On a generalization of the concept of S-permutable subgroup of a finite group

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Abstract. Let $\sigma = \{\pi_i | i \in I$ and $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j\}$ be a partition of the set of all primes into mutually disjoint subsets. In this paper we considered subgroups that permutes with given sets of π_i-maximal subgroups for all $\pi_i \in \sigma$. In particular we showed that such subgroups forms a sublattice of the lattice of all subgroups of a finite group. As corollaries we obtained some well known results about S-permutable subgroups.

Keywords. Finite groups; nilpotent groups; π-maximal subgroup; S-permutable subgroup; σ-permutable subgroup.

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Introduction and the results

All considered groups are finite. All through this paper we denote by $\sigma = \{\pi_i | i \in I$ and $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j\}$ some partition of the set of all primes $\mathbb{P}$ into mutually disjoint subsets.

A subgroup H of a group G is said to permute with a subgroup K if HK is a subgroup of G. H is said to be S-permutable in G if it permutes with every Sylow subgroup of G. According to O. H. Kegel [1] and W. E. Deskins [2] if H is S-permutable in G then H^G/H_G is nilpotent. Moreover, the set of all S-permutable subgroups of G forms a sublattice of the lattice of all subgroups of G (see [1, Statz 2]). P. Schmid [3] showed that if H is S-permutable subgroup of a group G then $N_G(H)$ is also S-permutable in G.

The concepts of S-permutable subgroup plays important role in the structural study of finite non-simple groups. That is why there were several attempts to generalize this concept. A. N. Skiba [4] suggested the following generalization of the concept of S-permutable subgroup. A subgroup H of a group G is called σ-permutable if G has a Hall π_i-subgroup P_i with $HP_i^x = P_i^x H$ for every $x \in G$ and every $\pi_i \in \sigma$. In particular, A. N. Skiba obtained the analogues of the results of O. H. Kegel and E. W. Deskins for groups with some complete sets of Hall subgroups.

The main disadvantage of the concept of σ-permutable subgroup is that it requires the existence of Hall subgroups. The aim of this paper is to extend the theory of σ-permutable subgroups to the class of all groups.

Let $\mathfrak{X}$ be a class of groups and G be a group. A subgroup H of G is said to be a $\mathfrak{X}$-projector of G if HN/N is $\mathfrak{X}$-maximal in G for every $N \triangleleft G$. Let π be a set of primes. Recall that $\mathfrak{G}_\pi$ is the class of all π-groups. According to [5, III, 3.10] $\mathfrak{G}_\pi$-projectors exist in every group.

Definition 1. We shall call a subgroup H of a group G $\sigma^{(1)}$-permutable if it permutes with every π_i-maximal subgroups of G for all $\pi_i \in \sigma$.

Definition 2. We shall call a subgroup H of a group G $\sigma^{(2)}$-permutable if it permutes with every $\mathfrak{G}_{\pi_i}$-projector of G for all $\pi_i \in \sigma$.

Definition 3. We shall call a subgroup H of a group G $\sigma^{(3)}$-permutable if for all $\pi_i \in \sigma$ there is a $\mathfrak{G}_{\pi_i}$-projector P of G such that $HP^x = P^x H$ for all $x \in G$.

Let $\sigma_1 = \{\{2\}, \{3\}, \{5\}, \dots\}$. Then the concepts of S-permutable and $\sigma_1^{(i)}$-permutable subgroups coincides for $i \in \{1, 2, 3\}$. Let π be a set of primes and G be a group. If H is a Hall π-subgroup of G then it is a $\mathfrak{G}_\pi$-projector of G. Therefore every σ-permutable subgroup is $\sigma^{(3)}$-permutable. Also it is clear that every $\sigma^{(1)}$-permutable subgroup is $\sigma^{(2)}$-permutable and every

$\sigma^{(2)}$-permutable subgroup is $\sigma^{(3)}$-permutable. As follows form theorems of Hall the concepts of σ-permutable subgroup and $\sigma^{(i)}$-permutable subgroup for $i \in \{1, 2, 3\}$ coincides for a soluble group. Moreover

Theorem 1. *A subgroup H of a group G is $\sigma^{(3)}$-permutable if and only if it is $\sigma^{(2)}$-permutable.*

Conjecture 1. *A subgroup H of a group G is $\sigma^{(1)}$-permutable if and only if it is $\sigma^{(2)}$-permutable.*

According to [4] a group is called σ-nilpotent if it is the direct product of its Hall π_i-subgroups for all $\pi_i \in \sigma$. Such classes of groups are the examples of lattice formations (see [6, Chapter 6]). The connection between the class $\mathfrak{N}_\sigma$ of all σ-nilpotent groups and $\sigma^{(3)}$-permutable subgroups is shown in

Theorem 2. *Let H be a $\sigma^{(3)}$-permutable subgroup of a group G. Then H^G/H_G is σ-nilpotent.*

Corollary 1 (Kegel [1] and Deskins [2])**.** *Let H be a S-permutable subgroup of a group G then H^G/H_G is nilpotent.*

Corollary 2 ([4, Theorem B(i)])**.** *Let G be a E_{π_i}-group for all $\pi_i \in \sigma$. If H is a σ-permutable subgroup of a group G then H^G/H_G is σ-nilpotent.*

Recall [4] that a subgroup H of a group G is called σ-subnormal in G if there is a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{j-1} is normal in H_j or $\pi(H_j/(H_{j-1})_{H_j}) \subseteq \pi_i$ for some $\pi_i \in \sigma$ and $j = 1, \ldots, n$. In fact, the concept of σ-subnormal subgroup is equivalent to the concept of K-$\mathfrak{N}_\sigma$-subnormal subgroup in the sense of [6, 6.1.4].

Corollary 3. *Let H be a $\sigma^{(3)}$-permutable subgroup of a group G. Then H is σ-subnormal in G.*

The following theorem shows that conjecture 1 is true for σ-nilpotent $\sigma^{(2)}$-subnormal subgroups.

Theorem 3. *Let H be a σ-nilpotent subgroup of a group G. Then*

(1) *If H is $\sigma^{(i)}$-permutable in G for some $i \in \{1, 2, 3\}$ then H is $\sigma^{(i)}$-permutable in G for all $i \in \{1, 2, 3\}$.*

(2) *H is $\sigma^{(3)}$-permutable in G if and only if every Hall π_i-subgroup of H is $\sigma^{(3)}$-permutable in G for all $\pi_i \in \sigma$.*

Corollary 4 (Schmid [3])**.** *Let H be a nilpotent subgroup of a group G. Then H is S-permutable in G if and only if every Sylow subgroup of H is S-permutable in G.*

The main result of this paper is

Theorem 4. *The set of all $\sigma^{(3)}$-permutable subgroups of a group G forms a sublattice of the lattice of all subgroups of G.*

Corollary 5 ([1, Satz 2])**.** *The set of all S-permutable subgroups of a group G forms a sublattice of the lattice of all subgroups of G.*

Corollary 6 ([4, Theorem C])**.** *Let every subgroup of a group G be a D_{π_i}-group for all $\pi_i \in \sigma$. Then the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all subgroups of G.*

Our final result concerns the $\sigma^{(3)}$-permutability of the normalizer of a $\sigma^{(3)}$-permutable subgroup.

Theorem 5. *If H is a $\sigma^{(3)}$-permutable subgroup of a group G then $N_G(H)$ is also $\sigma^{(3)}$-permutable in G.*

Corollary 7 (Schmid [3])**.** *If H is a S-permutable subgroup of G then $N_G(H)$ is also S-permutable in G.*

1 Preliminaries

All unexplained notations and terminologies are standard. The reader is referred to [5, 7] if necessary. Recall that $\mathrm{O}_\pi(G)$ is the unique largest normal π-subgroup of G; $\mathrm{O}^\pi(G)$ is the unique smallest normal subgroup of G for which the corresponding factor group is a π-group; H_G is the unique largest normal subgroup of G contained in H; H^G is the unique smallest normal subgroup of G containing H.

Let $\mathfrak{F}$ be a homomorph. It is known that if a subgroup P of a group G is an $\mathfrak{F}$-projector then PN/N is an $\mathfrak{F}$-projector of G/N. And if P/N is an $\mathfrak{F}$-projector of G/N then all $\mathfrak{F}$-projectors of P are $\mathfrak{F}$-projectors of G (see [5, III, 3.7]). It means that the set $\{PN/N \mid P$ is an $\mathfrak{F}$-projector of $G\}$ is the set of all $\mathfrak{F}$-projectors of G/N.

Lemma 1. *Let N be a normal subgroup of a group G and $i \in \{2,3\}$.*

(1) *If H is a $\sigma^{(i)}$-permutable subgroup of G then HN/N is a $\sigma^{(i)}$-permutable subgroup of G/N.*

(2) *If H/N is a $\sigma^{(i)}$-permutable subgroup of G/N then H is a $\sigma^{(i)}$-permutable subgroup of G.*

Proof. Assume that H is $\sigma^{(2)}$-permutable in G. Then $HP = PH$ for every $\mathfrak{G}_{\pi_i}$-projector P of G and every $\pi_i \in \sigma$. So $(HN/N)(PN/N) = HPN/N = (PN/N)(HN/N)$ for every $\mathfrak{G}_{\pi_i}$-projector P of G and every $\pi_i \in \sigma$. It means that HN/N permutes with all $\mathfrak{G}_{\pi_i}$-projectors G/N for all $\pi_i \in \sigma$. Thus HN/N is $\sigma^{(2)}$-permutable in G/N.

Assume that HN/N is $\sigma^{(2)}$-permutable in G/N. Then $(HN/N)(PN/N) = HPN/N = (PN/N)(HN/N)$ for every $\mathfrak{G}_{\pi_i}$-projector P of G and every $\pi_i \in \sigma$. It means that $(HN)(PN) = HPN = (PN)(HN)$ or $(HN)P = HPN = P(HN)$ for every $\mathfrak{G}_{\pi_i}$-projector P of G and every $\pi_i \in \sigma$. Thus HN is $\sigma^{(2)}$-permutable in G.

The proof for $\sigma^{(3)}$-permutable subgroups is analogues. □

Lemma 2 ([4, Lemma 2.6])**.** *Let H, K be a subgroups of a group G. Then*

(1) *If H is σ-subnormal in G then $H \cap K$ is σ-subnormal in K.*

(2) *If H is σ-subnormal in G and $|G : H|$ is a π_i-number for some $\pi_i \in \sigma$ then $\mathrm{O}^{\pi_i}(H) = \mathrm{O}^{\pi_i}(G)$.*

Lemma 3 ([7, 1.2.2])**.** *If a subgroup H of a group G permutes with the subgroups X and Y of G the it is also permutes with their join $\langle X, Y\rangle$.*

The following lemma directly follows from [6, 6.3.8].

Lemma 4. *Let H be a σ-subnormal π_i-subgroup for some $\pi_i \in \sigma$. Then $H \leq \mathrm{O}_{\pi_i}(G)$.*

2 Proofs of the results

Proof of theorem 2. Assume that (G, H) is a counterexample with $|G| + |G : H|$ minimum. From (1) of lemma 1 and inductive hypothesis it follows that $H_G = 1$.

Let $D = \bigcap_{x\in G\setminus N_G(H)} \langle H, H^x\rangle$. We have that $H \leq D$ and $\langle H, H^x\rangle$ is $\sigma^{(3)}$-permutable for all $x \in G$ by lemma 3. It is clear that $D^G = H^G$ and $D_G = \bigcap_{x\in G\setminus N_G(H)} \langle H, H^x\rangle_G$.

Assume that $D = H$. We see that $\langle H, H^x\rangle^G = H^G$ for all $x \in G \setminus N_G(H)$. By induction $\langle H, H^x\rangle^G/\langle H, H^x\rangle_G = H^G/\langle H, H^x\rangle_G$ is σ-nilpotent. Since the class of all σ-nilpotent groups is a formation, $H^G/\bigcap_{x\in G\setminus N_G(H)} \langle H, H^x\rangle_G = H^G/H_G$ is σ-nilpotent, a contradiction.

So H is a proper subgroup of D. From $H_G = 1$ and $H \neq 1$ it follows that $N_G(H) \neq G$. So there is a π_i-element $x \in G \setminus N_G(H)$ for some $\pi_i \in \sigma$. Note that if $N_G(H)$ contains some $\mathfrak{G}_{\pi_i}$-projector of G and all its conjugates in G then it contains all π_i-elements of G. It means that there is a $\mathfrak{G}_{\pi_i}$-projector T of G with $HT = TH$ and we may assume that $x \in T$. Then

$H < D \leq \langle H, H^x\rangle \leq \langle H, x\rangle \leq HT = TH$. Hence $|D : H|$ is a π_i-number. If there is a π_j-element $y \in N_G(H)$ for $i \neq j$ then the same argument shows that $|D : H|$ is a π_j-number. So $H = D$, a contradiction.

It means that all π_i'-elements of G lie in $N_G(H)$. Hence $O^{\pi_i}(G) \leq N_G(H)$. So $H^G = H^{HT}$. Thus $|H^G : H|$ is a π_i-number. From $H \triangleleft HO^{\pi_i}(G) \leq TO^{\pi_i}(G) = G$ it follows that H is σ-subnormal in G, and hence in H^G by (1) of lemma 2. So $O^{\pi_i}(H^G) = O^{\pi_i}(H)$ by (2) of lemma 2. Since $H_G = 1$, we see $O^{\pi_i}(H^G) = 1$. Thus H^G is a π_i-group, i.e. H^G/H_G is σ-nilpotent, the final contradiction. □

Proof of corollary 3. Assume that H is a $\sigma^{(3)}$-permutable subgroup of a group G. According to theorem 2 H^G/H_G is σ-nilpotent. Now H/H_G is σ-subnormal in $H^G/H_G \triangleleft G/H_G$. Hence H/H_G is σ-subnormal in G/H_G. Thus H is σ-subnormal in G. □

Lemma 5. *Let H be a $\sigma^{(3)}$-permutable subgroup of a group G. Then $\mathrm{O}^{\pi_i}(G) \leq N_G(\mathrm{O}_{\pi_i}(H))$ for every $\pi_i \in \sigma$.*

Proof. Let $\pi_n \in \sigma$ and P be a $\mathfrak{G}_{\pi_i}$-projector of G for some $\pi_i \in \sigma \setminus \{\pi_n\}$ with $P^xH = HP^x$ for all $x \in G$. According to corollary 3 H is σ-subnormal in G. Now H is σ-subnormal in P^xH by (1) of lemma 2. From (2) of lemma 2 it follows that $\mathrm{O}^{\pi_i}(H) = \mathrm{O}^{\pi_i}(P^xH)$. Hence $\mathrm{O}_{\pi_n}(H) char \mathrm{O}^{\pi_i}(H) = \mathrm{O}^{\pi_i}(P^xH)$. Thus $P^x \leq N_G(\mathrm{O}_{\pi_n}(H))$ for all $x \in G$. Since P is a $\mathfrak{G}_{\pi_i}$-projector of G, $\mathrm{O}^{\pi_i'}(G) \leq N_G(\mathrm{O}_{\pi_n}(H))$. Therefore $\mathrm{O}^{\pi_n}(G) \leq N_G(\mathrm{O}_{\pi_n}(H))$. □

Proof of theorem 3. Since H is σ-nilpotent, $\mathrm{O}_{\pi_i}(H)$ is the unique Hall π_i-subgroup of H for all $\pi_i \in \sigma$. Assume that H is $\sigma^{(3)}$-permutable in G. Then $\mathrm{O}^{\pi_i}(G) \leq N_G(\mathrm{O}_{\pi_i}(H))$ by lemma 5. Hence $\mathrm{O}_{\pi_i}(H)$ is σ-subnormal in G and $\mathrm{O}_{\pi_i}(H)P = P\mathrm{O}_{\pi_i}(H)$ for every π_j-maximal subgroup P of G and all $\pi_j \in \sigma \setminus \{\pi_i\}$. According to lemma 4 $\mathrm{O}_{\pi_i}(H) \leq \mathrm{O}_{\pi_i}(G)$. Therefore $\mathrm{O}_{\pi_i}(H)P = P\mathrm{O}_{\pi_i}(H) = P$ for every π_i-maximal subgroup P of G. Thus $\mathrm{O}_{\pi_i}(H)$ is $\sigma^{(1)}$-permutable (and hence $\sigma^{(3)}$-permutable) in G for all $\pi_i \in \sigma$. Therefore H is $\sigma^{(1)}$-permutable in G by lemma 3.

Assume that every Hall π_i-subgroup of H is $\sigma^{(3)}$-permutable in G for all $\pi_i \in \sigma$. By (1) every Hall π_i-subgroup of H is $\sigma^{(1)}$-permutable in G for all $\pi_i \in \sigma$. From lemma 3 it follows that H is $\sigma^{(1)}$-permutable (and hence $\sigma^{(3)}$-permutable) in G. □

Proof of theorem 1. We need only to prove that every $\sigma^{(3)}$-permutable subgroup is $\sigma^{(2)}$-permutable. Let H be a $\sigma^{(3)}$-permutable subgroup of a group G. Then H/H_G is $\sigma^{(2)}$-permutable in G/H_G by (1) of theorem 3. So H is $\sigma^{(2)}$-permutable in G by (2) of lemma 1. □

Proof of theorem 4. In fact, in view of lemma 3, we have only to show that if A and B are $\sigma^{(3)}$-permutable subgroups of G, then $C = A \cap B$ is $\sigma^{(3)}$-permutable in G. Assume that this statement is false and let a group G be minimal order counterexample. Then $A_G \cap B_G = 1$ by lemma 1. From theorem 2 it follows that A^G/A_G and B^G/B_G are σ-nilpotent. Hence $(A^G \cap B^G)/(A_G \cap B^G)$ and $(A^G \cap B^G)/(A^G \cap B_G)$ are σ-nilpotent. Thus $(A^G \cap B^G)/(A_G \cap B_G) \simeq (A^G \cap B^G)$ is σ-nilpotent. So C is σ-nilpotent.

From corollary 3 it follows that A and B are σ-subnormal in G. So C is σ-subnormal in G by (1) of lemma 2. Now every Hall π_i-subgroup C_{π_i} of C is σ-subnormal in G for all $\pi_i \in \sigma$. Hence $C_{\pi_i} \leq \mathrm{O}_{\pi_i}(G)$ by lemma 4 for all $\pi_i \in \sigma$. Thus C_{π_i} permutes with every $\mathfrak{G}_{\pi_i}$-projector of G for all $\pi_i \in \sigma$.

Since every $\sigma^{(3)}$-permutable subgroup is a $\sigma^{(2)}$-permutable subgroup by theorem 1, $AH \cap BH$ is a subgroup of G for every $\mathfrak{G}_{\pi_i}$-projector H of G and every $\pi_i \in \sigma$. From (2) of lemma 2 it follows that $\mathrm{O}^{\pi_i}(A) = \mathrm{O}^{\pi_i}(AH)$ and $\mathrm{O}^{\pi_i}(B) = \mathrm{O}^{\pi_i}(BH)$ for every $\mathfrak{G}_{\pi_i}$-projector H of G and every $\pi_i \in \sigma$. Thus $\mathrm{O}^{\pi_i}(AH \cap BH) \leq C$. Therefore $|(AH \cap BH) : C|$ is a π_i-number. So $\mathrm{O}^{\pi_i}(C) = \mathrm{O}^{\pi_i}(AH \cap BH)$ by lemma 2. Hence $H \leq N_G(C_{\pi_j})$ for every Hall π_j-subgroup C_{π_j} of C for all $\pi_j \in \sigma \setminus \{\pi_i\}$ and every $\mathfrak{G}_{\pi_i}$-projector H of G and every $\pi_i \in \sigma$. Thus $O^{\pi_i}(G) \leq N_G(C_{\pi_i})$ for every $\pi_i \in \sigma$. It means that C_{π_i} permutes with every $\mathfrak{G}_{\pi_j}$-projector for all $\pi_j \in \sigma \setminus \{\pi_i\}$

for every $\pi_i \in \sigma$. Hence C_{π_i} is $\sigma^{(3)}$-permutable subgroup of G for all $\pi_i \in \sigma$. Thus C is $\sigma^{(3)}$-permutable subgroup of G by theorem 3, the contradiction. $\square$

Proof of theorem 5. Assume that (G, H) is a counterexample with $|G|+|G : H|$ minimum. Applying lemma 1 we may assume that $H_G = 1$. So H is σ-nilpotent by theorem 2. According to (2) of theorem 3 every Hall π_i-subgroup of H is $\sigma^{(3)}$-permutable for all $\pi_i \in \sigma$. Suppose that every Hall π_i-subgroup of H is a proper subgroup of H for all $\pi_i \in \sigma$. Therefore $N_G(P)$ is $\sigma^{(3)}$-permutable in G for every Hall π_i-subgroup P of H for all $\pi_i \in \sigma$ by the choice of H and theorem 3. Now $N_G(H)$ is $\sigma^{(3)}$-permutable in G by theorem 4, a contradiction. Thus H is a π_i-group for some $\pi_i \in \sigma$. Hence $\mathrm{O}^{\pi_i}(G) \leq N_G(H)$ by lemma 5. So $N_G(H)P = G$ for every $\mathfrak{G}_{\pi_i}$-projector P of G and $N_G(H)P = N_G(H)$ for every $\mathfrak{G}_{\pi_j}$-projector P of G for all $\pi_j \in \sigma \setminus \{\pi_i\}$. Thus $N_G(H)$ is $\sigma^{(3)}$-permutable in G, the final contradiction. $\square$

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