ONE FORMATION OF FINITE GROUPS

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Abstract. It is shown that two formations of finite groups, one was introduced by V.S. Monakhov and V.N. Kniahina and another one was introduced by R. Brandl, are coincides.

In this note only finite groups are considered. V.S. Monakhov and V.N. Kniahina [1] studied the class $\mathfrak{X}$ of all groups whose cyclic primary subgroups are $\mathbb{P}$-subnormal. It was shown that $\mathfrak{X}$ is a hereditary saturated formation and $\mathfrak{U} \subset \mathfrak{X} \subset \mathfrak{D}$ where $\mathfrak{U}$ and $\mathfrak{D}$ are classes of all supersoluble groups and groups with Sylow tower of supersoluble type respectively. In [2, 3] R. Brandl studied groups satisfying some law $\ddot{u}_k(x,y) = 1$ where $\ddot{u}_1(x,y) = [x,y]$ and $\ddot{u}_{k+1}(x,y) = \ddot{u}_k(x,y)^{-k}[\ddot{u}_k(x,y),y]$ for $k > 1$. He showed that the class $\mathfrak{B} = (G|$ for all $x,y \in G$ there is a natural k such that $\ddot{u}_k(x,y) = 1)$ is a hereditary saturated formation containing $\mathfrak{U}$. Also he showed that this class coincides with the class of all groups whose subgroups with nilpotent derived subgroup are supersoluble. In this note will be proved that classes $\mathfrak{B}$ and $\mathfrak{X}$ are coincides.

The standard definitions and notation from [4] are used. Recall [5] that a subgroup H of a group G is called $\mathbb{P}$-subnormal if either $H = G$ or there is a maximal chain of subgroups $H = H_0 \subset H_1 \subset \dots \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $i = 1, \dots, n$.

Theorem. *For a group G the following statements are equivalent*:

(1) *All cyclic primary subgroups of G are $\mathbb{P}$-subnormal in G.*

(2) *All subgroups of G with nilpotent derived subgroup are supersoluble.*

(3) *For all $x,y \in G$ there is natural k such that $\ddot{u}_k(x,y) = 1$.*

Proof. The equivalence of Assertions (2) and (3) was established in [2]. Also there were shown that $\mathfrak{B} = LF(f)$ where $f(p)$ is the class of all soluble groups of exponent dividing $p-1$ for all primes p. It is well known that $f(p)$ is hereditary formation for all primes p.

Assume that $\mathfrak{X}$ is not contained in $\mathfrak{B}$. Let choose a group G of minimal order from $\mathfrak{X} \setminus \mathfrak{B}$. Note [5] that G is soluble. Since $\mathfrak{X}$ and $\mathfrak{B}$ are saturated formations, we see that $\Phi(G) = 1$. It is clear that G is $\mathfrak{B}$-critical group. Since $\mathfrak{B}$ is a formation, we note that G has unique minimal normal subgroup N. Now N is abelian p-subgroup and $N = C_G(N)$. Let M be a maximal subgroup G which does not contain N. Then $G = NM$. Since N is abelian, $N \cap M = 1$. Since G has Sylow tower of supersoluble type, p is the maximal prime divisor of $|G|$ and $(p, |M|) = 1$. Let H be a proper subgroup of M. From $N = C_G(N)$ it follows that $O_{p'}(NH) = 1$. Now $O_{p',p}(NH) = N$. From $NH \in \mathfrak{B}$ it follows that $H \simeq NH/O_{p',p}(NH) \in f(p)$. Hence M is $f(p)$-critical group. If M is not a cyclic primary subgroup then $M \in f(p)$, a contradiction. Thus M is a cyclic primary subgroup. By theorem B of [1] G is supersoluble, a contradiction. So $\mathfrak{X} \subseteq \mathfrak{B}$.

Assume now that $\mathfrak{B}$ is not contained in $\mathfrak{X}$. Let choose a group G of minimal order from $\mathfrak{B} \setminus \mathfrak{X}$. It is easy to show that $\Phi(G) = 1$ and G is an $\mathfrak{X}$-critical group. By [1] G is a biprimary minimal non-supersoluble group with unique minimal normal subgroup N which is the Sylow p-subgroup, a Sylow q-subgroup Q of G is cyclic. It means that the derived subgroup of G is nilpotent. Hence G is supersoluble. Now $G \in \mathfrak{X}$, the final contradiction. So $\mathfrak{B} \subseteq \mathfrak{X}$. Thus $\mathfrak{B} = \mathfrak{X}$. □

References

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