

ON GENERALIZATIONS OF BAER'S THEOREMS ABOUT THE HYPERCENTER OF A FINITE GROUP

V. I. Murashka

{mvimath@yandex.net}

Francisk Skorina Gomel State University, Gomel

Abstract. We investigate the intersection of normalizers and \mathfrak{F} -subnormalizers of different types of systems of subgroups (\mathfrak{F} -maximal, Sylow, cyclic primary). We described all formations $\mathfrak{F} = \times_{i \in I} \mathfrak{F}_{\pi_i}$ for which the intersection of normalizers of all \mathfrak{F}_i -maximal subgroups of G is the \mathfrak{F} -hypercenter of G for every group G . Also we described all formations \mathfrak{F} for which the intersection of \mathfrak{F} -subnormalizers of all Sylow (cyclic primary) subgroups of G is the \mathfrak{F} -hypercenter of G for every group G .

Keywords: saturated formation, hereditary formation, \mathfrak{F} -hypercenter, \mathfrak{F} -subnormalizer, intersection of subgroups.

Mathematic Subject Classification(2010): 20D25, 20F17, 20F19.

1 Introduction

All considered groups are finite. In [1] R. Baer showed that from one hand the hypercenter $Z_\infty(G)$ of a group G coincides with the intersection of all maximal nilpotent subgroups of G and from another hand $Z_\infty(G)$ coincides with the intersection of normalizers of all Sylow subgroups of G .

The concept of hypercenter was extended on classes of groups (see [2, p. 127–128] or [3, p. 6–8]). Let \mathfrak{X} be a class of groups. A chief factor H/K of a group G is called \mathfrak{X} -central if $(H/K) \rtimes G/C_G(H/K) \in \mathfrak{X}$. A normal subgroup N of G is said to be \mathfrak{X} -hypercentral in G if $N = 1$ or $N \neq 1$ and every chief factor of G below N is \mathfrak{X} -central. The \mathfrak{X} -hypercenter $Z_\mathfrak{X}(G)$ is the product of all normal \mathfrak{X} -hypercentral subgroups of G . So if $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups then $Z_\infty(G) = Z_\mathfrak{N}(G)$ for every group G .

In [4] A. V. Sidorov showed that for a soluble group G the intersection of all maximal subgroups of nilpotent length at most r is $Z_{\mathfrak{N}^r}(G)$. Beidleman and Heineken [5] studied the properties of the intersection $\text{Int}_\mathfrak{F}(G)$ of \mathfrak{F} -maximal subgroups of a group G in case when G is soluble and \mathfrak{F} is a hereditary saturated formation.

Let F be the canonical local definition of a local formation \mathfrak{F} . Then \mathfrak{F} is said to satisfy the boundary condition [6] if \mathfrak{F} contains every group G whose all maximal subgroups belong to $F(p)$ for some prime p .

A. N. Skiba [6] showed that the equality $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$ holds for every group G if and only if a hereditary saturated formations \mathfrak{F} satisfies the boundary condition. This and further results was included in the first chapter of [3].

The intersection of normalizers of different systems of subgroups is the main theme of many papers. In [7] Baer considered the intersection of normalizers of all subgroups of a group. Wielandt [8] studied the intersection of normalizers of all subnormal subgroups of a group. Li and Shen [9] considered the intersection of normalizers of all derived subgroups of all subgroups of a group.

Let $\sigma = \{\pi_i | i \in I\}$ be a partition of \mathbb{P} into disjoint subsets, \mathfrak{X}_i be a class of groups such that $\pi(\mathfrak{X}_i) = \pi_i$. Then $\times_{i \in I} \mathfrak{X}_{\pi_i} = (G = \times_{i \in I} O_{\pi_i}(G) | O_{\pi_i}(G) \in \mathfrak{X}_i)$. Recall that \mathfrak{G}_π is the class of all π -groups. Hence $\mathfrak{N} = \times_{p \in \mathbb{P}} \mathfrak{G}_p$.

In [10] author showed that if $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$ then for any group G the intersection of all normalizers of all π_i -maximal subgroups of G for all $i \in I$ coincides with the \mathfrak{F} -hypercenter. So the general problem is

Problem A. Let Σ be a subgroup functor. What can be said about the intersection of normalizers of subgroups from $\Sigma(G)$?

Recall [11, p. 206] that a subgroup functor is a function τ which assigns to each group G a possibly empty set $\tau(G)$ of subgroups of G satisfying $f(\tau(G)) = \tau(f(G))$ for any isomorphism $f : G \rightarrow G^*$.

Definition 1. Let \mathfrak{X} be a class of groups and G be a group. Then $\text{NI}_{\mathfrak{X}}(G)$ is the intersection of all normalizers of \mathfrak{X} -maximal subgroups of G .

The following proposition shows that if \mathfrak{F} is a hereditary saturated formation and $\pi(\mathfrak{F}) = \mathbb{P}$ then the equality $\text{NI}_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G)$ holds for every group G .

Proposition 1. *Let \mathfrak{F} be a hereditary saturated formation and $\pi = \pi(\mathfrak{F})$. Then $\text{O}^{\pi'}(\text{NI}_{\mathfrak{F}}(G)) = \text{Int}_{\mathfrak{F}}(G)$ for every group G .*

The following theorem generalizes two above mentioned Baer's theorems about the hypercenter:

Theorem A. *Let $\sigma = \{\pi_i | i \in I\}$ be a partition of \mathbb{P} into disjoint subsets and \mathfrak{F}_i be a hereditary saturated formation such that $\pi(\mathfrak{F}_i) = \pi_i$ and $\mathfrak{F} = \times_{i \in I} \mathfrak{F}_i$. The following statements are equivalent:*

- (1) \mathfrak{F}_i satisfies the boundary condition in the universe of all π_i -groups for all $i \in I$.
- (2) For every group G holds $\bigcap_{i \in I} \text{NI}_{\mathfrak{F}_i}(G) = \text{Z}_{\mathfrak{F}}(G)$.

Corollary A.1 [1]. *The hypercenter of a group G is the intersection of all normalizers of all Sylow subgroups of G .*

Corollary A.2 [10]. *Let $\sigma = \{\pi_i | i \in I\}$ be a partition of \mathbb{P} into disjoint subsets, $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$ and G be a group. Then the intersection of all normalizers of all π_i -maximal subgroups of G for all $i \in I$ is the \mathfrak{F} -hypercenter of G .*

From proposition 1 and theorem A when $|I| = 1$ it follows that our theorem A extends theorem A from [6]:

Corollary A.3. *Let \mathfrak{F} be a hereditary saturated formation and $\pi(\mathfrak{F}) = \mathbb{P}$. The equality $\text{NI}_{\mathfrak{F}}(G) = \text{Z}_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G)$ holds for every group G if and only if \mathfrak{F} satisfies the boundary condition.*

Corollary A.4 [1]. *The hypercenter of a group G is the intersection of all maximal nilpotent subgroups of G .*

Let \mathfrak{X} be a class of groups. Recall that a subgroup H of a group G is called \mathfrak{F} -subnormal if either $H = G$ or there is a maximal chain of subgroups $H = H_0 < H_1 < \dots < H_n = G$ such that $H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{X}$ for all $i = 1, \dots, n$.

Let \mathfrak{X} be a class of groups. A \mathfrak{X} -subnormalizer [12, p. 380] of a subgroup H of a group G is a subgroup T of G such that H is \mathfrak{X} -subnormal in T and if H is \mathfrak{X} -subnormal in M and $T \leq M$ then $T = M$. It is clear that a \mathfrak{X} -subnormalizer always exists but may be not unique.

Problem B. Let $\Sigma(G)$ be a subgroup functor and \mathfrak{F} be a formation. What can be said about the intersection $\text{SI}_{\Sigma}^{\mathfrak{F}}(G)$ of \mathfrak{F} -subnormalizers of subgroups from $\Sigma(G)$?

If $\Sigma(G)$ is the set of all maximal subgroups of G then this intersection coincides with $\Delta_{\mathfrak{F}}(G)$ where $\Delta_{\mathfrak{F}}(G)$ is the intersection of all \mathfrak{F} -abnormal maximal subgroups of G . According to [13, p. 96] if \mathfrak{F} is a hereditary saturated formation then $\Delta_{\mathfrak{F}}(G)/\Phi(G) = \text{Z}_{\mathfrak{F}}(G/\Phi(G))$.

Proposition 2. *Let \mathfrak{F} be a hereditary formation and Σ be a subgroup functor. Then $\text{SI}_{\Sigma}^{\mathfrak{F}}(G)$ is the product of normal subgroups N of a group G such that H is \mathfrak{F} -subnormal in HN for every $H \in \Sigma(G)$.*

A.F. Vasil'ev and T.I. Vasil'eva [14] studied a class of groups $w\mathfrak{F}$ whose all Sylow subgroups are \mathfrak{F} -subnormal for a given hereditary saturated formation \mathfrak{F} . Let us note that in this case

$Z_{w\mathfrak{F}}(G)$ lies in the intersection of all \mathfrak{F} -subnormalizers of all Sylow subgroups of a group G . Author [15] studied a class of groups $v\mathfrak{F}$ whose all cyclic primary subgroups are \mathfrak{F} -subnormal for a given hereditary saturated formation \mathfrak{F} . Again $Z_{v\mathfrak{F}}(G)$ lies in the intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroup of a group G .

In this paper we count the unit group as cyclic primary subgroup and also as Sylow subgroup.

Theorem B. *Let \mathfrak{F} be a hereditary saturated formation. The following statements are equivalent:*

- (1) *There exists a partition $\sigma = \{\pi_i | i \in I\}$ of $\pi(\mathfrak{F})$ into disjoint subsets such that $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$.*
- (2) *The intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroups of G is the \mathfrak{F} -hypercenter of G for every group G .*
- (3) *The intersection of all \mathfrak{F} -subnormalizers of all Sylow subgroups of G is the \mathfrak{F} -hypercenter of G for every group G .*

Note that in the universe of all soluble groups the concepts of a subnormal subgroup and a \mathfrak{N} -subnormal subgroup coincides. It is well known that if a Sylow subgroup P of G is subnormal in G then it is normal in G . Hence a \mathfrak{N} -subnormalizer of a Sylow subgroup P of a soluble group G is just the normalizer of P in G . So theorem B can be viewed as the generalization of R. Baer's theorem about the intersection of normalizers of Sylow subgroups.

Remark. Formations $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$ are lattice formations, i.e. formations were \mathfrak{F} -subnormal subgroups form a sublattice of the subgroup's lattice of every group (for example see chapter 6.3 of [11]). Also properties of the \mathfrak{F} -hypercenter and the \mathfrak{F} -residual for such formations was studied by author in [10]. A.N. Skiba extends the theory of nilpotent groups on such classes (for example see [16]).

2 Preliminaries

We use standard notation and terminology that if necessary can be found in [12]. Recall some of them that are important in this paper. By \mathbb{P} is denoted the set of all primes; $\pi(G)$ is the set of all prime divisors of the order of G ; $\pi(\mathfrak{F}) = \bigcup_{G \in \mathfrak{F}} \pi(G)$; a group G is called π -group if $\pi(G) \subseteq \pi$; Z_p is the cyclic group of order p ; $O_\pi(G)$ is the greatest normal π -subgroup G ; $O^\pi(G)$ is the smallest subgroup of G such that $\pi(G/O^\pi(G)) \subseteq \pi$; G' is the derived subgroup of G ; $G^\mathfrak{F}$ is the \mathfrak{F} -residual for a formation \mathfrak{F} ; $O_{p',p}(G)$ is the p -nilpotent radical of G for $p \in \mathbb{P}$, it also can be defined by $O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$; $\Phi(G)$ is the Frattini subgroup of a group G ; $G = N \rtimes M$ is the semidirect product of groups M and N ($N \triangleleft G$ and $N \cap M = 1$); \mathfrak{G}_π ($\mathfrak{S}_\pi, \mathfrak{N}_\pi$) is the class of (soluble, nilpotent) π -groups, where $\pi \subseteq \mathbb{P}$.

A class of groups \mathfrak{F} is called a formation if from $G \in \mathfrak{F}$ and $N \triangleleft G$ it follows that $G/N \in \mathfrak{F}$ and from $H/A \in \mathfrak{F}$ and $H/B \in \mathfrak{F}$ it follows that $H/A \cap B \in \mathfrak{F}$.

A class of groups \mathfrak{X} is called hereditary if from $G \in \mathfrak{X}$ and $H \leq G$ it follows that $H \in \mathfrak{X}$.

Let \mathfrak{F} and \mathfrak{K} be formations then $\mathfrak{F}\mathfrak{K} = (G | G^\mathfrak{K} \in \mathfrak{F})$ is also formation.

A class of groups \mathfrak{X} is called saturated if from $G/\Phi(G) \in \mathfrak{X}$ it follows that $G \in \mathfrak{X}$.

By well known Gashutz-Lubeseder-Shmid Theorem saturated formations are exactly local formations, i.e. formations $\mathfrak{F} = LF(f)$ defined by a formation function f : $LF(f) = \{G \in \mathfrak{G} | \text{if } H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K) \text{ then } G/C_G(H/K) \in f(p)\}$. Among all possible local definitions of a local formation \mathfrak{F} there is exactly one, denoted by F , such that F is integrated ($F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full ($\mathfrak{N}_p F(p) = F(p)$ for all $p \in \mathbb{P}$). F is called the canonical local definition of \mathfrak{F} .

Let \mathfrak{F} be a local formation, F be its canonical local definition and G be a group. Then a chief factor H/K of a group G is \mathfrak{F} -central if and only if $G/C_G(H/K) \in F(p)$ for all $p \in \pi(H/K)$ (see [3, p. 6]).

Let $\mathfrak{F} = LF(F)$ be a hereditary local formation, F be its canonical local definition and $\pi = \pi(\mathfrak{H})$. Then \mathfrak{F} is said to satisfy the boundary condition in the universe of all π -groups if \mathfrak{F} contains every π -group whose all maximal subgroups belong to $F(p)$ for some prime p .

The following lemma can be found in [13, p. 239]. For reader's convenience, we give a direct proof.

Lemma 2.1. *Let \mathfrak{X} be a saturated homomorph and N be a normal subgroup of a group G . Then for every \mathfrak{X} -subgroup H/N of G/N there exists a \mathfrak{X} -subgroup M of G such that $MN/N = H/N$.*

Proof. Let H/N be a \mathfrak{X} -subgroup of G/N . Let us show that there exists a \mathfrak{X} -subgroup K of G such that $KN/N = H/N$. Let M be a minimal subgroup of H such that $MN = H$ (i.e. if $M_1 < M$ then $M_1N < H$). Assume that there is a maximal subgroup M_1 of M such that $M_1(M \cap N) = M$. Then $M_1N = H$, a contradiction. Hence $M \cap N \leq \Phi(M)$. Since \mathfrak{X} is saturated and $H/N = MN/N \simeq M/M \cap N \in \mathfrak{X}$, we see that $M \in \mathfrak{X}$. It means that there is a \mathfrak{X} -subgroup M of G such that $H/N = MN/N$. \square

Lemma 2.2. *Let \mathfrak{F} be a hereditary saturated formation, N be a normal subgroup of a group G , H be a subgroup of G then*

- (1) $\text{NI}_{\mathfrak{F}}(G)N/N \leq \text{NI}_{\mathfrak{F}}(G/N)$.
- (2) $\text{NI}_{\mathfrak{F}}(G) \cap H \leq \text{NI}_{\mathfrak{F}}(H)$.
- (3) *Let $N \leq \text{Int}_{\mathfrak{F}}(G)$ then $N \leq \text{NI}_{\mathfrak{F}}(G)$ and $\text{NI}_{\mathfrak{F}}(G)/N = \text{NI}_{\mathfrak{F}}(G/N)$.*

Proof. (1) If K/N is a \mathfrak{F} -maximal subgroup of G/N then by lemma 2.1 there exists a \mathfrak{F} -maximal subgroup Q of G such that $QN/N = K/N$. If $x \in N_G(Q)$ then $xN \in N_{G/N}(QN/N) = N_{G/N}(K/N)$. Thus $\text{NI}_{\mathfrak{F}}(G)N/N \leq \text{NI}_{\mathfrak{F}}(G/N)$.

(2) If M is a \mathfrak{F} -maximal subgroup of H then there exists a \mathfrak{F} -maximal subgroup Q of G such that $Q \cap H = M$. So if $x \in \text{NI}_{\mathfrak{F}}(G) \cap H$ then $M^x = Q^x \cap H^x = Q \cap H = M$. Hence $x \in \text{NI}_{\mathfrak{F}}(H)$. Thus $\text{NI}_{\mathfrak{F}}(G) \cap H \leq \text{NI}_{\mathfrak{F}}(H)$.

(3) Let $N \leq \text{Int}_{\mathfrak{F}}(G)$. It is clear that $N \leq \text{NI}_{\mathfrak{F}}(G)$. Note that M is a \mathfrak{F} -maximal subgroup of G if and only if M/N is a \mathfrak{F} -maximal subgroup of G/N . Now $N_G(M)/N = N_{G/N}(M/N)$. Thus $\text{NI}_{\mathfrak{F}}(G)/N = \text{NI}_{\mathfrak{F}}(G/N)$. \square

Let \mathfrak{F} be a saturated formation. Then in every group exists a \mathfrak{F} -projector [12, p. 292]. Recall that a \mathfrak{F} -projector of a group G is a \mathfrak{F} -maximal subgroup H of G such that HN/N is a \mathfrak{F} -maximal subgroup of G/N for every normal subgroup N of G .

Recall that a group G is called semisimple if G is the direct product of simple groups. A chief factor of a group is the example of a semisimple group.

Lemma 2.3. *Let \mathfrak{F} be a hereditary saturated formation and a group $G = HK$ be a product of normal \mathfrak{F} -subgroups. If K is semisimple then $G \in \mathfrak{F}$.*

Proof. Assume the contrary. Let a group G be a counterexample of a minimal order. Then $G = HK$ is a product of normal \mathfrak{F} -subgroups H and K where K is semisimple. Let N be a normal subgroup of G . Then $G/N = (HN/N)(KN/N)$ where HN/N and KN/N are normal \mathfrak{F} -subgroups of G/N and KN/N is semisimple. So $G/N \in \mathfrak{F}$. Since \mathfrak{F} is a saturated formation, we see that $\Phi(G) = 1$ and G has an unique minimal normal subgroup that equals K . Now $K \leq H$. So $G = H \in \mathfrak{F}$, the contradiction. \square

The following lemma is well known.

Lemma 2.4. *Let \mathfrak{F} be a hereditary saturated formation and H be a \mathfrak{F} -subgroup of a group G . Then $Z_{\mathfrak{F}}(G)H \in \mathfrak{F}$.*

Recall that if \mathfrak{F} is a hereditary formation then a subgroup H of a group G is called \mathfrak{F} -subnormal if either $H = G$ or there is a chain of subgroups $H = H_0 < H_1 < \dots < H_n = G$ such that $H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, \dots, n$. We will need the following facts about \mathfrak{F} -subnormal subgroups.

Lemma 2.5 [11, p. 236]. *Let \mathfrak{F} be a hereditary formation, N be a normal subgroup of a group G and H, K be subgroups of G . Then:*

- (1) If H is \mathfrak{F} -subnormal in G then HN/N is \mathfrak{F} -subnormal in G/N .
- (2) If H/N is \mathfrak{F} -subnormal in G/N then H is \mathfrak{F} -subnormal in G .
- (3) If H is \mathfrak{F} -subnormal in K and K is \mathfrak{F} -subnormal in G then H is \mathfrak{F} -subnormal in G .

Lemma 2.6 [11, p. 239]. Let \mathfrak{F} be a saturated formation and a group $G = HF^*(G)$ where H is a \mathfrak{F} -subnormal \mathfrak{F} -subgroup of G . Then $G \in \mathfrak{F}$.

Lemma 2.7 [12, p. 390]. Let \mathfrak{F} be a hereditary saturated formation then $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$ for any group G .

3 Proves of the main results

3.1 Proof of proposition 1

Let \mathfrak{F} be a hereditary saturated formation and G be a group. According to lemma 2.2 all \mathfrak{F} -maximal subgroups of $NI_{\mathfrak{F}}(G)$ are normal in $NI_{\mathfrak{F}}(G)$. Among this \mathfrak{F} -maximal subgroups there is a \mathfrak{F} -projector H . Now $NI_{\mathfrak{F}}(G)/H$ does not contain any \mathfrak{F} -subgroup. Hence $NI_{\mathfrak{F}}(G)/H \in \pi(\mathfrak{F})'$.

It is clear that $\text{Int}_{\mathfrak{F}}(G) \leq O^{\pi'}(NI_{\mathfrak{F}}(G)) \in \mathfrak{F}$. Let us show by induction the the equality $\text{Int}_{\mathfrak{F}}(G) = O^{\pi'}(NI_{\mathfrak{F}}(G))$ holds. It is clear that it holds for the unit group. Assume that we prove our statement for groups whose order is less then the order of a group G . Let N be a minimal normal subgroup of G such that $N \leq O^{\pi'}(NI_{\mathfrak{F}}(G))$ and M be a \mathfrak{F} -maximal subgroup of G . So $M \triangleleft MN$ and N is a normal semisimple \mathfrak{F} -subgroup of MN . By lemma 2.3 $MN \in \mathfrak{F}$ and hence $MN = M$ for all \mathfrak{F} -maximal subgroups M of G . Hence $N \leq \text{Int}_{\mathfrak{F}}(G)$.

By induction $\text{Int}_{\mathfrak{F}}(G/N) = O^{\pi'}(NI_{\mathfrak{F}}(G/N))$. According to [6] $\text{Int}_{\mathfrak{F}}(G)/N = \text{Int}_{\mathfrak{F}}(G/N)$. By (3) of lemma 2.2 $NI_{\mathfrak{F}}(G)/N = NI_{\mathfrak{F}}(G/N)$. From $\pi(N) \subseteq \pi(\mathfrak{F})$ it follows that $O^{\pi'}(NI_{\mathfrak{F}}(G)/N) = O^{\pi'}(NI_{\mathfrak{F}}(G)/N)$. Now $\text{Int}_{\mathfrak{F}}(G)/N = \text{Int}_{\mathfrak{F}}(G/N) = O^{\pi'}(NI_{\mathfrak{F}}(G/N)) = O^{\pi'}(NI_{\mathfrak{F}}(G)/N) = O^{\pi'}(NI_{\mathfrak{F}}(G))/N$. Thus $\text{Int}_{\mathfrak{F}}(G) = O^{\pi'}(NI_{\mathfrak{F}}(G))$. \square

3.2 Proof of theorem A

The following result directly follows from the proof of the main result of [6].

Let \mathfrak{H} be a hereditary saturated formation and $\pi = \pi(\mathfrak{H})$. Then for every π -group G the intersection of all \mathfrak{H} -maximal subgroups of G is the \mathfrak{H} -hypercenter of G if and only if \mathfrak{H} satisfies the boundary condition in the universe of all π -groups.

According to [11, p. 96] \mathfrak{F} is a hereditary saturated formation. So \mathfrak{F} is a local formation. Let F be the canonical local definition of \mathfrak{F} .

(1) \Rightarrow (2) Assume that \mathfrak{F}_i satisfies the boundary condition in the universe of all π_i -groups for all $i \in I$. Let us show that $\bigcap_{i \in I} NI_{\mathfrak{F}_i}(G) = Z_{\mathfrak{F}}(G)$ holds for every group G .

Let G be a group and $D = \bigcap_{i \in I} NI_{\mathfrak{F}_i}(G)$. By proposition 1 $NI_{\mathfrak{F}_i}(G)$ has the Hall π_i -subgroup that belongs to \mathfrak{F}_i and is normal in G . Hence D has the normal Hall π_i -subgroup that belongs to \mathfrak{F}_i for every $i \in I$. Thus $D \in \mathfrak{F}$.

Let H/K be a chief factor of G below D . Then $\pi(H/K) \subseteq \pi_n$ for some $n \in I$.

(a) $O^{\pi_n}(G/K) \leq C_G(H/K)$.

By (1) of lemma 2.2 H/K normalizes all \mathfrak{F}_i -maximal subgroups of G/K . Hence H/K normalizes all \mathfrak{F}_i -projectors of G/K for all $i \in I \setminus \{n\}$. Let F/K be a \mathfrak{F}_i -projector of G/K for some $i \in I \setminus \{n\}$. From $\pi_n \cap \pi_i = \emptyset$ it follows that $F/K \cap H/K = K/K$. Let $hK \in H/K$ and $fK \in F/K$. Then from one hand $[fK, hK] = (fK)^{-1}(fK)^{(hK)} \in F/K$ and from another hand $[fK, hK] = ((hK)^{-1})^{(fK)}(hK) \in H/K$. So $[fK, hK] = 1$. Hence $[H/K, F/K] = 1$. Thus H/K centralizes all \mathfrak{F}_i -projectors of G/K for all $i \in I \setminus \{n\}$. Since \mathfrak{F}_i is a hereditary saturated

formation for all $i \in I$, we see that $G/C_G(H/K)$ does not contain any π_i -subgroups for all $i \in I \setminus \{n\}$. Thus $O^{\pi_n}(G/K) \leq C_G(H/K)$.

(b) $H/K \leq Z_{\mathfrak{F}_n}(R/K)$ for every \mathfrak{G}_{π_n} -maximal subgroup R/K of G/K .

Let R/K be a \mathfrak{G}_{π_n} -maximal subgroup of G/K . Then $\pi_n((R/K)(H/K)) \subseteq \pi_n$. Hence $(R/K)(H/K) = R/K$. So $H/K \subseteq R/K$. By (2) of lemma 2.2 H/K normalizes all \mathfrak{F}_n -maximal subgroups of R/K . Note that H/K is semisimple \mathfrak{F}_n -subgroup. So $(H/K)(F/K) \in \mathfrak{F}_n$ for every \mathfrak{F}_n -maximal subgroup F/K of R/K by lemma 2.3. Hence $(H/K)(F/K) = F/K$ for every \mathfrak{F}_n -maximal subgroup F/K of R/K . Thus $H/K \leq \text{Int}_{\mathfrak{F}_n}(R/K)$. Since \mathfrak{F}_n satisfies the boundary condition in the universe of all π_n -groups, $H/K \leq Z_{\mathfrak{F}_n}(R/K)$.

(c) Let R/K be a \mathfrak{G}_{π_n} -maximal subgroup of G/K such that $(R/K)O^{\pi_n}(G/K) = G/K$. Then H/K is a chief factor of R/K .

Assume that N/K is a minimal normal subgroup of R/K such that $K/K \neq N/K < H/K$. From $O^{\pi_n}(G/K) \leq C_G(H/K)$ it follows that $O^{\pi_n}(G/K) \leq C_G(N/K)$. From $(R/K)O^{\pi_n}(G/K) = G/K$ it follows that N/K is normal in G/K . Hence H/K is not a chief factor of G , a contradiction.

(d) $(R/K)^{F(p)} \leq C_G(H/K)$ for all $p \in \pi(H/K)$.

From $H/K \leq Z_{\mathfrak{F}_n}(R/K)$ and $\mathfrak{F}_n \subseteq \mathfrak{F}$ it follows that a chief factor H/K of R/K lies in $Z_{\mathfrak{F}}(R/K)$. Now $(R/K)/C_{R/K}(H/K) \in F(p)$ for all $p \in \pi(H/K)$. Thus $(R/K)^{F(p)} \leq C_G(H/K)$ for all $p \in \pi(H/K)$.

(e) H/K is a \mathfrak{F} -central chief factor of G .

From $O^{\pi_n}(G/K) \leq C_G(H/K)$, $(R/K)^{F(p)} \leq C_G(H/K)$ for all $p \in \pi(H/K)$ and $(R/K)O^{\pi_n}(G/K) = G/K$ it follows that $G/C_G(H/K) \in F(p)$ for all $p \in \pi(H/K)$. Thus H/K is a \mathfrak{F} -central chief factor of G .

(f) $D \leq Z_{\mathfrak{F}}(G)$.

We showed that every chief factor of G below D is \mathfrak{F} -central. Hence $D \leq Z_{\mathfrak{F}}(G)$.

(g) $D \geq Z_{\mathfrak{F}}(G)$ and hence $D = Z_{\mathfrak{F}}(G)$.

Let H be a \mathfrak{F}_i -maximal subgroup of G for some $i \in I$. Then $HZ_{\mathfrak{F}}(G) \in \mathfrak{F}$ by lemma 2.4. Since H is a \mathfrak{F}_i -maximal subgroup of G , H is a \mathfrak{F}_i -maximal subgroup of $HZ_{\mathfrak{F}}(G)$. So $H \triangleleft HZ_{\mathfrak{F}}(G)$. Hence $D \geq Z_{\mathfrak{F}}(G)$. Thus $D = Z_{\mathfrak{F}}(G)$.

(2) \Rightarrow (1) Suppose now that $\bigcap_{i \in I} \text{NI}_{\mathfrak{F}_i}(G) = Z_{\mathfrak{F}}(G)$ holds for every group G . Let us show that \mathfrak{F}_i satisfies the boundary condition in the universe of all π_i -groups for all $i \in I$.

Assume the contrary. Then some \mathfrak{F}_n does not satisfy the boundary condition in the universe of all π_n -groups. So there is π_n -group G such that $\text{Int}_{\mathfrak{F}_n}(G) \neq Z_{\mathfrak{F}_n}(G)$. Note that $\text{Int}_{\mathfrak{F}_n}(G) = \text{NI}_{\mathfrak{F}_n}(G)$ by proposition 1. Since G is a π_n -group, $\text{NI}_{\mathfrak{F}_n}(G) = \bigcap_{i \in I} \text{NI}_{\mathfrak{F}_i}(G)$. From $\mathfrak{G}_{\pi_n} \cap \mathfrak{F} = \mathfrak{F}_n$ it follows that $Z_{\mathfrak{F}_n}(G) = Z_{\mathfrak{F}}(G)$.

Hence $\bigcap_{i \in I} \text{NI}_{\mathfrak{F}_i}(G) = \text{NI}_{\mathfrak{F}_n}(G) = \text{Int}_{\mathfrak{F}_n}(G) \neq Z_{\mathfrak{F}_n}(G) = Z_{\mathfrak{F}}(G)$, the contradiction.

3.3 Proof of proposition 2

Let N be a normal subgroup of a group G such that H is \mathfrak{F} -subnormal in HN for every $H \in \Sigma(G)$. Let S be a \mathfrak{F} -subnormalizer in G of $H \in \Sigma(G)$. Then HN/N is \mathfrak{F} -subnormal in SN/N by (1) of lemma 2.5. So HN is \mathfrak{F} -subnormal in SN by (2) of lemma 2.5. Hence H is \mathfrak{F} -subnormal in SN by (3) of lemma 2.5. Thus $SN = N$. It means that $N \leq \text{SI}_{\Sigma}^{\mathfrak{F}}(G)$. So every normal subgroup of G that \mathfrak{F} -subnormalize all subgroups from $\Sigma(G)$ lies in $\text{SI}_{\Sigma}^{\mathfrak{F}}(G)$.

From the other hand $H\text{SI}_{\Sigma}^{\mathfrak{F}}(G)$ belongs to every \mathfrak{F} -subnormalizer of H in G for every $H \in \Sigma(G)$. Hence H is \mathfrak{F} -subnormal in $H\text{SI}_{\Sigma}^{\mathfrak{F}}(G)$ for every $H \in \Sigma(G)$.

3.4 Proof of theorem B

(1) \Rightarrow (2) Assume that there exists a partition $\sigma = \{\pi_i | i \in I\}$ of $\pi(\mathfrak{F})$ into disjoint subsets such that $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$. Let us show that the intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroups of a group G is the \mathfrak{F} -hypercenter of G for every group G .

Note that \mathfrak{F} is local formation with the canonical local definition F where $F(p) = \mathfrak{G}_{\pi_i}$ for $p \in \pi_i$ for all $i \in I$.

Let D be the intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroups of a group G and H/K be a chief factor of G below D .

(a) H/K lies in the intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroups of a group G/K .

Let CK/K be a cyclic primary subgroup of G/K . According to lemma 2.1 we may assume that C is a cyclic primary subgroup of G . Now C is \mathfrak{F} -subnormal in HC by proposition 2. So CK/K is \mathfrak{F} -subnormal in HC/K by (1) of lemma 2.5. Hence H/K lies in the intersection of \mathfrak{F} -subnormalizers of all cyclic primary subgroups of G/K .

(b) $H/K \in \mathfrak{F}$.

Now K/K is a \mathfrak{F} -subnormal \mathfrak{F} -subgroup of a quasinilpotent group H/K . By lemma 2.6 $H/K \in \mathfrak{F}$. Hence $\pi(H/K) \subseteq \pi_n$ for some $n \in I$.

(c) $C/K \leq C_G(H/K)$ for every cyclic primary $\pi(\mathfrak{F})'$ -subgroup of G/K .

Let C/K be a cyclic primary $\pi(\mathfrak{F})'$ -subgroup of G/K . Since C/K is a \mathfrak{F} -subnormal $\pi(\mathfrak{F})'$ -subgroup of HC/K , $(C/K)^{\mathfrak{F}} = (C/K)$ is subnormal in HC/K by (1) of lemma 6.1.9 [11, p. 237]. Since C/K is a subnormal Sylow subgroup of HC/K , we see that $C/K \triangleleft HC/K$. Now it is easy to see that $C/K \leq C_G(H/K)$.

(d) $C/K \leq C_G(H/K)$ for every cyclic primary $\pi(\mathfrak{F}) \cap (\pi'_n)$ -subgroup of G/K .

Let C/K be a cyclic primary $\pi(\mathfrak{F}) \cap (\pi'_n)$ -subgroup of G/K . Since C/K is \mathfrak{F} -subnormal in HC/K , $HC/K \in \mathfrak{F}$ by lemma 2.6. So $C/K \leq C_G(H/K)$.

(e) H/K is a \mathfrak{F} -central chief factor of G and $D \leq Z_{\mathfrak{F}}(G)$.

From (c) and (d) it follows that $O^{\pi_n}(G) \leq C_G(H/K)$. Hence $G/C_G(H/K) \in \mathfrak{G}_{\pi_n} = F(p)$ for all $p \in \pi(H/K)$. So H/K is a \mathfrak{F} -central chief factor of G . It means that $D \leq Z_{\mathfrak{F}}(G)$.

(f) $Z_{\mathfrak{F}}(G) \leq D$ and hence $D = Z_{\mathfrak{F}}(G)$.

Let C be a cyclic p -subgroup of a group G . If $p \in \pi(\mathfrak{F})$ then $CZ_{\mathfrak{F}}(G) \in \mathfrak{F}$ by lemma 2.4. Hence C is \mathfrak{F} -subnormal in $CZ_{\mathfrak{F}}(G)$. If $p \notin \pi(\mathfrak{F})$ then $C \leq G^{\mathfrak{F}}$. By lemma 2.7 $C \leq C_G(Z_{\mathfrak{F}}(G))$. Hence $(CZ_{\mathfrak{F}}(G))^{\mathfrak{F}} = C$. So C is \mathfrak{F} -subnormal in $CZ_{\mathfrak{F}}(G)$. Hence $Z_{\mathfrak{F}}(G) \leq D$. Thus $D = Z_{\mathfrak{F}}(G)$.

(2) \Rightarrow (3) Let P be a Sylow p -subgroup of G . If $p \in \pi(\mathfrak{F})$ then $P \in \mathfrak{F}$ and hence $PZ_{\mathfrak{F}}(G) \in \mathfrak{F}$ by lemma 2.4. So P is \mathfrak{F} -subnormal in $PZ_{\mathfrak{F}}(G)$.

If $p \notin \pi(\mathfrak{F})$ then $P \leq G^{\mathfrak{F}}$. By lemma 2.7 $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$. So $PZ_{\mathfrak{F}}(G) = P \times Z_{\mathfrak{F}}(G)$. Hence P is \mathfrak{F} -subnormal in $PZ_{\mathfrak{F}}(G)$.

Thus $Z_{\mathfrak{F}}(G)$ lies in the intersection D of all \mathfrak{F} -subnormalizers of all Sylow subgroups of G . Since the unit subgroup is \mathfrak{F} -subnormal in D , we see that $\pi(D) \subseteq \pi(\mathfrak{F})$.

Now let C be a cyclic primary p -subgroup of G . Then there is a Sylow p -subgroup P of G such that $C \leq P$. If $p \in \pi(\mathfrak{F})$ then C is \mathfrak{F} -subnormal in P and P is \mathfrak{F} -subnormal in PD . Hence C is \mathfrak{F} -subnormal in PD and also in CD by lemma 2.5.

If $p \notin \pi(\mathfrak{F})$ then C is subnormal in P and P is normal in PD . Hence C is subnormal in PD and also in CD . So C is the normal Sylow subgroup of CD . By our assumption the unit group is a Sylow subgroup. Hence 1 is \mathfrak{F} -subnormal in D . Now C/C is \mathfrak{F} -subnormal in CD/C . Hence C is \mathfrak{F} -subnormal in CD .

Thus D lies in the intersection of all \mathfrak{F} -subnormalizers of all cyclic primary subgroups of G . Hence $Z_{\mathfrak{F}}(G) \leq D \leq Z_{\mathfrak{F}}(G)$. Thus $D = Z_{\mathfrak{F}}(G)$.

Consider the following statement:

(4) \mathfrak{F} has the canonical local definition F such that for every prime p , $F(p)$ contains every group G whose all Sylow subgroups belong to $F(p)$.

(3) \Rightarrow (4) Let the intersection of all \mathfrak{F} -subnormalizers of all Sylow subgroups of G be the \mathfrak{F} -hypercenter of G for every group G . Assume that there exist a prime p and groups G such that $G \notin F(p)$ but for every Sylow subgroup P of G , $P \in F(p)$. Let us chose the minimal order group G from such groups.

It is clear that $O_p(G) = 1$ and G has an unique minimal normal subgroup. Then by lemma 2.6 from [6] there exists a faithful irreducible G -module N over the field F_p . Let H be the semidirect product of N and G . Note that $NP \in \mathfrak{F}$ for every Sylow subgroup P of H . Hence N lies in the intersection of all \mathfrak{F} -subnormalizers of Sylow subgroups of H by proposition 2. But $H/C_H(N) \notin F(p)$. So $N \not\subseteq Z_{\mathfrak{F}}(H)$, the contradiction.

(4) \Rightarrow (1). Assume that $Z_q \in F(p)$ for primes $p \neq q$. Suppose that $F(q) \cap \mathfrak{N}_p \neq \mathfrak{N}_p$. Let P be the minimal order p -group from $\mathfrak{N}_p \setminus (F(q) \cap \mathfrak{N}_p)$. Then P has an unique minimal normal subgroup and $P \in F(p)$. There exists a faithful irreducible P -module Q over the field F_q . Note that $Q \in F(p)$. Hence the semidirect product $G = Q \rtimes P \in F(p) \subseteq \mathfrak{F}$. Now $G/O_{q',q}(G) = G/Q \simeq P \in F(q)$, a contradiction.

So from $Z_q \in F(p)$ it follows that $F(q) \cap \mathfrak{N}_p = \mathfrak{N}_p$ and hence $F(p) \cap \mathfrak{N}_q = \mathfrak{N}_q$. So $\mathfrak{N}_{\pi(F(p))} \subseteq F(p)$. Let a group G be a s -critical for $F(p)$. Since $F(p)$ is hereditary, we see that G is r -group for some prime r . Now $r \notin \pi(F(p))$. Hence $G \simeq Z_r$. It means that $F(p) = \mathfrak{G}_{\pi(F(p))}$ for all $p \in \pi(\mathfrak{F})$.

Assume now that for three different primes p, q and r we have that $\{p, q\} \subseteq \pi(F(r))$. Let us show that $q \in \pi(F(p))$. By theorem 10.3B [12] there exists a faithful irreducible Z_q -module P over the field F_p . Let G be the semidirect product P and Z_q . Then $T \in F(r) \subseteq \mathfrak{F}$. Thus $G/O_{p',p}(G) = G/P \simeq Z_q \in F(p)$.

It means that there exists a partition $\sigma = \{\pi_i | i \in I\}$ of $\pi(\mathfrak{F})$ into disjoint subsets such that $F(p) = \mathfrak{G}_{\pi_i}$ for all $p \in \pi_i$ and for all $i \in I$. Now $\mathfrak{F} = \times_{i \in I} \mathfrak{G}_{\pi_i}$.

References

- [1] R. Baer, Group Elements of Prime Power Index, Trans. Amer. Math Soc. **75**(1) (1953), 20–47.
- [2] L. A. Shemetkov and A. N. Skiba, Formations of algebraic systems, Nauka, 1989.
- [3] Wenbin Guo, Structure theory for canonical classes of finite groups. Springer 2015.
- [4] A. V. Sidorov, On properties of \mathfrak{F} -hypercenter of a finite group, in Problems in Algebra. **10** (1996), 141–143. (In Russian)
- [5] J. C. Beidleman and H. Heineken, A note of intersection of maximal \mathfrak{F} -subgroups, J. Algebra. **333** (2010), 120–127.
- [6] A. N. Skiba, On the \mathfrak{F} -hypercenter and the intersection of all \mathfrak{F} -maximal subgroups of a finite group, Journal of Pure and Applied Algebra. **216**(4) (2012), 789–799.
- [7] R. Baer, Der Kern eine charakteristische Untergruppe, Compos. Math. **1** (1935), 254–285.
- [8] H. Wielandt, Über den Normalisator der subnormalen Untergruppen, Math. Z. **69** (1958), 463–465.
- [9] S. Li and Z. Shen, On the intersection of the normalizers of derived subgroups of all subgroups of a finite group, J. Algebra. **323**(5) (2010), 1349–1357.

- [10] V.I. Murashka, On one generalization of Baer's theorems on hypercenter and nilpotent residual, *Prob Fiz. Mat. Tech.* **16** (2013), 84–88.
- [11] A. Ballester-Bolinches and L.M. Ezquerro, *Classes of Finite Groups*, Springer, 2006.
- [12] K. Doerk and T. Hawkes, *Finite soluble groups*, Walter de Gruyter, 1992.
- [13] L. A. Shemetkov, *Formations of finite groups*. Nauka, 1978. (In Russian)
- [14] A.F. Vasil'ev and T.I. Vasil'eva, On finite groups with generalized subnormal Sylow subgroups, *Prob. Fiz. Mat. Tech.* **4** (2011), 86–91.
- [15] V.I. Murashka, Classes of finite groups with generalized subnormal cyclic primary subgroups, *Siberian J. Math.* **55**(6) (2014), 1353–1367.
- [16] A.N. Skiba, On σ -subnormal and σ -permutable subgroups of finite groups, *J. Algebra.* **436** (2015), 1–16.