

On $K\text{-}\mathbb{P}_t$ -subnormal subgroups of finite groups and related formations

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Abstract

Let t be a fixed natural number. A subgroup H of a group G will be called $K\text{-}\mathbb{P}_t$ -subnormal in G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \dots \leq H_{m-1} \leq H_m = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a some prime p and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes for every $i = 1, \dots, n$. In this work, properties of $K\text{-}\mathbb{P}_t$ -subnormal subgroups and classes of groups with Sylow $K\text{-}\mathbb{P}_t$ -subnormal subgroups are obtained.

Keywords: finite group, $K\text{-}\mathbb{P}_t$ -subnormal subgroup, $K\text{-}\mathbb{P}$ -subnormal subgroup, Sylow subgroup, supersoluble group, formation

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Introduction

All groups under consideration are finite.

The goal of this work is to study the following generalization of the concept of a subnormal subgroup in a group and to find some of its applications.

Definition 1. *Let t be a fixed natural number. A subgroup H of a group G will be called $K\text{-}\mathbb{P}_t$ -subnormal in G if there exists a chain of subgroups*

$$H = H_0 \leq H_1 \leq \dots \leq H_{n-1} \leq H_n = G \quad (1.1)$$

such that either $H_{i-1} \trianglelefteq H_i$ or $|H_i : H_{i-1}|$ is a some prime p and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes for every $i = 1, \dots, n$.

A subgroup H of a group G is said to be:

\mathbb{P} -subnormal in G [1] if either $H = G$ or there exists a chain of subgroups (1.1) such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$;

$K\text{-}\mathbb{P}$ -subnormal in G [2] if there exists a chain of subgroups (1.1) such that either $H_{i-1} \trianglelefteq H_i$ or $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$.

In recent years, the concepts \mathbb{P} -subnormal and $K\text{-}\mathbb{P}$ -subnormal subgroups have been used in the works of many authors to solve various problems in group theory (see, for example, [3–17]).

It is clear that every $K\text{-}\mathbb{P}_t$ -subnormal subgroup in G is $K\text{-}\mathbb{P}$ -subnormal. The converse statement does not hold in the general case. For example, let $t = 3$ and let $G = AB$ be a group, where $A \cong Z_{17}$ and $B \cong \text{Aut}(Z_{17}) \cong Z_{16}$. In G , there is a chain of subgroups

$$H \trianglelefteq B < G,$$

with $|H| = 2$. Then H is $K\mathbb{P}$ -subnormal and \mathbb{P} -subnormal in G , but H is not $K\mathbb{P}_3$ -subnormal in G , since $|G : B| = 17$, $17 - 1 = 2^4$.

Let \mathfrak{F} be a non-empty formation. A subgroup H of a group G is said to be:

\mathfrak{F} -subnormal in G [18] if either $H = G$ or there exists a chain of subgroups (1.1) such that $H_i^{\mathfrak{F}} \leq H_{i-1}$ for every $i = 1, \dots, n$. Here $G^{\mathfrak{F}}$ is the \mathfrak{F} -residual of G , i.e. the least normal subgroup of G for which $G/G^{\mathfrak{F}} \in \mathfrak{F}$;

$K\mathfrak{F}$ -subnormal in G [18] if there exists a chain of subgroups (1.1) such that either $H_{i-1} \trianglelefteq H_i$ or $H_i^{\mathfrak{F}} \leq H_{i-1}$ for every $i = 1, \dots, n$.

Let \mathfrak{U}_k be the class of all supersoluble groups in which exponents are not divided by the $(k+1)$ th powers of primes, where k is a natural number. In [19], V.S. Monakhov and I.L. Sochor showed that \mathfrak{U}_k is a hereditary formation, and studied the class of groups $w\mathfrak{U}_k$ in which every Sylow subgroup is \mathfrak{U}_k -subnormal.

In a soluble group G every $K\mathbb{P}_t$ -subnormal subgroup is \mathfrak{U}_t -subnormal in G (Lemma 2.4). The converse does not hold (Example 2.1).

Note that in G , a $K\mathbb{P}_t$ -subnormal subgroup is not \mathfrak{U}_t -subnormal in general case. For example, let $t = 2$ and let $G \cong A_5$ is an alternating group of degree 5. The Sylow 2-subgroup H of G is $K\mathbb{P}_2$ -subnormal, but is not \mathfrak{U}_2 -subnormal in G , since $H \trianglelefteq H_1 < G$, where $H_1 \cong A_4$, $5 - 1 = 2^2$, $H_1^{\mathfrak{U}_2} = H$, $G^{\mathfrak{U}_2} = G$.

In this work, properties of $K\mathbb{P}_t$ -subnormal subgroups and classes of groups with Sylow $K\mathbb{P}_t$ -subnormal subgroups are obtained.

1. Preliminary results

We use the notation and terminology from [18, 20]. We recall some concepts significant in the paper.

Let G be a group. If H is a subgroup of G , we write $H \leq G$ and if $H \neq G$, we write $H < G$. We denote by $|G|$ the order of G ; by $\pi(G)$ the set of all distinct prime divisors of the order of G ; by $\text{Syl}_p(G)$ the set of all Sylow p -subgroups of G ; by $\text{Syl}(G)$ the set of all Sylow subgroups of G ; by $\text{Core}_G(M)$ the core of subgroup M in G , i.e. $\text{Core}_G(M) = \bigcap M^x$ for all $x \in G$; by $F(G)$ the Fitting subgroup of G ; by $|G : H|$ the index of H in G ; by $\pi(G : H)$ the set of all different prime divisors of $|G : H|$; by Z_n the cyclic group of order n ; by \mathbb{P} the set of all primes; by \mathfrak{S} the class of all soluble groups; by \mathfrak{U} the class of all supersoluble groups; by \mathfrak{N} the class of all nilpotent groups; by \mathfrak{N}_p the class of all p -groups for $p \in \mathbb{P}$; by $\mathfrak{A}(p-1)$ the class of all abelian groups of exponent dividing $p-1$.

A group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ (where p_i is a prime, $i = 1, 2, \dots, n$) is called *Ore dispersive* or *Sylow tower group* whenever $p_1 > p_2 > \cdots > p_n$ and G has a normal subgroup of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$ for every $i = 1, 2, \dots, n$.

A class of groups \mathfrak{F} is called a *formation* if the following conditions hold: (a) every quotient group of a group lying in \mathfrak{F} also lies in \mathfrak{F} ; (b) if $G/N_i \in \mathfrak{F}$, $N_i \trianglelefteq G$, $i = 1, 2$ then $G/N_1 \cap N_2 \in \mathfrak{F}$. A formation \mathfrak{F} is called *hereditary* whenever \mathfrak{F} together with every group contains all its subgroups, and *saturated*, if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

A function $f : \mathbb{P} \rightarrow \{\text{formations}\}$ is called a *local function*. A formation \mathfrak{F} is called *local*, if there exists a local function f such that $\mathfrak{F} = LF(f) = (G | G/C_G(H/K) \in f(p) \text{ for every chief factor } H/K \text{ of a group } G \text{ and for all primes } p \in \pi(H/K))$.

Lemma 1.1 [21, Chap. I, Theorem 1.4]. *Let H/K be a chief p -factor of a group G . $|H/K| = p$ if and only if $\text{Aut}_G(H/K)$ is abelian group of exponent dividing $p-1$.*

Lemma 1.2 [2, Lemma 3.6, Corollary 4.3.1]. *The class $\overline{w}\mathfrak{A}$ of all groups in which any Sylow subgroup is $K\text{-}\mathbb{P}$ -subnormal consists of Ore dispersive groups, and forms a hereditary saturated formation.*

Lemma 1.3 [22, Theorem 2]. *A group G is supersoluble if and only if G can be represented as the product of two nilpotent \mathbb{P} -subnormal subgroups.*

We will need some properties of \mathfrak{F} -subnormal subgroups (see, for example, [18, Chap. 6]). In what follows, \mathfrak{F} means a non-empty formation.

Lemma 1.4. *Let H and U be subgroups of a group G and let $N \trianglelefteq G$. Suppose that \mathfrak{F} is a formation. Then the following statements hold.*

(1) *If H is \mathfrak{F} -subnormal in U and U is \mathfrak{F} -subnormal in G then H is \mathfrak{F} -subnormal in G .*

(2) *If $N \leq U$ and U/N is \mathfrak{F} -subnormal in G/N then U is \mathfrak{F} -subnormal in G .*

(3) *If H is \mathfrak{F} -subnormal in G then HN/N is \mathfrak{F} -subnormal in G/N .*

Suppose that \mathfrak{F} is a hereditary formation. Then the following statements hold.

(4) *If $G^{\mathfrak{F}} \leq H$ then H is \mathfrak{F} -subnormal in G .*

(5) *If H is \mathfrak{F} -subnormal in G then $H \cap U$ is \mathfrak{F} -subnormal in U .*

According to [23], class of groups $w\mathfrak{F} = (G | \pi(G) \subseteq \pi(\mathfrak{F}))$ and every Sylow subgroup of G is \mathfrak{F} -subnormal in G). Here $\pi(\mathfrak{F})$ is the set of all distinct prime divisors $|G|$ for $G \in \mathfrak{F}$.

Lemma 1.5 [23, Lemma 1.6]. *If \mathfrak{F} is a hereditary formation and $\mathfrak{F} \subseteq \mathfrak{G}$ then $w\mathfrak{F} \subseteq \mathfrak{G}$.*

Lemma 1.6 [23, Theorem B]. *If \mathfrak{F} is a hereditary saturated formation then $w\mathfrak{F}$ is a hereditary saturated formation.*

2. Properties of $K\text{-}\mathbb{P}_t$ -subnormal subgroups of groups

Further t means a fixed natural number.

Lemma 2.1. *Let H be a subgroup of a group G . Then the following statements hold.*

(1) *If H is $K\text{-}\mathbb{P}_t$ -subnormal in G , then H^x is $K\text{-}\mathbb{P}_t$ -subnormal in G for all $x \in G$.*

(2) *If $H \leq R \leq G$, H is $K\text{-}\mathbb{P}_t$ -subnormal in R and R is $K\text{-}\mathbb{P}_t$ -subnormal in G , then H is $K\text{-}\mathbb{P}_t$ -subnormal in G .*

Proof. Statements (1) and (2) follow from Definition 1. □

Lemma 2.2. *Let H be a subgroup of a group G and $N \trianglelefteq G$. Then the following statements hold.*

(1) *If H is $K\text{-}\mathbb{P}_t$ -subnormal in G , then $(H \cap N)$ is $K\text{-}\mathbb{P}_t$ -subnormal in N and HN/N is $K\text{-}\mathbb{P}_t$ -subnormal in G/N .*

(2) *If $N \leq H$ and H/N is $K\text{-}\mathbb{P}_t$ -subnormal in G/N , then H is $K\text{-}\mathbb{P}_t$ -subnormal in G .*

(3) *The subgroup HN is $K\text{-}\mathbb{P}_t$ -subnormal in G if and only if HN/N is $K\text{-}\mathbb{P}_t$ -subnormal in G/N .*

(4) *If HN_i is $K\text{-}\mathbb{P}_t$ -subnormal in G and $N_i \trianglelefteq G$, $i = 1, 2$, then $(HN_1 \cap HN_2)$ is $K\text{-}\mathbb{P}_t$ -subnormal in G .*

Proof. (1) Suppose that H is $K\text{-}\mathbb{P}_t$ -subnormal in G . We can assume that $H \neq G$. There is a chain of subgroups (1.1). Consider the chains of subgroups

$$H \cap N = H_0 \cap N \leq H_1 \cap N \leq \cdots \leq H_{n-1} \cap N \leq H_n \cap N = N,$$

$$HN/N = H_0N/N \leq H_1N/N \leq \cdots \leq H_{n-1}N/N \leq H_nN/N = G/N.$$

If $H_{i-1} \cap N \trianglelefteq H_i \cap N$ for all $i = 1, \dots, n$, then $H \cap N$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in N .

Let's assume that $H_{i-1} \cap N \not\trianglelefteq H_i \cap N$ for some $i \in \{1, \dots, n\}$. In that case $H_{i-1} \cap N \neq H_i \cap N$ and $H_{i-1} \not\trianglelefteq H_i$. Then $|H_i : H_{i-1}|$ is a some prime p and $p-1$ is not divisible by the $(t+1)$ th powers of primes. Then H_{i-1} is maximal in H_i and $H_i = (H_i \cap N)H_{i-1}$. We have $|H_i \cap N : H_{i-1} \cap N| = |H_i : H_{i-1}| = p$. Thus, $H \cap N$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in N .

If $H_{i-1}N/N \trianglelefteq H_iN/N$ for all $i = 1, \dots, n$, then HN/N is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G/N . Let's assume that $H_{j-1}N/N \not\trianglelefteq H_jN/N$ for some $j \in \{1, \dots, n\}$. Then $H_{j-1} \not\trianglelefteq H_j$. By Definition 1 $|H_j : H_{j-1}|$ is a some prime q and $q-1$ is not divisible by the $(t+1)$ th powers of primes. Hence H_{j-1} is maximal in H_j and $H_{j-1} = (H_j \cap N)H_{j-1}$. We have $|H_jN/N : H_{j-1}N/N| = |H_j : H_{j-1}| = q$. Therefore HN/N is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G/N .

(2) Assume that H/N is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G/N . Then there exists a chain of subgroups

$$H/N = H_0/N \leq H_1/N \leq \cdots \leq H_{n-1}/N \leq H_n/N = G/N$$

such that either $H_{i-1}/N \trianglelefteq H_i/N$ or $|H_i/N : H_{i-1}/N|$ is a some prime p and $p-1$ is not divisible by the $(t+1)$ th powers of primes for every $i = 1, \dots, n$. Therefore either $H_{i-1} \trianglelefteq H_i$ or $|H_i/N : H_{i-1}/N| = |H_i : H_{i-1}|$ and H is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G .

(3) The statement follows from statements (1) and (2).

(4) Let HN_i is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G and $N_i \trianglelefteq G$, $i = 1, 2$. Then there is a chain of subgroups

$$HN_1 = K_0 \leq K_1 \leq \cdots \leq K_{s-1} \leq K_s = G$$

such that either K_{i-1} is normal in K_i or $|K_i : K_{i-1}|$ is a some prime p and $p-1$ is not divisible by the $(t+1)$ th powers of primes for every $i = 1, \dots, s$. If $(K_{i-1} \cap HN_2) \trianglelefteq (K_i \cap HN_2)$ for every $i = 1, \dots, s$, then $HN_1 \cap HN_2$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G . Let there be $j \in \{1, \dots, s\}$ such that $(K_{j-1} \cap HN_2) \not\trianglelefteq (K_j \cap HN_2)$. Then $K_{j-1} \not\trianglelefteq K_j$ and $|K_j : K_{j-1}|$ is a prime p and $p-1$ is not divisible by the $(t+1)$ th powers of primes. Since $K_{j-1} \leq (K_j \cap HN_2)K_{j-1} \leq K_j$ and $(K_j \cap HN_2)K_{j-1} = (K_j \cap N_2)K_{j-1}$ is a subgroup of K_j , it follows that $(K_j \cap HN_2)K_{j-1} = K_j$. Therefore, $|K_j \cap HN_2 : K_{j-1} \cap HN_2| = |(K_j : K_{j-1})| = p$. Thus, considering the chain of subgroups

$$HN_1 \cap HN_2 = K_0 \cap HN_2 \leq K_1 \cap HN_2 \leq \cdots \leq K_{s-1} \cap HN_2 \leq K_s \cap HN_2 = HN_2$$

and taking into account the $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormality of HN_2 in G , we see that $(HN_1 \cap HN_2)$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G . \square

Lemma 2.3. *Let G a soluble group. Let H and U be subgroups of G . Then the following statements hold.*

(1) *If H is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G , then $(H \cap U)$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in U .*

(2) *If H is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G and U is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G , then $(H \cap U)$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G .*

Proof. (1) Let H be $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in G .

If $H_{i-1} \cap U \trianglelefteq H_i \cap U$ for all $i = 1, \dots, n$ and H_i is from the chain (1.1), then $(H \cap U)$ is $\mathbb{K}\text{-}\mathbb{P}_t$ -subnormal in U .

Suppose that there exists $j \in \{1, \dots, n\}$ such that $H_{j-1} \cap U \not\trianglelefteq H_j \cap U$. Then $H_{j-1} \not\trianglelefteq H_j$ and $|H_j : H_{j-1}|$ is a some prime p and $p-1$ is not divisible by the $(t+1)$ th powers of primes.

Let's denote $L = \text{Core}_{H_j}(H_{j-1})$. By [20, Chap. A, Theorem 15.2(4)] $H_j/L = N/L \cdot H_{j-1}/L$, where N/L is a minimal normal subgroup of H_j/L , $N/L = C_{H_j/L}(N/L)$ and $(N/L) \cap (H_{j-1}/L) = L/L$. Then $|N/L| = |H_j : H_{j-1}| = p$ and $H_{j-1}/L \cong H_j/L/C_{H_j/L}(N/L) \cong \text{Aut}_{H_j}(N/L)$ is isomorphic to a subgroup from Z_{p-1} . Therefore $N/L \in \text{Syl}_p(G)$.

Let us first assume that p does not divide $|(H_j \cap U)L/L|$. Since G is soluble, by Hall's theorem [20, Chap. A, Theorem 3.3] $(H_j \cap U)L/L \leq (H_{j-1}/L)^{xL}$ for some $x \in H_j$. Then $(H_j \cap U)/(U \cap L) \cong (H_j \cap U)L/L$ is cyclic, we have $(H_{j-1} \cap U)/(U \cap L) \trianglelefteq (H_j \cap U)/(U \cap L)$. We have obtained a contradiction with the assumption $H_{j-1} \cap U \not\leq H_j \cap U$.

Thus p divides $|(H_j \cap U)L/L|$. Since $N/L \trianglelefteq H_j/L$, by Sylow's theorem $N/L \leq (H_j \cap U)L/L$. Then

$$(H_j \cap U)L/L = (H_j \cap U)L/L \cap N/L \cdot H_{j-1}/L = N/L \cdot (H_{j-1} \cap U)L/L$$

and $p = |N/L| = |(H_j \cap U) : (H_{j-1} \cap U)|$. Therefore we have the chain of subgroups

$$H \cap U = H_0 \cap U \leq H_1 \cap U \leq \dots \leq H_{n-1} \cap U \leq H_n \cap U = U$$

such that either $(H_{i-1} \cap U) \trianglelefteq (H_i \cap U)$ or $|(H_i \cap U) : (H_{i-1} \cap U)|$ is a some prime p and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes for every $i = 1, \dots, n$.

Statement (2) follows from (1) and Lemma 2.1(2). \square

Lemma 2.4. *If G is a soluble group and H is $\text{K-}\mathbb{P}_t$ -subnormal in G , then H is \mathfrak{U}_t -subnormal in G .*

Proof. By Definition 1 there is a chain of subgroups (1.1). Assume that $H \neq G$. Then $H_{i-1} \neq H_i$ for some $i \in \{1, \dots, n\}$.

Assume that $H_{i-1} \trianglelefteq H_i$. Since G is soluble, we draw a composition series through H_{i-1} and H_i :

$$H_{i-1} = R_0 < R_1 < \dots < R_{m-1} < R_m = H_i,$$

there R_{j-1} is normal in R_j and $|R_j : R_{j-1}|$ is some prime q , $j = 1, \dots, m$. Since $R_{j-1} = \text{Core}_{R_j}(R_{j-1})$, we have $R_j/R_{j-1} \in \mathfrak{U}_t$. Thus $R_j^{\mathfrak{U}_t} \leq R_{j-1}$.

Assume that $H_{i-1} \not\trianglelefteq H_i$. Then $|H_i : H_{i-1}|$ is a some prime p and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes. Since $L = \text{Core}_{H_i}(H_{i-1}) \neq H_{i-1}$, we have $H_i/L = N/L \cdot H_{i-1}/L$, where N/L is a minimal normal subgroup of H_i/L , $N/L = C_{H_i/L}(N/L)$ and $(N/L) \cap (H_{i-1}/L) = L/L$ by [20, Chap. A, Theorem 15.2(4)]. Then $|N/L| = |H_i : H_{i-1}| = p$ and $H_{i-1}/L \cong H_i/L/C_{H_i/L}(N/L) \cong \text{Aut}_{H_i}(N/L)$ is isomorphic to a subgroup from Z_{p-1} . Thus $H_i/L \in \mathfrak{U}_t$ and $H_i^{\mathfrak{U}_t} \leq L \leq H_{i-1}$.

Therefore H is \mathfrak{U}_t -subnormal in G . \square

Note that the converse of Lemma 2.4 does not always hold.

Example 2.1. Let $t = 1$ and let G be a non-abelian group of order 39. In G , the Sylow 3-subgroup H is \mathfrak{U}_1 -subnormal, but not $\text{K-}\mathbb{P}_1$ is subnormal, since $G^{\mathfrak{U}_1} = 1 \leq H$, $|G : H| = 13$ and $13 - 1 = 2^2 \cdot 3$.

3. Classes of groups with $\text{K-}\mathbb{P}_t$ -subnormal Sylow subgroups

Lemma 3.1. *Let G be a group. Let $H \in \text{Syl}_p(G)$ and $N \trianglelefteq G$. If H is $\text{K-}\mathbb{P}_t$ -subnormal in G , then every Sylow p -subgroup of N is $\text{K-}\mathbb{P}_t$ -subnormal in N and every Sylow p -subgroup of G/N is $\text{K-}\mathbb{P}_t$ -subnormal in G/N .*

Proof. Let $P \in \text{Syl}_p(N)$ and $R/N \in \text{Syl}_p(G/N)$. By Sylow's theorem $P \leq H^x$ for some $x \in G$ and $R/N = (HN/N)^{yN}$ for some $y \in G$. Then $P = H^x \cap N$ is $\text{K-}\mathbb{P}_t$ -subnormal in N and $R/N = H^y N/N$ is $\text{K-}\mathbb{P}_t$ -subnormal in G/N by Lemmas 2.1(1), 2.2(1). \square

By Lemma 1.2 the class of groups $w_t\mathfrak{U}$ consists of Ore dispersive groups, therefore $w_t\mathfrak{U} \subseteq \mathfrak{S}$.

Theorem 3.1. *Let \mathfrak{H} be the class of all groups in which every Sylow subgroup is $\text{K-}\mathbb{P}_t$ -subnormal. Let G be a group. Then the following assertions hold.*

- (1) $\mathfrak{N} \subseteq \mathfrak{H}$ and \mathfrak{H} consists of Ore dispersive groups.
- (2) If $G \in \mathfrak{H}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{H}$.
- (3) If $G/N_1 \in \mathfrak{H}$ and $G/N_2 \in \mathfrak{H}$ for any $N_i \trianglelefteq G$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathfrak{H}$.
- (4) A direct product of groups from \mathfrak{H} lies in \mathfrak{H} .
- (5) If $G \in \mathfrak{H}$ and U is a subgroup of G , then $U \in \mathfrak{H}$.
- (6) If $G/\Phi(G) \in \mathfrak{H}$, then $G \in \mathfrak{H}$.
- (7) The class \mathfrak{H} is a hereditary saturated formation.

Proof. Statement (1) follows from Definition 1 and Lemma 1.2.

Statement (2) follows from Lemma 3.1.

(3) Let G be a group of the least order such that $G/N_i \in \mathfrak{H}$, $N_i \trianglelefteq G$, $i = 1, 2$, but $G/N_1 \cap N_2 \notin \mathfrak{H}$.

If $N = N_1 \cap N_2 \neq 1$, then $G/N/N_i/N \cong G/N_i \in \mathfrak{H}$ for $i = 1, 2$. It follows from $|G/N| < |G|$ that $G/N/(N_1/N \cap N_2/N) \cong G/N_1 \cap N_2 \in \mathfrak{H}$. This contradicts the choice of G .

Thus, $N_1 \cap N_2 = 1$. Let $Q \in \text{Syl}_q(G)$. Then $QN_i/N_i \in \text{Syl}_q(G/N_i)$ and QN_i/N_i is $\text{K-}\mathbb{P}_t$ -subnormal in G/N_i for $i = 1, 2$. By Lemma 2.2(2) QN_i is $\text{K-}\mathbb{P}_t$ -subnormal in G for $i = 1, 2$. Since G is soluble, by Lemma 2.3(2) $QN_1 \cap QN_2$ is $\text{K-}\mathbb{P}_t$ -subnormal in G . By [20, Chap. A, Theorem 6.4(b)] $QN_1 \cap QN_2 = Q(N_1 \cap N_2) = Q$ is $\text{K-}\mathbb{P}_t$ -subnormal in G . Therefore $G = G/N_1 \cap N_2 \in \mathfrak{H}$. The contradiction thus obtained completes the proof of Statement (3).

Statement (4) follows from (3).

(5) Let $G \in \mathfrak{H}$ and $U \leq G$. Let's take $P \in \text{Syl}_p(U)$. By Sylow's theorem $P \leq P_1$ for some $P_1 \in \text{Syl}_p(G)$. Since G is soluble and P_1 is $\text{K-}\mathbb{P}_t$ -subnormal in G , by Lemma 2.3(1) $P = P_1 \cap U$ is $\text{K-}\mathbb{P}_t$ -subnormal in U . Therefore $U \in \mathfrak{H}$. Statement (5) has been proven.

(6) Let G be a group of the least order for which $G/\Phi(G) \in \mathfrak{H}$ and $G \notin \mathfrak{H}$. Then G is soluble, since $G/\Phi(G)$ and $\Phi(G)$ are soluble. Let N be a minimal normal subgroup of G . We note that N is a p -group for some prime p . By [20, Chap. A, Theorem 9.2(e)] $\Phi(G)N/N \leq \Phi(G/N)$. Since $G/\Phi(G)N \in \mathfrak{H}$, we have $(G/N)/\Phi(G/N) \in \mathfrak{H}$. From $|G/N| < |G|$, it follows $G/N \in \mathfrak{H}$.

By (2) and (3) \mathfrak{H} is a formation. Then N is a unique minimal normal subgroup of G . Thus, $N \leq \Phi(G) \leq F(G)$ and $F(G)$ is p -groups. By [20, Chap. A, Theorem 10.6(c)] $\Phi(G) < F(G)$. Let $Q \in \text{Syl}_q(G)$. Since $QN/N \in \text{Syl}_q(G/N)$ and $QN/N \not\leq \Phi(G/N)$, we have $QN/N \neq N/N$ and QN/N is $\text{K-}\mathbb{P}_t$ -subnormal in G/N . By Lemma 2.2(2) QN is $\text{K-}\mathbb{P}_t$ -subnormal in G .

If $q = p$, then $QN = Q$ is $\text{K-}\mathbb{P}_t$ -subnormal in G .

Let $q \neq p$. Write $H/N = QF(G)/N$. By Lemma 2.3(1) QN/N is $\text{K-}\mathbb{P}_t$ -subnormal in H/N . Let's consider two cases.

1. Assume that QN/N is subnormal in H/N . Since QN/N is pronormal in H/N , by [20, Chap. A, Lemma 6.3(d)] QN/N is normal in H/N . Thus, QN is normal in

H . Since $\Phi(F(G)) \text{ char } F(G) \trianglelefteq G$, we have $\Phi(F(G)) \trianglelefteq G$. Then $N \leq \Phi(F(G))$. From $F(G) \trianglelefteq H$ and by [20, Chap. A, Theorem 9.2(e)] it follows that $\Phi(F(G)) \leq \Phi(H)$. Thus, $N \leq \Phi(F(G)) \leq \Phi(H)$. By Frattini argument $H = N_H(Q)QN = N_H(Q)$. Then we have $Q \trianglelefteq QN$ and Q is $K\text{-}\mathbb{P}_t$ -subnormal in G .

2. Assume that QN/N is not subnormal in H/N . From $N \leq \Phi(G) < F(G)$ and $Q \in \text{Syl}_q(H)$ we have $QN/N \neq H/N$. By Definition 1 it follow that there exists a chain of subgroups $QN/N = R_0/N \leq R_1/N \leq \dots \leq R_{m-1}/N \leq R_m/N = H/N$ such that either $R_{i-1}/N \trianglelefteq R_i/N$ or $|R_i/N : R_{i-1}/N| = p$ and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes for every $i = 1, \dots, m$. Since $G/\Phi(G) \in \mathfrak{H} \subseteq \overline{\mathfrak{WU}}$ and by Lemma 1.2 $\overline{\mathfrak{WU}}$ is a hereditary saturated formation, we have $G \in \overline{\mathfrak{WU}}$. Then Q is $K\text{-}\mathbb{P}$ -subnormal in G and Q is $K\text{-}\mathbb{P}$ -subnormal in QN . Since $|QN : Q| = |N|$ and $p - 1$ is not divisible by the $(t + 1)$ th powers of primes, this implies that Q is $K\text{-}\mathbb{P}_t$ -subnormal in G . Therefore, $G \in \mathfrak{H}$. We arrive at a contradiction to the choice of G . Statement (6) has been proven.

Statement (7) follows from (2), (3) and (5). \square

In the work [24], local definitions of the formation of groups whose Sylow subgroups are \mathfrak{F} -subnormal ($K\text{-}\mathfrak{F}$ -subnormal, respectively) were studied. Next we solve a similar problem for the case of $K\text{-}\mathbb{P}_t$ -subnormal Sylow subgroups.

Since $\mathfrak{N}_p\mathfrak{A}(p - 1)$ is a hereditary formation, the following result is easily verified.

Lemma 3.2. *Let p be a prime number and $p - 1$ is not divisible by the $(t + 1)$ th powers of prime numbers. The class of groups $(G \mid \text{Syl}(G) \subseteq \mathfrak{N}_p\mathfrak{A}(p - 1))$ is a hereditary formation.*

Теорема 3.2. *Let \mathfrak{H} be the class of all groups in which every Sylow subgroup is $K\text{-}\mathbb{P}_t$ -subnormal. Then \mathfrak{H} is a hereditary saturated formation that is defined by a local function F such that $F(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{N}_p\mathfrak{A}(p - 1))$ if $p - 1$ is not divisible by the $(t + 1)$ th powers of prime numbers; $F(p) = \mathfrak{N}_p$ if $p - 1$ is divisible by the $(t + 1)$ th power of some prime number.*

Proof. By Theorem 3.1 \mathfrak{H} is a hereditary saturated formation. Since $F(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{N}_p\mathfrak{A}(p - 1))$ is a formation and $F(p) = \mathfrak{N}_p$ is a formation, F is a local function. Let $\mathfrak{F} = LF(F)$. By [20, Chap. IV, Proposition 3.14 and Theorem 4.6] \mathfrak{F} is a hereditary saturated formation.

Show that $\mathfrak{F} \subseteq \mathfrak{H}$. Let G be a group of least order in $\mathfrak{F} \setminus \mathfrak{H}$. Let N be a minimal normal subgroup of G . Since $\mathfrak{F} \subseteq \mathfrak{S}$, N is an p -group for some prime p . From $\mathfrak{N}_p \subseteq \mathfrak{F} \cap \mathfrak{H}$ it follows that $G \neq N$. We have $G/N \in \mathfrak{H}$, $\Phi(G) = 1$ and N is a unique minimal normal subgroup of G , since \mathfrak{F} and \mathfrak{H} are saturated formation. Then $G = NM$, where M is a maximal subgroup of G , $\text{Core}_G(M) = 1$, $M \cap N = 1$ and $N = C_G(N)$. Let $Q \in \text{Syl}_q(G)$. Then $QN/N \in \text{Syl}_q(G/N)$. From $G/N \in \mathfrak{H}$ by Lemma 2.2(3) we have that QN is $K\text{-}\mathbb{P}_t$ -subnormal in G .

If $q = p$ then $QN = Q$ is $K\text{-}\mathbb{P}_t$ -subnormal in G .

Let $q \neq p$. If $G \neq QN$, then $QN \in \mathfrak{H}$ since \mathfrak{F} and \mathfrak{H} are hereditary. Thus Q is $K\text{-}\mathbb{P}_t$ -subnormal in QN and Q is $K\text{-}\mathbb{P}_t$ -subnormal in G . Now suppose that $G = QN$. Then $Q = M$. We have that $Q \cong G/N = G/C_G(N) \in F(p)$ because $G/N \in \mathfrak{F}$. Hence $F(p) \neq \mathfrak{N}_p$. Therefore $p - 1$ is not divisible by the $(t + 1)$ th powers of prime numbers and $Q \in \mathfrak{A}(p - 1)$. Then G is supersoluble and $|N| = p$. We conclude that $|G : Q| = p$ and by Definition 1, we have that Q is $K\text{-}\mathbb{P}_t$ -subnormal in G . Hence $G \in \mathfrak{H}$. It is a contradiction to the choice of G . Thus, $\mathfrak{F} \subseteq \mathfrak{H}$.

Prove that $\mathfrak{H} \subseteq \mathfrak{F}$. Let G be a group of least order in $\mathfrak{H} \setminus \mathfrak{F}$. From $G \in \mathfrak{H}$, we have that G is soluble. We denote by N a minimal normal subgroup of G . If $G = N$ then $G/C_G(N) = G/N \cong 1 \in \mathfrak{F}$. It is a contradiction to the choice of G . Hence $G \neq N$. Since \mathfrak{H} and \mathfrak{F} are saturated formations, $\Phi(G) = 1$. In G , N is the unique minimal normal subgroup, $N = C_G(N) = F(G)$ and $|N| = p^\alpha$ for some prime p . The group $G = NM$, where M is a maximal subgroup of G , $N \cap M = 1$ and $\text{Core}_G(M) = 1$. Since G is Ore dispersive, we have $N \leq P \trianglelefteq G$, $P \in \text{Syl}_p(G)$ and p is the largest prime number in $\pi(G)$. By [20, cp. A, Theorem 15.6(b)] it follows that $O_p(M) = 1$. From $P \cap M \leq O_p(M)$ we have that $P = (M \cap P)N = N \in \text{Syl}_p(G)$ and M is a p' -group.

Let $R \in \text{Syl}_q(M)$. Then $R \in \text{Syl}_q(G)$.

Suppose that $G \neq RN$. Let $H = RN$. Since \mathfrak{H} is hereditary, we have that $H \in \mathfrak{H}$ and $H \in \mathfrak{F}$. Note that $C_H(N) = N$. Then $R \cong H/C_H(N) \in F(p)$. From $q \neq p$ it follows that $F(p) \neq \mathfrak{N}_p$ and $p-1$ is not divisible by the $(t+1)$ th powers of prime numbers. Thus $R \in \mathfrak{A}(p-1)$. Since R is chosen arbitrarily, we have that $G/C_G(N) = G/N \cong M \in F(p)$. Hence $G \in \mathfrak{F}$. We have a contradiction to the choice of G .

Now suppose that $G = RN$. By [22] G is supersoluble. Then $|N| = p$. Hence $R = M \cong G/C_G(N)$ is isomorphic to a subgroup from Z_{p-1} . On the other hand, R is $\text{K-}\mathbb{P}_t$ -subnormal in G , since $G \in \mathfrak{H}$. From $|G : R| = p$ it follows that $p-1$ is not divisible by the $(t+1)$ th powers of prime numbers. Therefore $G/C_G(N) \in F(p)$ and $G \in \mathfrak{F}$. This contradicts the choice of G . Thus $\mathfrak{H} \subseteq \mathfrak{F}$. This means that the equality $\mathfrak{H} = \mathfrak{F}$ is proven. \square

Definition 3.1. Denote by \mathfrak{U}_t^0 the class of all supersoluble groups in which every Sylow subgroup is $\text{K-}\mathbb{P}_t$ -subnormal.

Theorem 3.3. The class of groups \mathfrak{U}_t^0 is a hereditary saturated formation that is defined by a local function X such that $X(p) = \mathfrak{N}_p \mathfrak{A}(p-1)$ if $p-1$ is not divisible by the $(t+1)$ th powers of prime numbers; $X(p) = \mathfrak{N}_p$ if $p-1$ is divisible by the $(t+1)$ th power of some prime number.

Proof. We have that $\mathfrak{U}_t^0 = \mathfrak{U} \cap \mathfrak{H}$.

Let $G \in \mathfrak{U}_t^0$ and let H/K be its any chief factor. Then $|H/K| = p$ for some prime p . Suppose that $p-1$ is divisible by the $(t+1)$ th power of some prime number. From $G \in \mathfrak{H}$ by Theorem 3.2, we have that $G/C_G(H/K) \in F(p) = \mathfrak{N}_p = X(p)$. Now suppose that $p-1$ is not divisible by the $(t+1)$ th power of some prime number. Since $G \in \mathfrak{U}$ we have that $G/C_G(H/K) \in \mathfrak{A}(p-1) \subseteq \mathfrak{N}_p \mathfrak{A}(p-1) = X(p)$. This means that $G \in LF(X)$, i.e. $\mathfrak{U}_t^0 \subseteq LF(X)$.

Now let $G \in LF(X)$. Since $X(p) \subseteq \mathfrak{S}$ we have that $LF(X) \subseteq \mathfrak{S}$. Let H/K be any chief factor of G . Then H/K is abelian and $|H/K| = p^\alpha$ for some prime p .

Suppose that $p-1$ is divisible by the $(t+1)$ th power of some prime number. Then $G/C_G(H/K) \in X(p) = \mathfrak{N}_p$. By Theorem 3.2 $G \in \mathfrak{H}$. On the other hand, by [20, Chap. A, Lemma 13.6(b)] $O_p(G/C_G(H/K)) = 1$ it follows that $G/C_G(H/K) = 1 \in \mathfrak{A}(p-1)$. Therefore $G \in \mathfrak{U}$. Thus $G \in \mathfrak{U} \cap \mathfrak{H}$.

Let $p-1$ is not divisible by the $(t+1)$ th powers of prime numbers. Then $G/C_G(H/K) \in X(p) = \mathfrak{N}_p \mathfrak{A}(p-1)$. We have that $G/C_G(H/K) \in F(p)$ and $G \in \mathfrak{H}$ by Theorem 3.2. From [20, Chap. A, Lemma 13.6(b)] we conclude that $G/C_G(H/K) \in \mathfrak{A}(p-1)$. Thus $G \in \mathfrak{U}$. Therefore $G \in \mathfrak{U}_t^0$ and $LF(X) \subseteq \mathfrak{U}_t^0$.

Thus $LF(X) = \mathfrak{U}_t^0$. \square

Remark 3.1. If $t = 1$ then from Example 2.1 it follows that $\mathfrak{U}_1^0 \subseteq s\mathfrak{U}$ and $\mathfrak{U}_1^0 \neq s\mathfrak{U}$. Here $s\mathfrak{U}$ is a class of all supersoluble groups in which each Sylow subgroup is submodular. The properties of submodular subgroups and the class $s\mathfrak{U}$ were studied in [25] and [26].

Теорема 3.4. *The class \mathfrak{H} of all groups in which every Sylow subgroup is $K\text{-}\mathbb{P}_t$ -subnormal coincides with the class of all groups in which every Sylow subgroup is \mathfrak{U}_t^0 -subnormal, i.e. $\mathfrak{H} = w\mathfrak{U}_t^0$.*

Proof. From $\mathfrak{N} \subseteq \mathfrak{U}_t^0$ we have that $\pi(\mathfrak{U}_t^0) = \mathbb{P}$. Since \mathfrak{U}_t^0 is a hereditary saturated formation, by Lemma 1.5 $w\mathfrak{U}_t^0$ is a hereditary saturated formation and $w\mathfrak{U}_t^0 \subseteq \mathfrak{S}$.

Show that $\mathfrak{H} \subseteq w\mathfrak{U}_t^0$. Suppose that $\mathfrak{H} \setminus w\mathfrak{U}_t^0 \neq \emptyset$. Let G be a group of the least order in $\mathfrak{H} \setminus w\mathfrak{U}_t^0$. Then G has the unique minimal normal subgroup N and $\Phi(G) = 1$, since \mathfrak{H} and $w\mathfrak{U}_t^0$ are saturated formations. From $G \in \mathfrak{H} \subseteq \mathfrak{S}$ it follows that N is a p -group for some prime p . Then $G = NM$, where M is some maximal in G subgroup with $\text{Core}_G(M) = 1$, $N \cap M = 1$ and $N = C_G(N)$.

Let Q be an arbitrary Sylow q -subgroup of G . Since $G/N \in w\mathfrak{U}_t^0$ we have that QN/N is \mathfrak{U}_t^0 -subnormal in G/N . By Lemma 1.4(2) QN is \mathfrak{U}_t^0 -subnormal in G . If $q = p$, then $QN = Q$ is \mathfrak{U}_t^0 -subnormal in G .

Let $q \neq p$. Suppose that $QN \neq G$. From $QN \in w\mathfrak{U}_t^0$ we conclude that Q is \mathfrak{U}_t^0 -subnormal in QN . By Lemma 1.4(1) Q is \mathfrak{U}_t^0 -subnormal in G . Let $QN = G$. Then $Q = M$. By Theorem 3.2 $Q \cong G/C_G(N) \in F(p)$. Since $G \notin \mathfrak{N}$, we have that $F(p) = (G \in \mathfrak{S} \mid \text{Syl}(G) \subseteq \mathfrak{N}_p\mathfrak{A}(p-1))$ if $p-1$ is not divisible by the $(t+1)$ th powers of prime numbers. Hence $Q \in \mathfrak{A}(p-1)$ and $G \in \mathfrak{U}$. Thus $G \in w\mathfrak{U}_t^0$ and Q is \mathfrak{U}_t^0 -subnormal in G . Consequently, $G \in w\mathfrak{U}_t^0$. We get a contradiction. Hence $\mathfrak{H} \subseteq w\mathfrak{U}_t^0$.

Prove that $w\mathfrak{U}_t^0 \subseteq \mathfrak{H}$. Suppose that $w\mathfrak{U}_t^0 \setminus \mathfrak{H} \neq \emptyset$. Choose a group G of the least order in $w\mathfrak{U}_t^0 \setminus \mathfrak{H}$. It is clear that G has the unique minimal normal subgroup $N = G^{\mathfrak{S}} = F(G)$ and $\Phi(G) = 1$. From $\mathfrak{H} \subseteq \mathfrak{S}$ it follows that $|N| = p^\alpha$ for some prime p . Let $S \in \text{Syl}_q(G)$. By choice of G we have $G \neq N$. From $G/N \in \mathfrak{H}$ and $SN/N \in \text{Syl}_q(G/N)$ it follows that SN/N is $K\text{-}\mathbb{P}_t$ -subnormal in G/N . By Lemma 2.2(2) SN is $K\text{-}\mathbb{P}_t$ -subnormal in G .

If $q = p$, then $SN = S$ is $K\text{-}\mathbb{P}_t$ -subnormal in G .

Suppose that $q \neq p$. If $SN \neq G$ then $SN \in \mathfrak{H}$ by choice of G . Consequently S is $K\text{-}\mathbb{P}_t$ -subnormal in SN . By Lemma 2.1(2) S is $K\text{-}\mathbb{P}_t$ -subnormal in G . Let $SN = G$. Then S is maximal in G and $\text{Core}_G(S) = 1$. From $G \in w\mathfrak{U}_t^0$ it follows that S is \mathfrak{U}_t^0 -subnormal in G . Therefore $G^{\mathfrak{U}_t^0} \leq S$. Then $G^{\mathfrak{U}_t^0} \leq \text{Core}_G(S) = 1$ and $G \in \mathfrak{U}_t^0$. Thus S is $K\text{-}\mathbb{P}_t$ -subnormal in G . Due to the arbitrary choice of S , we received a contradiction $G \in \mathfrak{H}$. Consequently $w\mathfrak{U}_t^0 \subseteq \mathfrak{H}$. Thus $w\mathfrak{U}_t^0 = \mathfrak{H}$. \square

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References

- [1] Vasil'ev A.F., Vasil'eva T.I., Tyutyaynov V.N. On the finite groups of supersoluble type // Siberian Math. J. 2010. Vol. 51, No. 6. P. 1004–1012.
- [2] Vasil'ev A.F., Vasil'eva T.I., Tyutyaynov V.N. On $K - \mathbb{P}$ -Subnormal Subgroups of Finite Groups // Math. Notes. 2014. Vol. 95, No. 4. P. 471–480.

- [3] *Kniahina V.N., Monakhov V.S.* On supersolvability of finite groups with \mathbb{P} -subnormal subgroups // Internal. J. of Group Theory. 2013. Vol.2, No. 4. P. 21–29.
- [4] *Vasil'ev A.F., Vasil'eva T.I., Myslovets E.N.* Finite widely c -supersoluble groups and their mutually permutable products // Siberian Math. J. 2016. Vol. 57, No. 3. pp. 476–485.
- [5] *Vasil'ev A.F., Vasil'eva T.I., Parfenkov K.L.* Finite groups with three given subgroups // Siberian Math. J. 2018. Vol. 59, No.1. P. 50–58.
- [6] *Ballester-Bolinches A., Li Y., Pedraza-Aguilera M.C., Su N.* On Factorised Finite Groups // Mediterr. J. Math. 2020. Vol. 17, No. 2. P. 65.
- [7] *Murashka V.I.* Finite Groups With Given Sets of \mathfrak{F} -Subnormal Subgroups // Asian-Eur. J. Math. 2020. Vol. 13, No. 4. P. 2050073.
- [8] *Lucchini A., Nemmi D.* The Non- \mathfrak{F} Graph of a Finite Group // Math. Nachr. 2021. Vol. 294, No. 10. P. 1912–1921.
- [9] *Trofimuk A.A.* Finite factorizable groups with restrictions on factors. Minsk: BSU Publishing Center, 2021 (In Russian).
- [10] *Vasilyeva T.I.* Subgroups of the Fan of Sylow Subgroups and the Supersolvability of a Finite Group // Math. Notes. 2021. Vol. 110, No. 2. P. 186–195.
- [11] *Ballester-Bolinches A., Madanha S.Y., Shumba T.M.M., Pedraza-Aguilera M.C.* On Certain Products of Permutable Subgroups // Bull. Aust. Math. Soc. 2022. Vol. 105, No. 2. P. 278–285.
- [12] *Ballester-Bolinches A., Madanha S.Y., Pedraza-Aguilera M.C., Wu X.* On some products of finite groups // Proc. Edinburgh Math. Soc. 2023. Vol. 66, No. 1. P. 89.
- [13] *Chen R. Zhao X. Li X.* \mathbb{P} -Subnormal Subgroups and the Structure of Finite Groups // Ric. Mat. 2023. Vol. 72. P. 771–778.
- [14] *Vasilyeva T.I., Koranchuk A.G.* On Finite Groups with \mathbb{P}_π -Subnormal Subgroups // Math. Notes. 2023. Vol. 114, No. 4. P. 421–432.
- [15] *Murashka V.I.* Formations of finite groups in polynomial time: \mathfrak{F} -residuals and \mathfrak{F} -subnormality // Journal of Symbolic Computation. 2024. Vol. 122. P. 102271.
- [16] *Yi X., Xu Z., Kamornikov S.F.* Finite groups with \mathbb{P} -subnormal Schmidt subgroups // Trudy Instituta Matematiki i Mekhaniki UrO RAN. 2024. Vol. 30, No. 1. 100–108.
- [17] *Lisi F.* A Jordan–Holder type theorem for finite groups // Annali di Matematica. 2024. <https://doi.org/10.1007/s10231-024-01456-w>
- [18] *Ballester-Bolinches A., Ezquerro L.M.* Classes of Finite Groups, in Math. Appl. (Springer) Dordrecht: Springer, 2006. Vol. 584.

- [19] *Monakhov V.S., Sokhor I.L.* Finite groups with formational subnormal primary subgroups of bounded exponent // Siberian Electronic Mathematical News. 2023. Vol. 20, No. 2. P. 785–796.
- [20] *Doerk K., Hawkes T.* Finite Soluble Groups. Berlin-New York: Walter de Gruyter, 1992.
- [21] Between Nilpotent and Solvable / H.G. Bray [and others]; edited by M. Weinsten. Passaic: Polugonal Publishing House, 1982.
- [22] *Vasil'ev A.F.* New properties of finite dinilpotent groups // Vestsi Nats. Akad. Navuk Belarusi. Ser. Fiz.-Mat. Navuk. 2004. No. 2. P. 29–33 (In Russian).
- [23] *Vasil'ev A.F., Vasil'eva T.I.* On finite groups with generally subnormal Sylow subgroups // Problems of physics, mathematics and technics. 2011. No. 4 (9). P. 86–91 (In Russian).
- [24] *Vasil'ev A.F., Vasil'eva T.I., Vegera A.S.* Finite groups with generalized subnormal embedding of Sylow subgroups // Siberian Math. J. 2016. Vol. 57, No. 2. P. 200–212.
- [25] *Zimmermann I.* Submodular subgroups in finite groups // Math. Z. 1989. Vol. 202. P. 545–557.
- [26] *Vasilyev V.A.* Finite groups with submodular Sylow subgroups // Siberian Math. J. 2015. Vol. 56, No. 6. P. 1019–1027.

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