

# On the quasi- $\mathfrak{F}$ -hypercenter of a finite group

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**Abstract.** In this paper some properties of the  $\mathfrak{F}^*$ -hypercenter of a finite group are studied where  $\mathfrak{F}^*$  is the class of all finite quasi- $\mathfrak{F}$ -groups for a hereditary saturated formation  $\mathfrak{F}$  of finite groups. In particular, it is shown that the quasinilpotent hypercenter of a finite group coincides with the intersection of all maximal quasinilpotent subgroups.

**Keywords.** Finite groups; quasinilpotent groups; quasi- $\mathfrak{F}$ -groups; hereditary saturated formation; hypercenter of a group.

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## Introduction and the results

All considered groups are finite. In [1] R. Baer showed that the hypercenter  $Z_\infty(G)$  of a group  $G$  coincides with the intersection of all maximal nilpotent subgroups of  $G$ . The concept of hypercenter was extended on classes of groups (see [11, p. 127–128] or [5, 1, 2.2]). Let  $\mathfrak{X}$  be a class of groups. A chief factor  $H/K$  of a group  $G$  is called  $\mathfrak{X}$ -central if  $(H/K) \rtimes G/C_G(H/K) \in \mathfrak{X}$  otherwise it is called  $\mathfrak{X}$ -eccentric. A normal subgroup  $N$  of  $G$  is said to be  $\mathfrak{X}$ -hypercentral in  $G$  if  $N = 1$  or  $N \neq 1$  and every chief factor of  $G$  below  $N$  is  $\mathfrak{X}$ -central. The  $\mathfrak{X}$ -hypercenter  $Z_\mathfrak{X}(G)$  is the product of all normal  $\mathfrak{X}$ -hypercentral subgroups of  $G$ . So if  $\mathfrak{X} = \mathfrak{N}$  is the class of all nilpotent groups then  $Z_\infty(G) = Z_\mathfrak{N}(G)$  for every group  $G$ .

L. A. Shemetkov posed the following problem on the Gomel Algebraic Seminar in 1995: “Describe all hereditary saturated formations  $\mathfrak{F}$  such that  $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$  holds for every group  $G$ ”. This problem was solved by A. N. Skiba in [12]. Let  $F$  be the canonical local definition of a nonempty local formation  $\mathfrak{F}$ . Then  $\mathfrak{F}$  is said to satisfy the boundary condition [12] if  $\mathfrak{F}$  contains every group  $G$  whose all maximal subgroups belong to  $F(p)$  for some prime  $p$ .

**Theorem 1** ([12, Theorem A]). *Let  $\mathfrak{F}$  be a hereditary saturated formation. Then  $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$  holds for every group  $G$  if and only if  $\mathfrak{F}$  satisfies the boundary condition.*

The natural generalization of a saturated formation is a solubly saturated formation. Recall that a formation  $\mathfrak{F}$  is called solubly saturated if from  $G/\Phi(G_\mathfrak{E}) \in \mathfrak{F}$  it follows that  $G \in \mathfrak{F}$  where  $G_\mathfrak{E}$  is the soluble radical of a group  $G$ . Every saturated formation is solubly saturated. The class  $\mathfrak{N}^*$  of all quasinilpotent groups is the example of a solubly saturated and not saturated formation. Hence the following problem seems natural:

**Problem 1.** Describe all normally hereditary solubly saturated formations  $\mathfrak{F}$  such that  $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$  holds for every group  $G$ .

**Remark 1.** The solution of Problem 1 when  $\mathfrak{F}$  is a normally hereditary saturated formation is not known at the time of writing.

Recall [3, 3.4.8] that every solubly saturated formation  $\mathfrak{F}$  contains the greatest saturated subformation  $\mathfrak{F}_l$  with respect to inclusion. A. F. Vasil’ev suggested the following problem:

**Problem 2** (A. F. Vasil’ev). (1) Let  $\mathfrak{H}$  be a saturated formation. Assume that  $\text{Int}_\mathfrak{H}(G) = Z_\mathfrak{H}(G)$  holds for every group  $G$ . Describe all normally hereditary solubly saturated formations  $\mathfrak{F}$  with  $\mathfrak{F}_l = \mathfrak{H}$  such that  $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$  holds for every group  $G$ .

(2) Let  $\mathfrak{F}$  be a normally hereditary solubly saturated formation. Assume that  $\text{Int}_\mathfrak{F}(G) = Z_\mathfrak{F}(G)$  holds for every group  $G$ . Does  $\text{Int}_{\mathfrak{F}_l}(G) = Z_{\mathfrak{F}_l}(G)$  hold for every group  $G$ ?

Let  $\mathfrak{F}$  be a normally hereditary solubly saturated formation. The following example shows that if  $\text{Int}_{\mathfrak{F}_i}(G) = Z_{\mathfrak{F}_i}(G)$  holds for every group  $G$  then it is not necessary that  $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$  holds for every group  $G$ .

**Example 1.** Recall that  $\mathfrak{N}_{ca}$  is the class of groups whose abelian chief factors are central and non-abelian chief factors are simple groups. According to [14]  $\mathfrak{N}_{ca}$  is a normally hereditary solubly saturated formation. As follows from [3, 3.4.5] the greatest saturated subformation with respect to inclusion of  $\mathfrak{N}_{ca}$  is  $\mathfrak{N}$ . Recall that  $\text{Int}_{\mathfrak{N}}(G) = Z_{\mathfrak{N}}(G)$  holds for every group  $G$ .

Let  $G \simeq D_4(2)$  be a Chevalley orthogonal group and  $H$  be a  $\mathfrak{N}_{ca}$ -maximal subgroup of  $\text{Aut}(G)$ . We may assume that  $G \simeq \text{Inn}(G)$  is a normal subgroup of  $\text{Aut}(G)$ . Since  $G$  is simple and  $HG/G \in \mathfrak{N}_{ca}$ ,  $HG \in \mathfrak{N}_{ca}$  by the definition of  $\mathfrak{N}_{ca}$ . Hence  $HG = H$ . So  $G$  lies in the intersection of all  $\mathfrak{N}_{ca}$ -maximal subgroups of  $\text{Aut}(G)$ . From [9] it follows that  $\text{Aut}(G)/C_{\text{Aut}(G)}(G) \simeq \text{Aut}(G)$ . If  $G \leq Z_{\mathfrak{N}_{ca}}(\text{Aut}(G))$  then  $G \times (\text{Aut}(G)/C_{\text{Aut}(G)}(G)) \simeq G \times \text{Aut}(G) \in \mathfrak{N}_{ca}$ . Note that  $\text{Out}(G) \simeq S_3 \notin \mathfrak{N}_{ca}$  is the quotient group of  $G \times \text{Aut}(G)$ . Therefore  $G \times \text{Aut}(G) \notin \mathfrak{N}_{ca}$ . Thus  $G \not\leq Z_{\mathfrak{N}_{ca}}(\text{Aut}(G))$  and  $\text{Int}_{\mathfrak{N}_{ca}}(\text{Aut}(G)) \neq Z_{\mathfrak{N}_{ca}}(\text{Aut}(G))$ .

In [6, 7] W. Guo and A. N. Skiba introduced the concept of quasi- $\mathfrak{F}$ -group for a saturated formation  $\mathfrak{F}$ . Recall that a group  $G$  is called quasi- $\mathfrak{F}$ -group if for every  $\mathfrak{F}$ -eccentric chief factor  $H/K$  and every  $x \in G$ ,  $x$  induces inner automorphism on  $H/K$ . We use  $\mathfrak{F}^*$  to denote the class of all quasi- $\mathfrak{F}$ -groups. If  $\mathfrak{N} \subseteq \mathfrak{F}$  is a normally hereditary saturated formation then  $\mathfrak{F}^*$  is a normally hereditary solubly saturated formation by [6, Theorem 2.6].

**Theorem 2.** *Let  $\mathfrak{F}$  be a hereditary saturated formation containing all nilpotent groups. Then  $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$  holds for every group  $G$  if and only if  $\text{Int}_{\mathfrak{F}^*}(G) = Z_{\mathfrak{F}^*}(G)$  holds for every group  $G$ .*

**Corollary 1.** *The intersection of all maximal quasinilpotent subgroups of a group  $G$  is  $Z_{\mathfrak{N}^*}(G)$ .*

**Remark 2.** Let  $\mathfrak{N} \subseteq \mathfrak{F}$  be a hereditary saturated formation. As follows from [3, 3.4.5] and [5, 1.3, 3.6] the greatest saturated subformation with respect to inclusion of  $\mathfrak{F}^*$  is  $\mathfrak{F}$ .

**Remark 3.** Note that every  $\mathfrak{N}$ -central chief factor is central. From (4) and (5) of the proof of Theorem 2 it follows that  $Z_{\mathfrak{N}^*}(G)$  is the greatest normal subgroup of  $G$  such that every element of  $G$  induces an inner automorphism on every chief factor of  $G$  below  $Z_{\mathfrak{N}^*}(G)$ .

## 1 Preliminaries

The notation and terminology agree with the books [3, 5]. We refer the reader to these books for the results on formations. Recall that  $\pi(G)$  is the set of all prime divisors of a group  $G$ ,  $\pi(\mathfrak{F}) = \bigcup_{G \in \mathfrak{F}} \pi(G)$  and  $\mathbb{P}$  is the set of all primes.

Recall that if  $\mathfrak{F}$  is a saturated formation, there exists a unique formation function  $F$ , defining  $\mathfrak{F}$ , which is integrated ( $F(p) \subseteq \mathfrak{F}$  for every  $p \in \mathbb{P}$ , the set of all primes) and full ( $F(p) = \mathfrak{N}_p F(p)$  for every  $p \in \mathbb{P}$ ) [3, IV, 3.8].  $F$  is called the canonical local definition of  $\mathfrak{F}$ .

**Lemma 1** ([5, 1, 1.15]). *Let  $\mathfrak{F}$  be a saturated formation,  $F$  be its canonical local definition and  $G$  be a group. Then a chief factor  $H/K$  of  $G$  is  $\mathfrak{F}$ -central if and only if  $G/C_G(H/K) \in F(p)$  for all  $p \in \pi(H/K)$ .*

The following lemma directly follows from [8, X, 13.16(a)].

**Lemma 2.** *Let a normal subgroup  $N$  of a group  $G$  be a direct product of isomorphic simple non-abelian groups. Then  $N$  is a direct product of minimal normal subgroups of  $G$ .*

**Lemma 3.** *Let  $\mathfrak{F}$  be a hereditary saturated formation. Then  $Z_{\mathfrak{F}^*}(G) \leq \text{Int}_{\mathfrak{F}^*}(G)$ .*

*Proof.* Let  $\mathfrak{F}$  be a hereditary saturated formation with the canonical local definition  $F$ ,  $M$  be a  $\mathfrak{F}^*$ -maximal subgroup of  $G$  and  $N = MZ_{\mathfrak{F}^*}(G)$ . Let show that  $N \in \mathfrak{F}^*$ . It is sufficient to show that for every chief factor  $H/K$  of  $N$  below  $Z_{\mathfrak{F}^*}(G)$  either  $H/K$  is  $\mathfrak{F}$ -central in  $N$  or every  $x \in N$  induces an inner automorphism on  $H/K$ . Let  $1 = Z_0 \triangleleft Z_1 \triangleleft \cdots \triangleleft Z_n = Z_{\mathfrak{F}^*}(G)$  be a chief series of  $G$  below  $Z_{\mathfrak{F}^*}(G)$ . Then we may assume that  $Z_{i-1} \leq K \leq H \leq Z_i$  for some  $i$  by Jordan-Hölder theorem.

If  $Z_i/Z_{i-1}$  is an  $\mathfrak{F}$ -central chief factor of  $G$  then  $G/C_G(Z_i/Z_{i-1}) \in F(p)$  for all  $p \in \pi(F(p))$  by Lemma 1. Since  $F(p)$  is hereditary by [3, IV, 3.16],  $NC_G(Z_i/Z_{i-1})/C_G(Z_i/Z_{i-1}) \simeq N/C_N(Z_i/Z_{i-1}) \in F(p)$  for all  $p \in \pi(F(p))$ . Note that  $N/C_N(H/K)$  is a quotient group of  $N/C_N(Z_i/Z_{i-1})$ . Thus  $H/K$  is an  $\mathfrak{F}$ -central chief factor of  $N$  by Lemma 1.

If  $Z_i/Z_{i-1}$  is a  $\mathfrak{F}$ -eccentric chief factor of  $G$  then every element of  $G$  induced an inner automorphism on it. It means that  $Z_i/Z_{i-1}$  is a simple group. Hence it is also a chief factor of  $N$ . From  $N \leq G$  it follows that every element of  $N$  induces an inner automorphism on  $Z_i/Z_{i-1}$ .

Thus  $N \in \mathfrak{F}^*$ . So  $N = MZ_{\mathfrak{F}^*}(G) = M$ . Therefore  $Z_{\mathfrak{F}^*}(G) \leq M$  for every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$ .  $\square$

## 2 Proof of Theorem 2

Assume that  $\mathfrak{F}$  is a hereditary saturated formation containing all nilpotent groups with the canonical local definition  $F$ . Then  $F(p)$  is a hereditary formation for all primes  $p$  (see [3, IV, 3.14]).

Suppose that  $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$  holds for every group  $G$ . Let show that  $\text{Int}_{\mathfrak{F}^*}(G) = Z_{\mathfrak{F}^*}(G)$  also holds for every group  $G$ .

Let  $D$  be the intersection of all  $\mathfrak{F}^*$ -maximal subgroups of  $G$  and let  $H/K$  be a chief factor of  $G$  below  $D$ .

(1) *If  $H/K$  is abelian then  $MC_G(H/K)/C_G(H/K) \in F(p)$  for every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$ .*

If  $H/K$  is abelian then it is an elementary abelian  $p$ -group for some prime  $p$  and  $H/K \in \mathfrak{F}$ . Let  $M$  be a  $\mathfrak{F}^*$ -maximal subgroup of  $G$  and  $K = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = H$  be a part of a chief series of  $M$ . If  $H_i/H_{i-1}$  is  $\mathfrak{F}$ -eccentric for some  $i$  then every element of  $M$  induces an inner automorphism on  $H_i/H_{i-1}$ . So  $M/C_M(H_i/H_{i-1}) \simeq 1 \in F(p)$ . Therefore  $H_i/H_{i-1}$  is an  $\mathfrak{F}$ -central chief factor of  $M$ , a contradiction. Hence  $H_i/H_{i-1}$  is an  $\mathfrak{F}$ -central chief factor of  $M$  for all  $i = 1, \dots, n$ . So  $M/C_M(H_i/H_{i-1}) \in F(p)$  by Lemma 1 for all  $i = 1, \dots, n$ . Therefore  $M/C_M(H/K) \in \mathfrak{N}_p F(p) = F(p)$  by [13, Lemma 1]. Now  $MC_G(H/K)/C_G(H/K) \simeq M/C_M(H/K) \in F(p)$  for every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$ .

(2) *If  $H/K \in \mathfrak{F}$  is non-abelian then  $MC_G(H/K)/C_G(H/K) \in F(p)$  for every  $p \in \pi(H/K)$  and every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$ .*

If  $H/K \in \mathfrak{F}$  is non-abelian then it is a direct product of isomorphic non-abelian simple  $\mathfrak{F}$ -groups. Let  $M$  be a  $\mathfrak{F}^*$ -maximal subgroup of  $G$ . By Lemma 2  $H/K = H_1/K \times \cdots \times H_n/K$  is a direct product of minimal normal subgroups  $H_i/K$  of  $M/K$ . From  $H_i/K \in \mathfrak{F}$  it follows that  $(H_i/K)/O_{p',p}(H_i/K) \simeq H_i/K \in F(p)$  for all  $p \in \pi(H_i/K)$  and all  $i = 1, \dots, n$ . Assume that  $H_i/K$  is a  $\mathfrak{F}$ -eccentric chief factor of  $M/K$  for some  $i$ . It means that every element of  $M$  induces an inner automorphism on  $H_i/K$ . So  $M/C_M(H_i/K) \simeq H_i/K \in F(p)$ , a contradiction.

Now  $H_i/K$  is  $\mathfrak{F}$ -central in  $M/K$  for all  $i = 1, \dots, n$ . Therefore  $M/C_M(H_i/K) \in F(p)$  for all  $p \in \pi(H_i/K)$  by Lemma 1. Note that  $C_M(H/K) = \bigcap_{i=1}^n C_M(H_i/K)$ . Since  $F(p)$  is a formation,  $M/\bigcap_{i=1}^n C_M(H_i/K) = M/C_M(H/K) \in F(p)$  for all  $p \in \pi(H/K)$ . It means that  $MC_G(H/K)/C_G(H/K) \simeq M/C_M(H/K) \in F(p)$  for every  $p \in \pi(H/K)$  and every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$ .

(3) If  $H/K \in \mathfrak{F}$  then all  $\mathfrak{F}$ -subgroups of  $G/C_G(H/K)$  are  $F(p)$ -groups.

Let  $Q/C_G(H/K)$  be a  $\mathfrak{F}$ -maximal subgroup of  $G/C_G(H/K)$ . Then there exists a  $\mathfrak{F}$ -maximal subgroup  $N$  of  $G$  with  $NC_G(H/K)/C_G(H/K) = Q/C_G(H/K)$  by [5, 1, 5.7]. From  $\mathfrak{F} \subseteq \mathfrak{F}^*$  it follows that there exists a  $\mathfrak{F}^*$ -maximal subgroup  $L$  of  $G$  with  $N \leq L$ . So  $Q/C_G(H/K) \leq LC_G(H/K)/C_G(H/K) \in F(p)$  by (1) and (2). Since  $F(p)$  is hereditary,  $Q/C_G(H/K) \in F(p)$ . It means that all  $\mathfrak{F}$ -maximal subgroups of  $G/C_G(H/K)$  are  $F(p)$ -groups. Hence all  $\mathfrak{F}$ -subgroups of  $G/C_G(H/K)$  are  $F(p)$ -groups.

(4) If  $H/K \in \mathfrak{F}$  then it is  $\mathfrak{F}$ -central in  $G$ .

Assume now that  $H/K$  is not  $\mathfrak{F}$ -central in  $G$ . So  $G/C_G(H/K) \notin F(p)$  for some  $p \in \pi(H/K)$  by Lemma 1. It means that  $G/C_G(H/K)$  contains a  $s$ -critical for  $F(p)$  subgroup  $S/C_G(H/K)$  for some  $p \in \pi(H/K)$ . Since  $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$  holds for every group  $G$ ,  $S/C_G(H/K) \in \mathfrak{F}$  by Theorem 1. Therefore  $S/C_G(H/K) \in F(p)$  by (4), the contradiction. Thus  $H/K$  is  $\mathfrak{F}$ -central in  $G$ .

(5) If  $H/K \notin \mathfrak{F}$  is non-abelian then every element of  $G$  induces an inner automorphism on it.

Let  $M$  be a  $\mathfrak{F}^*$ -maximal subgroup of  $G$ . By Lemma 2  $H/K = H_1/K \times \cdots \times H_n/K$  is a direct product of minimal normal subgroups  $H_i/K$  of  $M/K$ . Since  $H_i/K \notin \mathfrak{F}$  for all  $i = 1, \dots, n$ , it is an  $\mathfrak{F}$ -eccentric chief factor of  $M$  for all  $i = 1, \dots, n$ . So every element of  $M$  induces an inner automorphism on  $H_i/K$  for all  $i = 1, \dots, n$ . It means that for all  $a \in M$  and for all  $i = 1, \dots, n$  there is  $x(a, i) \in H_i/K$  and  $(H_i/K)^a = (H_i/K)^{x(a, i)}$ . So for all  $a \in M$  there is  $x(a) = x(a, 1) \dots x(a, n) \in H/K$  such that  $(H/K)^a = (H_1/K)^a \times \cdots \times (H_n/K)^a = (H_1/K)^{x(a, 1)} \times \cdots \times (H_n/K)^{x(a, n)} = (H_1/K)^{x(a)} \times \cdots \times (H_n/K)^{x(a)} = (H/K)^{x(a)}$ .

It means that for every  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$  every element of  $M$  induces an inner automorphism on  $H/K$ . Since  $\mathfrak{N} \subseteq \mathfrak{F}$ ,  $\langle x \rangle \in \mathfrak{F}$  for every  $x \in G$ . From  $\mathfrak{F} \subseteq \mathfrak{F}^*$  it follows that for every element  $x$  of  $G$  there is a  $\mathfrak{F}^*$ -maximal subgroup  $M$  of  $G$  with  $x \in M$ . Thus every element of  $G$  induces an inner automorphism on  $H/K$ .

(7) *The final step.*

If  $H/K \in \mathfrak{F}$  then from  $\mathfrak{F} \subseteq \mathfrak{F}^*$  and (4) it follows that  $H/K$  is  $\mathfrak{F}^*$ -central in  $G$ . Assume that  $H/K \notin \mathfrak{F}$ . By (5) every element of  $G$  induces an inner automorphism on it. Hence  $H/K$  is a simple non-abelian group. Since  $G/C_G(H/K)$  is isomorphic to some subgroup of  $\text{Inn}(H/K)$ , we see that  $G/C_G(H/K) \simeq H/K$ . It is straightforward to check that  $H/K \rtimes G/C_G(H/K) \simeq H/K \times H/K$ . From  $H/K \in \mathfrak{F}^*$  it follows that  $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}^*$ . Hence  $H/K$  is  $\mathfrak{F}^*$ -central in  $G$ .

Thus every chief factor of  $G$  below  $D$  is  $\mathfrak{F}^*$ -central in  $G$ . Hence  $D \leq Z_{\mathfrak{F}^*}(G)$ . According to Lemma 3  $Z_{\mathfrak{F}^*}(G) \leq D$ . Therefore  $Z_{\mathfrak{F}^*}(G) = D$ .

Suppose that  $\text{Int}_{\mathfrak{F}^*}(G) = Z_{\mathfrak{F}^*}(G)$  holds for every group  $G$ . Let show that  $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$  also holds for every group  $G$ . Assume the contrary. Then there exists a  $s$ -critical for  $F(p)$  group  $G \notin \mathfrak{F}$  for some  $p \in \mathbb{P}$  by Theorem 1. We may assume that  $G$  is a minimal group with this property. Then  $O_p(G) = \Phi(G) = 1$  and  $G$  has an unique minimal normal subgroup by [12, Lemma 2.10]. Note that  $G$  is also  $s$ -critical for  $\mathfrak{F}$ .

Assume that  $G \notin \mathfrak{F}^*$ . Then there exists a simple  $\mathbb{F}_p G$ -module  $V$  which is faithful for  $G$  by [3, 10.3B]. Let  $T = V \rtimes G$ . Note that  $T \notin \mathfrak{F}^*$ . Let  $M$  be a maximal subgroup of  $T$ . If  $V \leq M$  then  $M = M \cap VG = V(M \cap G)$  where  $M \cap G$  is a maximal subgroup of  $G$ . From  $M \cap G \in F(p)$  and  $F(p) = \mathfrak{N}_p F(p)$  it follows that  $V(M \cap G) = M \in F(p) \subseteq \mathfrak{F} \subseteq \mathfrak{F}^*$ . Hence  $M$  is an  $\mathfrak{F}^*$ -maximal subgroup of  $G$ . If  $V \not\leq M$  then  $M \simeq T/V \simeq G \notin \mathfrak{F}$ . Now it is clear that the sets of all maximal  $\mathfrak{F}$ -subgroups and all  $\mathfrak{F}^*$ -maximal subgroups of  $T$  coincide. Therefore  $V$  is the intersection of all  $\mathfrak{F}^*$ -maximal subgroups of  $T$ . From  $T \simeq V \rtimes T/C_T(V) \notin \mathfrak{F}^*$  it follows that  $V \not\leq Z_{\mathfrak{F}^*}(T)$ , a contradiction.

Assume that  $G \in \mathfrak{F}^*$ . Let  $N$  be a minimal normal subgroup of  $G$ . If  $N < G$  then  $N \in \mathfrak{F}$ . As follows from (4)  $N$  is an  $\mathfrak{F}$ -central chief factor of  $G$ . So  $N \leq Z_{\mathfrak{F}}(G)$ . Since  $N$  is an unique

minimal normal subgroup of  $s$ -critical for  $\mathfrak{F}$ -group  $G$  and  $\Phi(G) = 1$ , we see that  $G/N \in \mathfrak{F}$ . Hence  $G \in \mathfrak{F}$ , a contradiction. Thus  $N = G$  is a simple group. From  $\mathfrak{N} \subseteq \mathfrak{F}$  it follows that  $G$  is non-abelian.

Let  $p \in \pi(G)$ . According to [10] there is a simple Frattini  $\mathbb{F}_p G$ -module  $A$  which is faithful for  $G$ . By known Gaschütz theorem [4], there exists a Frattini extension  $A \twoheadrightarrow R \twoheadrightarrow G$  such that  $A \stackrel{G}{\simeq} \Phi(R)$  and  $R/\Phi(R) \simeq G$ . Since  $\Phi(R) \rtimes R/C_R(\Phi(R)) \simeq A \rtimes G \notin \mathfrak{F}$ , we see that  $R \notin \mathfrak{F}^*$ . Let  $M$  be a maximal subgroup of  $R$ . Then  $M/\Phi(R)$  is isomorphic to a maximal subgroup of  $G$ . So  $M/\Phi(R) \in F(p)$ . From  $\mathfrak{N}_p F(p) = F(p)$  it follows that  $M \in F(p) \subseteq \mathfrak{F} \subseteq \mathfrak{F}^*$ . Hence the sets of maximal and  $\mathfrak{F}^*$ -maximal subgroups of  $R$  coincide. Thus  $\Phi(R) = Z_{\mathfrak{F}^*}(R)$ . From  $R/Z_{\mathfrak{F}^*}(R) \simeq G \in \mathfrak{F}^*$  it follows that  $R \in \mathfrak{F}^*$ , the final contradiction.

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