrak{F}(G)$. Since $MC_\mathfrak{F}(G)/C_\mathfrak{F}(G) \in v^*\mathfrak{F}$, we see that $C_\mathfrak{F}(G)C/C_\mathfrak{F}(G)$ K-$\mathfrak{F}$-$sn\, MC_\mathfrak{F}(G)/C_\mathfrak{F}(G)$. Hence $C_\mathfrak{F}(G)C$ K-$\mathfrak{F}$-$sn\, MC_\mathfrak{F}(G)$ by (2) of Lemma 2. Note that C K-$\mathfrak{F}$-$sn\, C_\mathfrak{F}(G)C$ by Proposition 5. So C K-$\mathfrak{F}$-$sn\, MC_\mathfrak{F}(G)$ by (3) of Lemma 2. Thus $MC_\mathfrak{F}(G) \in v^*\mathfrak{F}$ by the definition of $v^*\mathfrak{F}$. Hence $MC_\mathfrak{F}(G) = M$. Therefore $C_\mathfrak{F}(G) \leq \mathrm{Int}_{v^*\mathfrak{F}}(G)$. Thus $\mathrm{Int}_{v^*\mathfrak{F}}(G) = C_\mathfrak{F}(G)$. The proof of that equality $\mathrm{Int}_{\overline{w}\mathfrak{F}}(G) = S_\mathfrak{F}(G)$ holds in every group is the same.

Since every cyclic primary subgroup is subnormal in some Sylow subgroup, we see that P K-$\mathfrak{F}$-$sn\, PS_\mathfrak{F}(G)$ for every cyclic primary subgroup P of G. So $S_\mathfrak{F}(G) \leq C_\mathfrak{F}(G)$ holds for every group G by Proposition 5. □

Proof of Theorem 1. (1) ⇒ (2). Since $\mathfrak{F} \subseteq \overline{w}\mathfrak{F}$ by Proposition 1, we see that $\mathrm{Z}_\mathfrak{F}(G) \leq \mathrm{Z}_{\overline{w}\mathfrak{F}}(G)$ for every group G. Note that $\mathrm{Z}_{\overline{w}\mathfrak{F}}(G) \leq \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$ for every group G by (3) of Proposition 3 and Proposition 6. According to Proposition 6, $S_\mathfrak{F}(G) = \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$ and $S_\mathfrak{F}(G) \leq C_\mathfrak{F}(G)$ for every group G. From these and (1) it follows that

$$\mathrm{Z}_\mathfrak{F}(G) \leq \mathrm{Z}_{\overline{w}\mathfrak{F}}(G) \leq \mathrm{Int}_{\overline{w}\mathfrak{F}}(G) = S_\mathfrak{F}(G) \leq C_\mathfrak{F}(G) = \mathrm{Z}_\mathfrak{F}(G)$$

for every group G. Thus $\mathrm{Z}_\mathfrak{F}(G) = S_\mathfrak{F}(G)$ for every group G.

(2) ⇒ (3). The proof consists of the following steps:

(a) *We may assume that $\mathfrak{N} \subseteq \mathfrak{F}$ is Z-saturated.*

According to Proposition 4 Statements (2) and (3) mean the same for $\mathfrak{F}$ and $Z\mathfrak{F}$. Note that $Z\mathfrak{F} = Z(Z\mathfrak{F})$ by Proposition 4. ***Therefore without lose of generality we may assume that $\mathfrak{F}$ is Z-saturated in the proof of*** (2) ⇒ (3). Since in every nilpotent group every Sylow subgroup is subnormal and $Z\mathfrak{F} = \mathfrak{F}$ we see that $\pi(\mathfrak{F}) = \mathbb{P}$ and $\mathfrak{N} \subseteq \mathfrak{F}$.

(b) *Assume that a group G has faithful irreducible module L over $\mathbb{F}_p$, $T = L \rtimes G$ and $L \leq S_\mathfrak{F}(T)$. Then $G \in \mathfrak{F}$.*

Note that $L \leq S_\mathfrak{F}(G) = \mathrm{Z}_\mathfrak{F}(T)$. Hence $L \rtimes (T/C_T(L)) \in \mathfrak{F}$. Thus $G \simeq T/C_T(L) \in \mathfrak{F}$, the contradiction.

(c) *Let $\pi(p) = \{q \in \mathbb{P} \mid (p,q) \in \Gamma_{Nc}(\mathfrak{F})\} \cup \{p\}$. Then $\mathfrak{F}$ contains every q-closed $\{p,q\}$-group for every $q \in \pi(p)$.*

Assume the contrary. Let G be a minimal order counterexample. Since $\mathfrak{F}$ and the class of all q-closed groups are hereditary formations, we see that G is an $\mathfrak{F}$-critical group, G has a unique minimal normal subgroup N and $G/N \in \mathfrak{F}$. Let P be a Sylow p-subgroup of G. If $NP < G$, then $NP \in \mathfrak{F}$. Hence P K-$\mathfrak{F}$-$sn\, PN$ and PN/N K-$\mathfrak{F}$-$sn\, G/N$. From Lemma 2 it follows that P K-$\mathfrak{F}$-$sn\, G$. Since G is a q-closed $\{p,q\}$-group, we see that every Sylow subgroup of G is K-$\mathfrak{F}$-subnormal. So $G \in Z\mathfrak{F} = \mathfrak{F}$, a contradiction.

Now N is a Sylow q-subgroup and $\mathrm{O}_p(G) = 1$. By Lemma 5 G has a faithful irreducible module L over $\mathbb{F}_p$. Let $T = L \rtimes G$. Therefore for every chief factor H/K of NL a group

$(H/K) \rtimes C_{NL}(H/K)$ is isomorphic to one of the following groups Z_p, Z_q and a Schmidt (p,q)-group with the trivial Frattini subgroup. Note that all these groups belong $\mathfrak{F}$. So $NL \in Z\mathfrak{F} = \mathfrak{F}$. Note that $L \leq \mathrm{O}_p(T)$. Hence $L \leq S_{\mathfrak{F}}(T)$ by Proposition 5. Thus $G \in \mathfrak{F}$ by (b), a contradiction.

From (c) it follows that

(d) $\Gamma_{Nc}(\mathfrak{F})$ *is undirected, i.e* $(p,q) \in \Gamma_{Nc}(\mathfrak{F})$ *iff* $(q,p) \in \Gamma_{Nc}(\mathfrak{F})$.

(e) *Let* p, q *and* r *be different primes. If* $(p,r),(q,r) \in \Gamma_{Nc}(\mathfrak{F})$, *then* $(p,q) \in \Gamma_{Nc}(\mathfrak{F})$.

Note that the cyclic group Z_q of order q has a faithful irreducible module P over $\mathbb{F}_p$ by Lemma 5. Let $G = P \rtimes Z_q$. Then G has a faithful irreducible module R over $\mathbb{F}_r$ by Lemma 5. Let $T = R \rtimes G$. From (c) it follows that $\mathfrak{F}$-contains all r-closed $\{p,r\}$-groups and $\{q,r\}$-groups. Hence $R \leq S_{\mathfrak{F}}(T)$ by Proposition 5. Thus $G \in \mathfrak{F}$ by (b). Note that G is a Schmidt (p,q)-group. It means that $(p,q) \in \Gamma_{Nc}(\mathfrak{F})$ by the definition of N-critical graph.

(f) $\mathfrak{F} = \mathfrak{N}_\sigma$ *for some partition* σ *of* $\mathbb{P}$.

From (d) and (e) it follows that $\Gamma_{Nc}(\mathfrak{F})$ is a disjoint union of complete (directed) graphs Γ_i, $i \in I$. Let $\pi_i = V(\Gamma_i)$. Then $\sigma = \{\pi_i \mid i \in I\}$ is a partition of $\mathbb{P}$. From Proposition 2 it follows that every $\mathfrak{F}$-group G has normal Hall π_i-subgroups for every $i \in I$ with $\pi_i \cap \pi(G) \neq \emptyset$. So G is σ-nilpotent. Hence $\mathfrak{F} \subseteq \mathfrak{N}_\sigma$.

Let show that the class $\mathfrak{G}_{\pi_i}$ of all π_i-groups is a subset of $\mathfrak{F}$ for every $i \in I$. It is true if $|\pi_i| = 1$. Assume now $|\pi_i| > 1$. Suppose the contrary and let a group G be a minimal order group from $\mathfrak{G}_{\pi_i} \setminus \mathfrak{F}$. Then G has a unique minimal normal subgroup, $\pi(G) \subseteq \pi_i$ and $|\pi(G)| > 1$. Note that $\mathrm{O}_q(G) = 1$ for some $q \in \pi(G)$. Hence G has a faithful irreducible module N over $\mathbb{F}_q$ by Lemma 5. Let $T = N \rtimes G$. Hence $NP \in \mathfrak{F}$ for every Sylow subgroup P of T by (c). Now $N \leq S_{\mathfrak{F}}(T)$ by Proposition 5. So $G \in \mathfrak{F}$ by (b), the contradiction.

Since a formation is closed under taking direct products, we see that $\mathfrak{N}_\sigma \subseteq \mathfrak{F}$. Thus $\mathfrak{F} = \mathfrak{N}_\sigma$.

(3) $\Rightarrow$ (1). Recall that the class of all σ-nilpotent groups is saturated. Hence it is Z-saturated. According to Proposition 4 Statements (3) and (1) mean the same for $\mathfrak{F}$ and $Z\mathfrak{F}$. Hence we may assume that $\mathfrak{F} = \mathfrak{N}_\sigma$ for some partition $\sigma = \{\pi_i \mid i \in I\}$ of $\mathbb{P}$. Then $\mathfrak{N}_\sigma$ has the lattice property for K-$\mathfrak{F}$-subnormal subgroups (see [33, Lemma 2.6(3)] or [8, Chapter 3]). According to [24, Theorem B and Corollary E.2] $v^*\mathfrak{F} = \mathfrak{F}$. By [32, Theorem A and Proposition 4.2] $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group G. By Proposition 6, $C_{\mathfrak{F}}(G) = \mathrm{Int}_{v^*\mathfrak{F}}(G)$ for every group G. Thus $C_{\mathfrak{F}}(G) = \mathrm{Int}_{v^*\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ for every group G.

(3) $\Rightarrow$ (4) Statement (3) means that $Z\mathfrak{F} = \mathfrak{N}_\sigma$ and $\pi(\mathfrak{F}) = \pi(Z\mathfrak{F}) = \mathbb{P}$. From $\mathfrak{F} \subseteq Z\mathfrak{F}$ it follows that $\mathfrak{F} = \times_{i \in I}\mathfrak{F}_{\pi_i}$ where $\mathfrak{F}_{\pi_i}$ is a hereditary formation with $\pi(\mathfrak{F}_{\pi_i}) = \pi_i$.

Assume that $\pi_i \in \sigma$ and $|\pi_i| \geq 2$. Let choose a minimal order π_i-group G from $Z\mathfrak{F} \setminus \mathfrak{F}_{\pi_i}$. Since $Z\mathfrak{F} = \mathfrak{N}_\sigma$ and $\mathfrak{F}_{\pi_i} = \mathfrak{F} \cap \mathfrak{G}_{\pi_i}$ are formations, we see that G has a unique minimal normal subgroup N. From $|\pi_i| \geq 2$ it follows that there exists $p \in \pi_i$ such that N is not a p-group. Therefore G has a faithful irreducible module V over $\mathbb{F}_p$ by Lemma 5. Let $T = V \rtimes G$. Since T is a π_i-group, $T \in \mathfrak{N}_\sigma = Z\mathfrak{F}$. Hence $R = V \rtimes (T/C_T(V)) \in \mathfrak{F} \cap \mathfrak{G}_{\pi_i} = \mathfrak{F}_{\pi_i}$ and $T/C_T(V) \simeq G$. Now $G \in \mathfrak{F}_{\pi_i}$ as a quotient group of R, a contradiction. It means that $\mathfrak{F} \cap \mathfrak{G}_{\pi_i} = Z\mathfrak{F} \cap \mathfrak{G}_{\pi_i} = \mathfrak{G}_{\pi_i}$.

(4) $\Rightarrow$ (3) Assume that $\mathrm{Z}_{\mathfrak{F}}(G) \neq \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ for some group G. It means that there exists a primitive $\mathfrak{N}_\sigma$-group H with $H \notin \mathfrak{F}$. Since H is a primitive $\mathfrak{N}_\sigma$-group, we see that H is a π_i-group for some $i \in I$. If $|\pi_i| \geq 2$, then $H \in \mathfrak{G}_{\pi_i} \subseteq \mathfrak{F}$, a contradiction. Hence $|\pi_i| = 1$. So H is a p-group for some $p \in \mathbb{P}$. Therefore H is a cyclic group of order p. Thus $H \in \mathfrak{F}$, the final contradiction. □

Proof of Corollary 1.1. Let D be the intersection of normalizers in G of all subgroups from $\mathcal{M}$. From (a) it follows that $D \trianglelefteq G$. Let P be a Sylow subgroup of G and H be a subgroup from $\mathcal{M}$ with $P \leq H$. Note that $H \in \mathfrak{N}_\sigma$. Now P K-$\mathfrak{N}_\sigma$-$sn\, H \trianglelefteq HD$. So P K-$\mathfrak{N}_\sigma$-$sn\, HD$. Hence P K-$\mathfrak{N}_\sigma$-$sn\, PD$ by Lemma 3. It means that D K-$\mathfrak{N}_\sigma$-subnormalizes all Sylow subgroups of G. Thus $D \leq S_{\mathfrak{N}_\sigma}(G)$ by Proposition 5.

From the proof of Theorem 1 it follows that $S_{\mathfrak{N}_\sigma}(G) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G) = \mathrm{Int}_{\mathfrak{N}_\sigma}(G)$. Let $H \in \mathcal{M}$. Now $HS_{\mathfrak{N}_\sigma}(G) \in \mathfrak{N}_\sigma$. Since H is a π_i-maximal subgroup of G, H is a π_i-maximal subgroup of

$HS_{\mathfrak{N}_\sigma}(G)$. It means that $H \trianglelefteq HS_{\mathfrak{N}_\sigma}(G)$. So $S_{\mathfrak{N}_\sigma}(G)$ normalizes all subgroups from $\mathcal{M}$. Hence $S_{\mathfrak{N}_\sigma}(G) \leq D$. Thus $D = S_{\mathfrak{N}_\sigma}(G) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ by Theorem 1. $\square$

Proof of Corollary 1.6. From Corollary 1.2 it follows that every π_i-element of $\mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ permutes with every π_i'-element of G.

Let A be the set of all π_i-elements of G that permute with all π_i'-elements of G and $H = \langle A \rangle$. So all elements of H permute with all π_i'-elements of G. Since $\mathrm{O}^{\pi_i}(H)$ is generated by all π_i'-elements of H, we see that $\mathrm{O}^{\pi_i}(H) \leq \mathrm{Z}(H)$. Hence all π_i'-elements of $\mathrm{O}^{\pi_i}(H)$ form a subgroup. So $\mathrm{O}^{\pi_i}(H)$ is a π_i'-group. Let K be a $\mathfrak{G}_{\pi_i}$-projector of H. Then $K\mathrm{O}^{\pi_i}(H) = H$. So $K \trianglelefteq H$. It means that π_i-elements of H form a subgroup. Thus $H = A$.

Now A is a normal π_i-subgroup of G. Hence it lies in every π_i-maximal subgroup of G. Note that A lies in the normalizer of every π_i'-subgroup by its definition. Thus $A \leq \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ by Corollary 1.2. $\square$

4 Applications

R. Baer [3] proved that the hypercenter of a group coincides with the intersection of all its maximal nilpotent subgroups. L. A. Shemetkov possed a question at the Gomel Algebraic Seminar in 1995 that can be formulated in the following way: For what non-empty (normally) hereditary (solubly) saturated formations $\mathfrak{F}$ does the intersection of all $\mathfrak{F}$-maximal subgroups coincides with the $\mathfrak{F}$-hypercenter in every group? A. N. Skiba [32] answered on this question for hereditary saturated formations $\mathfrak{F}$ (for the soluble case, see also J. C. Beidleman and H. Heineken [9]). From Theorem 1 follows a solution of this question for a family of hereditary not necessary saturated formations.

Theorem 2. *Let $\mathfrak{F}$ be a hereditary formation.*

(1) $\mathfrak{F} = \overline{w}\mathfrak{F}$ *if and only if* $S_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group.*

(2) $\mathfrak{F} = v^*\mathfrak{F}$ *if and only if* $C_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group.*

(3) *Assume that* $\mathfrak{F} = \overline{w}\mathfrak{F}$ *or* $\mathfrak{F} = v^*\mathfrak{F}$. *Then* $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group if and only if there is a partition* σ *of* $\mathbb{P}$ *such that* $\mathfrak{F}$ *is the class of all* σ*-nilpotent groups.*

Proof. From Proposition 6 it follows that $S_{\mathfrak{F}}(G) = \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$. Now (1) follows from the fact that $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$ holds for every group if and only if $\mathfrak{F} = \overline{w}\mathfrak{F}$. The proof of (2) is the same.

(3) Assume that $\mathfrak{F} = \overline{w}\mathfrak{F}$. Now $\mathfrak{F}$ is Z-saturated by Proposition 6 and $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$ holds for every group G. From Proposition 6 it follows that $S_{\mathfrak{F}}(G) = \mathrm{Int}_{\overline{w}\mathfrak{F}}(G)$ holds for every group G. Now $\mathrm{Int}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group if and only if $S_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ holds for every group G. From (3) of Theorem 1 it follows that the last equality holds for every group if and only if there is a partition σ of $\mathbb{P}$ such that $Z\mathfrak{F} = \mathfrak{N}_\sigma$. Hence $\mathfrak{F} = \mathfrak{N}_\sigma$. From this theorem is also follows that $\mathfrak{N}_\sigma = \overline{w}\mathfrak{N}_\sigma$.

The proof of (3) for $\mathfrak{F} = v^*\mathfrak{F}$ is the same. $\square$

Remark 2. There is a rather important family of not necessary saturated hereditary formations $\mathfrak{F}$ with $v^*\mathfrak{F} = \mathfrak{F}$ and $\overline{w}\mathfrak{F} = \mathfrak{F}$. Recall that a formation $\mathfrak{F}$ has the Shemetkov property if every $\mathfrak{F}$-critical group is either a Schmidt group of a cyclic group of prime order. The family of hereditary formations with the Shemetkov property contains non-saturated formations (see [8, Chapter 6.4]). For example let $\mathfrak{F}$ be a class of groups all whose Schmidt subgroups are Schmidt (p, q)-groups for $(p, q) \in \{(2, 3), (3, 2), (5, 2)\}$. Then $\mathfrak{F}$ has the Shemetkov property by [35, Theorem 3.5] and $\pi(\mathfrak{F}) = \mathbb{P}$. Let G be the alternating group of degree 5. Hence $G \in \mathfrak{F}$.

According to [14] there is a Frattini $\mathbb{F}_3G$-module T which is faithful for G. By the Gaschütz theorem (see [12, Appendix β]), there exists a Frattini extension $T \rightarrowtail R \twoheadrightarrow G$ such that $T \overset{G}{\simeq} \Phi(R)$ and $R/\Phi(R) \simeq G$. Let $K/\Phi(R)$ be a cyclic subgroup of $G/\Phi(G)$ of order 5. Since T is faithful for G, we see that K is a non-nilpotent group with a normal Sylow 3-subgroup. Hence its contains a Schmidt $(5,3)$-subgroup. It means that $G \not\in \mathfrak{F}$, i.e. $\mathfrak{F}$ is not saturated.

As follows from [24, 34] and [25, Corollaries 3.9 and 3.10] $v^*\mathfrak{F} = \mathfrak{F}$ and $\overline{w}\mathfrak{F} = \mathfrak{F}$ for every hereditary formation $\mathfrak{F}$ with the Shemetkov property and $\pi(\mathfrak{F}) = \mathbb{P}$.

Let give another application of Theorem 1. Recall that a formation $\mathfrak{F}$ is called regular [26], if for every group G holds

$$\mathcal{I}_{\mathfrak{F}}(G) = \{x \in G \mid \langle x, y\rangle \in \mathfrak{F}\ \forall y \in G\} = \mathrm{Int}_{\mathfrak{F}}(G).$$

The regular formations of soluble groups were studied in [26]. Here we give examples of such formations of non-necessary soluble groups.

Recall (see [26]) that the non-$\mathfrak{F}$-graph $\Gamma_{\mathfrak{F}}(G)$ of a group G is the graph whose vertex set is $G \setminus \mathcal{I}_{\mathfrak{F}}(G)$ and two vertices x and y are connected if $\langle x, y\rangle \not\in \mathfrak{F}$. This type of graphs can be traced back to P. Erdős who considered non-commuting (non-abelian) graph. A. Abdollahi and M. Zarrin [1] asked to find the bounds for diameters of non-nilpotent graphs. The final answer on this question was obtained by A. Lucchini and D. Nemmi [21].

Theorem 3. *The formation of all σ-nilpotent groups is regular and $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ holds for every group G. Moreover the graph $\Gamma_{\mathfrak{N}_\sigma}(G)$ is connected and $\mathrm{diam}(\Gamma_{\mathfrak{N}_\sigma}(G)) \leq 3$ for every group G.*

Proof. Let $x \in G$. Denote by G_p and x_p a Sylow p-subgroup of G and $x^{|G|/|G_p|}$ respectively. Note that if $\langle x_p, y\rangle \not\in \mathfrak{N}_\sigma$, then $\langle x, y\rangle \not\in \mathfrak{N}_\sigma$.

(1) *$\mathfrak{N}_\sigma$ is regular and $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ holds for every group G*

Let $y \in G$. Then $\langle y\rangle \in \mathfrak{N}_\sigma$. It means that $\langle y\rangle \mathrm{Z}_{\mathfrak{N}_\sigma}(G) \in \mathfrak{N}_\sigma$. Hence $\langle x, y\rangle \in \mathfrak{N}_\sigma$ for all $x \in \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ and $y \in G$. It means that $\mathrm{Z}_{\mathfrak{N}_\sigma}(G) \subseteq \mathcal{I}_{\mathfrak{N}_\sigma}(G)$.

Let $x \in \mathcal{I}_{\mathfrak{F}}(G)$. Note that $x = \prod_{p\in\pi(G)} x_p$. From $\langle x_p, y\rangle \leq \langle x, y\rangle \in \mathfrak{N}_\sigma$ it follows that $x_p \in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$ for all $p \in \pi(G)$.

Let $q \in \pi(G)$. Since σ is a partition of $\mathbb{P}$, there exists a unique $\pi_i \in \sigma$ with $q \in \pi_i$. Let y be a π_i'-element of G. Now $\langle x_q, y\rangle \in \mathfrak{N}_\sigma$. It means that $x_q y = y x_q$. So a π_i-element x_q permutes with all π_i'-elements of G. Thus $x_q \in \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$ by Corollary 1.6. Therefore $x \in \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$. So $\mathcal{I}_{\mathfrak{N}_\sigma}(G) \subseteq \mathrm{Z}_{\mathfrak{N}_\sigma}(G)$. Hence $\mathcal{I}_{\mathfrak{N}_\sigma}(G) = \mathrm{Z}_{\mathfrak{N}_\sigma}(G) = \mathrm{Int}_{\mathfrak{N}_\sigma}(G)$. Thus $\mathfrak{N}_\sigma$ is regular.

(2) *$\Gamma_{\mathfrak{N}_\sigma}(G)$ is connected and $\mathrm{diam}(\Gamma_{\mathfrak{N}_\sigma}(G)) \leq 3$ for every group G.*

If $|G\setminus\mathcal{I}_{\mathfrak{N}_\sigma}(G)| < 2$, then there is nothing to prove. So we may assume that $|G\setminus\mathcal{I}_{\mathfrak{N}_\sigma}(G)| \geq 2$. Assume that G is a counterexample to (2). Hence there are elements $x, y \in G$ such that they are not connected or the lengths of all paths connecting them are greater than 3.

If $x_p \in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$ for all $p \in \pi(G)$, then $x = \prod_{p\in\pi(G)} x_p \in \mathrm{Z}_{\mathfrak{N}_\sigma}(G) = \mathcal{I}_{\mathfrak{N}_\sigma}(G)$, a contradiction. It means that there exist $p, q \in \pi(G)$ with $x_p, y_q \not\in \mathcal{I}_{\mathfrak{N}_\sigma}(G)$. Hence there exist $\pi_i, \pi_j \in \sigma$, π_i-element w and π_j-element z with $p \not\in \pi_i$, $q \not\in \pi_j$, $\langle x_p, w\rangle \not\in \mathfrak{N}_\sigma$ and $\langle y_q, z\rangle \not\in \mathfrak{N}_\sigma$.

If $\langle w, z\rangle \not\in \mathfrak{N}_\sigma$, then (x, w, z, y) is the path connecting x and y and its length is not greater than 3, a contradiction. Now $\langle w, z\rangle \in \mathfrak{N}_\sigma$. Assume that $i \neq j$. So $wz = zw$ and $\langle zw\rangle = \langle z, w\rangle$. Now (x, wz, y) is the path connecting x and y of length 2, a contradiction. So $i = j$. If $\langle x_p, z\rangle \not\in \mathfrak{N}_\sigma$, then (x, z, y) is the path connecting x and y of length 2, a contradiction. Hence $\langle x_p, z\rangle \in \mathfrak{N}_\sigma$. Since $p \not\in \pi_i = \pi_j$, we see that $x_p z = z x_p$ and $\langle z x_p\rangle = \langle z, x_p\rangle$. Now $(x, w, x_p z, y)$ is the path connecting x and y and its length is not greater than 3, the final contradiction. ☐

Corollary 3.1 ([21, Theorem 1.1])**.** $\Gamma_{\mathfrak{N}}(G)$ *is connected and* $\mathrm{diam}(\Gamma_{\mathfrak{N}}(G)) \leq 3$ *for every group G.*

Corollary 3.2 ([1, Theorem 5.1])**.** $\Gamma_{\mathfrak{N}}(G)$ *is connected and* $\mathrm{diam}(\Gamma_{\mathfrak{N}}(G)) \leq 6$ *for every group G.*

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