

On the Generalized Fitting Height and Nonsoluble Length of the Mutually Permutable Products of Finite Groups*

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Abstract

The generalized Fitting height $h^*(G)$ of a finite group G is the least number h such that $F_h^*(G) = G$, where $F_{(0)}^*(G) = 1$, and $F_{(i+1)}^*(G)$ is the inverse image of the generalized Fitting subgroup $F^*(G/F_{(i)}^*(G))$. Let p be a prime, $1 = G_0 \leq G_1 \leq \dots \leq G_{2h+1} = G$ be the shortest normal series in which for i odd the factor G_{i+1}/G_i is p -soluble (possibly trivial), and for i even the factor G_{i+1}/G_i is a (non-empty) direct product of nonabelian simple groups. Then $h = \lambda_p(G)$ is called the non- p -soluble length of a group G . We proved that if a finite group G is a mutually permutable product of subgroups A and B then $\max\{h^*(A), h^*(B)\} \leq h^*(G) \leq \max\{h^*(A), h^*(B)\} + 1$ and $\max\{\lambda_p(A), \lambda_p(B)\} = \lambda_p(G)$. Also we introduced and studied the non-Frattini length.

Keywords: Finite group; generalized Fitting subgroup; mutually permutable product of groups; generalized Fitting height; non- p -soluble length; Plotkin radical.

1 Introduction and the Main Results

All groups considered here are finite. E.I. Khukhro and P. Shumyatsky introduced and studied interesting invariants of a group: the generalized Fitting height and the nonsoluble length [11–13]. The first one is the extension of the well known Fitting height to the class of all groups and the second one implicitly appeared in [8, 20].

Definition 1.1 (Khukhro, Shumyatsky). (1) *The generalized Fitting height $h^*(G)$ of a finite group G is the least number h such that $F_h^*(G) = G$, where $F_{(0)}^*(G) = 1$, and $F_{(i+1)}^*(G)$ is the inverse image of the generalized Fitting subgroup $F^*(G/F_{(i)}^*(G))$.*

(2) *Let p be a prime, $1 = G_0 \leq G_1 \leq \dots \leq G_{2h+1} = G$ be the shortest normal series in which for i odd the factor G_{i+1}/G_i is p -soluble (possibly trivial), and for i even the factor G_{i+1}/G_i is a (non-empty) direct product of nonabelian simple groups. Then $h = \lambda_p(G)$ is called the non- p -soluble length of a group G .*

(3) *Recall that $\lambda_2(G) = \lambda(G)$ is the nonsoluble length of a group G .*

In [12] E.I. Khukhro and P. Shumyatsky showed that in the general case the generalized Fitting height of a factorized group is not bounded in terms of the generalized Fitting heights of factors. The same situation is also for the nonsoluble length.

Recall [1, Definition 4.1.1] that a group G is called a mutually permutable product of its subgroups A and B if $G = AB$, A permutes with every subgroup of B and B permutes with

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every subgroup of A . The products of mutually permutable subgroups is the very interesting topic of the theory of groups (for example, see [1, Chapter 4]).

The main result of our paper is

Theorem 1.1. *Let a group G be the product of the mutually permutable subgroups A and B . Then*

- (1) $\max\{h^*(A), h^*(B)\} \leq h^*(G) \leq \max\{h^*(A), h^*(B)\} + 1$.
- (2) $\max\{\lambda_p(A), \lambda_p(B)\} = \lambda_p(G)$ for any prime p . In particular, $\max\{\lambda(A), \lambda(B)\} = \lambda(G)$.

If a group G is soluble, then $h^*(G) = h(G)$ is the Fitting height of a group G .

Corollary 1.2 ([10]). *If a soluble group G is the product of the mutually permutable subgroups A and B , then $\max\{h(A), h(B)\} \leq h(G) \leq \max\{h(A), h(B)\} + 1$.*

Example 1.1. *Note that the symmetric group \mathbb{S}_3 of degree 3 is the mutually permutable product of the cyclic groups Z_2 and Z_3 of orders 2 and 3 respectively. Hence $h^*(\mathbb{S}_3) = \max\{h^*(Z_2), h^*(Z_3)\} + 1 = \max\{h(Z_2), h(Z_3)\} + 1$.*

2 The Functorial Method

According to B.I. Plotkin [15] a functorial is a function γ which assigns to each group G its characteristic subgroup $\gamma(G)$ satisfying $f(\gamma(G)) = \gamma(f(G))$ for any isomorphism $f : G \rightarrow G^*$.

We are interested in functorials with some properties:

- (F1) $f(\gamma(G)) \subseteq \gamma(f(G))$ for every epimorphism $f : G \rightarrow G^*$.
- (F2) $\gamma(N) \subseteq \gamma(G)$ for every $N \trianglelefteq G$.
- (F3) $\gamma(G) \cap N \subseteq \gamma(N)$ for every $N \trianglelefteq G$.

Remark 2.1. (0) *Functions F^* and R_p that assign to every group respectively its the generalized Fitting subgroup and the p -soluble radical are examples of functorials. It is well known that they satisfy (F1), (F2), (F3).*

(1) *Recall that a functorial γ is called a Plotkin radical if it satisfies (F1), idempotent (i.e. $\gamma(\gamma(G)) = \gamma(G)$) and $N \subseteq \gamma(G)$ for every $\gamma(N) = N \trianglelefteq G$ [5, p. 28].*

(2) *A functorial that satisfies (F3) is often called hereditary (nevertheless, the same word means different in the theory of classes of groups).*

(3) *A functorial γ is a hereditary Plotkin radical if and only if it satisfies (F1), (F2), (F3). Let prove it. Assume that γ is a hereditary Plotkin radical. We need only to prove that it satisfies (F2). If $N \trianglelefteq G$, then $\gamma(N) \text{ char } N \trianglelefteq G$. So $\gamma(N) \trianglelefteq G$. Now $\gamma(N) = \gamma(\gamma(N)) \subseteq \gamma(G)$. Thus a hereditary Plotkin radical satisfies (F1), (F2), (F3). Assume that γ satisfies (F1), (F2), (F3). We need only to prove that it is idempotent. By (F3) we have $\gamma(G) = \gamma(G) \cap G \subseteq \gamma(\gamma(G)) \subseteq \gamma(G)$. Thus $\gamma(\gamma(G)) = \gamma(G)$.*

(4) *The functorial Φ which assigns to every group G its Frattini subgroup $\Phi(G)$ satisfies (F1) and (F2) but not (F3).*

(5) *If γ satisfies (F2) and (F3), then $\gamma(G) \cap N = \gamma(N)$ for every group G and $N \trianglelefteq G$.*

Lemma 2.1. *If γ satisfies (F1) and (F2), then $\gamma(G_1 \times G_2) = \gamma(G_1) \times \gamma(G_2)$ for any groups G_1 and G_2 .*

Proof. From $G_i \trianglelefteq G_1 \times G_2$ it follows that $\gamma(G_i) \subseteq \gamma(G_1 \times G_2)$ by (F2) for $i \in \{1, 2\}$. Note that $\gamma(G_1 \times G_2)G_i/G_i \subseteq \gamma((G_1 \times G_2)/G_i) = (\gamma(G_i) \times G_i)/G_i$ by (F1) for $i \in \{1, 2\}$. Now

$$\begin{aligned} \gamma(G_1 \times G_2) &\subseteq (\gamma(G_1 \times G_2)G_2) \cap (\gamma(G_1 \times G_2)G_1) \subseteq \\ &(\gamma(G_1) \times G_2) \cap (G_1 \times \gamma(G_2)) = \gamma(G_1) \times \gamma(G_2). \end{aligned}$$

Thus $\gamma(G_1 \times G_2) = \gamma(G_1) \times \gamma(G_2)$. □

Recall [15] that for functorials γ_1 and γ_2 the upper product $\gamma_2 \star \gamma_1$ is defined by

$$(\gamma_2 \star \gamma_1)(G)/\gamma_2(G) = \gamma_1(G/\gamma_2(G)).$$

Proposition 2.2. *Let γ_1 and γ_2 be functorials. If γ_1 and γ_2 satisfy (F1) and (F2), then $\gamma_2 \star \gamma_1$ satisfies (F1) and (F2). Moreover if γ_1 and γ_2 also satisfy (F3), then $\gamma_2 \star \gamma_1$ satisfies (F3).*

Proof. (1) $\gamma_2 \star \gamma_1$ satisfies (F1).

Let $f : G \rightarrow f(G)$ be an epimorphism. From $f(\gamma_2(G)) \subseteq \gamma_2(f(G))$ it follows that the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{f} & f(G) \\ \downarrow f_1 & \searrow f_4 & \downarrow f_3 \\ G/\gamma_2(G) & \xrightarrow{f_2} & f(G)/\gamma_2(f(G)) \end{array}$$

Let $X = \gamma_1(G/\gamma_2(G))$ and $Y = \gamma_1(f(G)/\gamma_2(f(G)))$. Note that $(\gamma_2 \star \gamma_1)(G) = f_1^{-1}(X)$ and $(\gamma_2 \star \gamma_1)(f(G)) = f_3^{-1}(Y)$ by the definition of $\gamma_2 \star \gamma_1$. Since γ_1 satisfies (F1), we see that $f_2(X) \subseteq Y$. Hence $X \subseteq f_2^{-1}(Y)$. Now $(\gamma_2 \star \gamma_1)(G) \subseteq f_1^{-1}(f_2^{-1}(Y)) = f_4^{-1}(Y)$. So

$$f((\gamma_2 \star \gamma_1)(G)) \subseteq f(f_4^{-1}(Y)) = f_3^{-1}(Y) = (\gamma_2 \star \gamma_1)(f(G)).$$

Thus $\gamma_2 \star \gamma_1$ satisfies (F1).

(2) $\gamma_2 \star \gamma_1$ satisfies (F2).

Let $N \trianglelefteq G$. From $\gamma_2(N) \text{ char } N \trianglelefteq G$ it follows that $\gamma_2(N) \trianglelefteq G$. Since γ_2 satisfies (F2), we see that $\gamma_2(N) \subseteq \gamma_2(G)$. So the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{f_1} & G/\gamma_2(N) \\ & \searrow f_3 & \downarrow f_2 \\ & & G/\gamma_2(G) \end{array}$$

Let $X = \gamma_1(G/\gamma_2(N))$, $Y = \gamma_1(N/\gamma_2(N))$ and $Z = \gamma_1(G/\gamma_2(G))$. Note that $(\gamma_2 \star \gamma_1)(G) = f_3^{-1}(Z)$ and $(\gamma_1 \star \gamma_2)(N) \subseteq f_1^{-1}(Y)$. Since γ_1 satisfies (F1) and (F2), we see that $f_2(X) \subseteq Z$ and $Y \subseteq X$. Now

$$(\gamma_2 \star \gamma_1)(N) \subseteq f_1^{-1}(Y) \subseteq f_1^{-1}(X) \subseteq f_1^{-1}(f_2^{-1}(Z)) = f_3^{-1}(Z) = (\gamma_2 \star \gamma_1)(G).$$

Hence $\gamma_2 \star \gamma_1$ satisfies (F2).

(3) If γ_1 and γ_2 also satisfy (F3), then $\gamma_2 \star \gamma_1$ satisfies (F3).

Assume that γ_1 and γ_2 satisfy (F2) and (F3). Let $N \trianglelefteq G$.

Since $N\gamma_2(G)/\gamma_2(G) \cap (\gamma_2 \star \gamma_1)(G)/\gamma_2(G) \trianglelefteq (\gamma_2 \star \gamma_1)(G)/\gamma_2(G) = \gamma_1(G/\gamma_2(G))$, we see by (5) of Remark 2.1 that

$$\gamma_1((N\gamma_2(G) \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(G)) = (N\gamma_2(G) \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(G).$$

Note that

$$\begin{aligned} (N\gamma_2(G) \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(G) &= \\ &= (N \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(G) \simeq (N \cap (\gamma_2 \star \gamma_1)(G))/(N \cap \gamma_2(G)) \\ &= (N \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(N) \trianglelefteq N/\gamma_2(N). \end{aligned}$$

It means that $(N \cap (\gamma_2 \star \gamma_1)(G))/\gamma_2(N) \subseteq \gamma_1(N/\gamma_2(N))$. Thus $N \cap (\gamma_2 \star \gamma_1)(G) \subseteq (\gamma_2 \star \gamma_1)(N)$, i.e $\gamma_2 \star \gamma_1$ satisfies (F3). \square

Here we introduce the height $h_\gamma(G)$ of a group G which corresponds to a given functorial γ .

Definition 2.1. *Let γ be a functorial. Then the γ -series of G is defined starting from $\gamma_{(0)}(G) = 1$, and then by induction $\gamma_{(i+1)}(G) = (\gamma_{(i)} \star \gamma)(G)$ is the inverse image of $\gamma(G/\gamma_{(i)}(G))$. The least number h such that $\gamma_{(h)}(G) = G$ is defined to be γ -height $h_\gamma(G)$ of G . If there is no such number, then $h_\gamma(G) = \infty$.*

The following Lemma directly follows from Proposition 2.2.

Lemma 2.3. *Let γ be a functorial. If γ satisfies (F1) and (F2), then $\gamma_{(n)}$ satisfies (F1) and (F2) for all natural n . Moreover if γ satisfies (F3), then $\gamma_{(n)}$ satisfies (F3) for all natural n .*

Lemma 2.4. *Let γ be a functorial. If γ satisfies (F1) and (F2), then $h_\gamma(G/N) \leq h_\gamma(G) \leq h_\gamma(N) + h_\gamma(G/N)$ for every $N \trianglelefteq G$. Moreover, if γ also satisfies (F3), then $h_\gamma(N) \leq h_\gamma(G)$.*

Proof. Note that $\gamma_{(n)}$ satisfies (F1) and (F2) for every n by Lemma 2.3.

Since $\gamma_{(n)}$ satisfies (F1), $G/N = \gamma_{h_\gamma(G)}(G)/N \leq \gamma_{(h_\gamma(G))}(G/N) \leq G/N$. So $\gamma_{(h_\gamma(G))}(G/N) = G/N$. Thus $h_\gamma(G/N) \leq h_\gamma(G)$.

Since $\gamma_{(n)}$ satisfies (F2), we see that $N = \gamma_{(h_\gamma(N))}(N) \subseteq \gamma_{(h_\gamma(N))}(G)$. Note that $h_\gamma(G/\gamma_{(h_\gamma(N))}(G)) \leq h_\gamma(G/N)$. Thus $h_\gamma(G) \leq h_\gamma(N) + h_\gamma(G/N)$.

Assume that γ also satisfies (F3). Then $\gamma_{(n)}$ satisfies (F3) by Lemma 2.3. Now $N = G \cap N = \gamma_{(h_\gamma(G))}(G) \cap N \subseteq \gamma_{(h_\gamma(G))}(N) \leq N$. So $\gamma_{(h_\gamma(G))}(N) = N$. Thus $h_\gamma(N) \leq h_\gamma(G)$. \square

If $\gamma = F^*$, then $h_\gamma(G) = h^*(G)$ for every group G . The non- p -soluble length can also be defined with the help of functorials. Here by $R_p(G)$ we denote the p -soluble radical of a group G .

Lemma 2.5. *Let $\overline{F}_p = R_p \star F^* \star R_p$ and G be a non- p -soluble group. Then $\lambda_p(G)$ is the smallest natural i with $\overline{F}_{p(i)}(G) = G$.*

Proof. Let $1 = G_0 \leq G_1 \leq \dots \leq G_{2h+1} = G$ be the shortest normal series in which for i odd the factor G_{i+1}/G_i is p -soluble (possibly trivial), and for i even the factor G_{i+1}/G_i is a (non-empty) direct product of nonabelian simple groups.

Note that $G_1 \leq R_p(G)$ and G_2/G_1 is quasinilpotent. Hence $G_2 R_p(G)/R_p(G)$ is quasinilpotent. It means that $G_2 R_p(G)/R_p(G) \leq F^*(G/R_p(G))$. Hence $G_2 \leq (R_p \star F^*)(G)$. Since G_3/G_2 is p -soluble, we see that $G_3(R_p \star F^*)(G)/(R_p \star F^*)(G)$ is p -soluble. Hence $G_3(R_p \star F^*)(G)/(R_p \star F^*)(G) \leq R_p(G/(R_p \star F^*)(G))$. It means that $G_3 \leq \overline{F}_p(G) = \overline{F}_{p(1)}(G)$.

Assume that we proved $G_{2i+1} \leq \overline{F}_{p(i)}(G)$. Let prove that $G_{2(i+1)+1} \leq \overline{F}_{p(i+1)}(G)$.

From $G_{2i+1} \leq \overline{F}_{p(i)}(G)$ it follows that $G_{2i+1} \leq (\overline{F}_{p(i)} \star R_p)(G)$. Note that G_{2i+2}/G_{2i+1} is quasinilpotent. It means that $G_{2i+2}(\overline{F}_{p(i)} \star R_p)(G)/(\overline{F}_{p(i)} \star R_p)(G)$ is quasinilpotent. Hence $G_{2i+2} \leq ((\overline{F}_{p(i)} \star R_p) \star F^*)(G)$. Since $G_{2(i+1)+1}/G_{2i+2}$ is p -soluble, we see that $G_{2(i+1)+1}(\overline{F}_{p(i)} \star R_p \star F^*)(G)/(\overline{F}_{p(i)} \star R_p \star F^*)(G)$ is p -soluble. Hence $G_{2(i+1)+1}(\overline{F}_{p(i)} \star R_p \star F^*)(G)/((\overline{F}_{p(i)} \star R_p \star F^*)(G) \leq R_p(G/(\overline{F}_{p(i)} \star R_p \star F^*)(G))$. It means that $G_{2(i+1)+1} \leq (\overline{F}_{p(i)} \star R_p \star F^* \star R_p)(G) = \overline{F}_{p(i+1)}(G)$.

Therefore $\lambda_p(G) \geq n$ where n is the smallest integer with $\overline{F}_{p(n)}(G) = G$. Since $R_p \star R_p = R_p$, we see that $\overline{F}_{p(n)}(G)$ presents a normal series $1 \leq F_1 \leq F_2 \leq \dots \leq F_{2n+1}$ in which for i odd the factor F_{i+1}/F_i is p -soluble (possibly trivial), and for i even the factor F_{i+1}/F_i is a (non-empty) direct product of nonabelian simple groups. So $\lambda_p(G) \leq n$. Thus $\lambda_p(G) = n$. \square

Now we are able to estimate the γ -height of the direct product subgroups and of the join of subnormal subgroups:

Theorem 2.6. *Let γ be a functorial with $\gamma(H) > 1$ for every group H that satisfies (F1) and (F2).*

(1) *If $G = \times_{i=1}^n A_i$ is the direct product of its normal subgroups A_i , then $h_\gamma(G) = \max\{h_\gamma(A_i) \mid 1 \leq i \leq n\}$.*

(2) *Let $G = \langle A_i \mid 1 \leq i \leq n \rangle$ be the join of its subnormal subgroups A_i . Then $h_\gamma(G) \leq \max\{h_\gamma(A_i) \mid 1 \leq i \leq n\}$. If γ satisfies (F3), then $h_\gamma(G) = \max\{h_\gamma(A_i) \mid 1 \leq i \leq n\}$.*

Proof. Note that $\gamma_{(n)}$ satisfies (F1) and (F2) for every n by Proposition 2.2.

(1) From Lemma 2.1 it follows that if $G = \times_{i=1}^n A_i$, then $\gamma_{(n)}(G) = \times_{i=1}^n \gamma_{(n)}(A_i)$. It means that $h_\gamma(G) = \max\{h_\gamma(A_i) \mid 1 \leq i \leq n\}$.

(2) Assume that $G = \langle A_i \mid 1 \leq i \leq n \rangle$ is the join of its subnormal subgroups A_i , $h_1 = \max\{h_\gamma(A_i) \mid 1 \leq i \leq n\}$ and $h_2 = h_\gamma(G)$. Since $\gamma_{(n)}$ satisfies (F2), we see that $\gamma_{(n)}(N) \subseteq \gamma_{(n)}(G)$ for every subnormal subgroup N of G and every n . Now

$$G = \langle A_i \mid 1 \leq i \leq n \rangle = \langle \gamma_{(h_1)}(A_i) \mid 1 \leq i \leq n \rangle \subseteq \gamma_{(h_1)}(G) \subseteq G.$$

Hence $\gamma_{(h_1)}(G) = G$. It means that $h_2 \leq h_1$.

Suppose that γ satisfies (F3). Now $\gamma_{(n)}$ satisfies (F3) for every n by Proposition 2.2. From (5) of Remark 2.1 it follows that $\gamma_{(n)}(G) \cap N = \gamma_{(n)}(N)$ for every subnormal subgroup N of G . Now $A_i = A_i \cap G = A_i \cap \gamma_{(h_2)}(G) = \gamma_{(h_2)}(A_i)$. It means that $h_\gamma(A_i) \leq h_2$ for every i . Hence $h_1 \leq h_2$. Thus $h_1 = h_2$. \square

Corollary 2.7. *Let a group $G = \langle A_i \mid 1 \leq i \leq n \rangle$ be the join of its subnormal subgroups A_i . Then $h^*(G) = \max\{h^*(A_i) \mid 1 \leq i \leq n\}$ and $\lambda_p(G) = \max\{\lambda_p(A_i) \mid 1 \leq i \leq n\}$.*

3 The Classes of Groups Method

Recall that a *formation* is a class \mathfrak{F} of groups with the following properties: (a) every homomorphic image of an \mathfrak{F} -group is an \mathfrak{F} -group, and (b) if G/M and G/N are \mathfrak{F} -groups, then also $G/(M \cap N) \in \mathfrak{F}$. Recall that the \mathfrak{F} -*residual* of a group G is the smallest normal subgroup $G^\mathfrak{F}$ of G with $G/G^\mathfrak{F} \in \mathfrak{F}$.

A formation is called *Fitting* if (a) from $N \trianglelefteq G \in \mathfrak{F}$ it follows that $N \in \mathfrak{F}$ and (b) a group $G \in \mathfrak{F}$ whenever it is a product of normal \mathfrak{F} -subgroups. Recall that the \mathfrak{F} -*radical* $G_\mathfrak{F}$ of a group G is the greatest normal \mathfrak{F} -subgroup.

The classes \mathfrak{N}^* of all quasinilpotent groups and \mathfrak{S}^p of all p -soluble groups are Fitting formations.

From [3, IX, Remarks 1.11 and Theorem 1.12] and [3, IV, Theorem 1.8] follows

Lemma 3.1. *Let \mathfrak{F} and \mathfrak{H} be non-empty Fitting formations. Then*

$$\mathfrak{F}\mathfrak{H} = (G \mid G^\mathfrak{F} \in \mathfrak{H}) = (G \mid G/G_\mathfrak{H} \in \mathfrak{F})$$

is a Fitting formation.

Corollary 3.2. *The class $\mathfrak{H}_p = (G \mid \overline{F}_p(G) = G)$ is a Fitting formation.*

It is straightforward to check that for a Fitting formation \mathfrak{F} , the \mathfrak{F} -radical can be considered as a functorial γ which satisfies (F1), (F2) and (F3). For convenience in this case denote h_γ by $h_\mathfrak{F}$. Now $h^*(G) = h_{\mathfrak{F}^*}(G) = h_{\mathfrak{N}^*}(G)$ and for a non- p -soluble group $\lambda_p(G) = h_{\overline{F}_p}(G) = h_{\mathfrak{S}_p}(G)$.

Lemma 3.3. *Let \mathfrak{F} be a Fitting formation. If $H \neq 1$ and $h_\mathfrak{F}(H) < \infty$, then $h_\mathfrak{F}(H^\mathfrak{F}) = h_\mathfrak{F}(H) - 1$.*

Proof. Let prove that if $H \neq 1$ and $h_\mathfrak{F}(H) < \infty$, then $h_\mathfrak{F}(H^\mathfrak{F}) = h_\mathfrak{F}(H) - 1$. Let $h_\mathfrak{F}(H) = n$ and $h_\mathfrak{F}(H^\mathfrak{F}) = k$. Then $H_{\mathfrak{F}(n-1)}(H) < H$ and $H/H_{\mathfrak{F}(n-1)} \in \mathfrak{F}$. It means that $H^\mathfrak{F} \leq H_{\mathfrak{F}(n-1)}$. Since $H_{\mathfrak{F}(n-1)}$ satisfies (F3), we see that $(H^\mathfrak{F})_{\mathfrak{F}(n-1)} = H^\mathfrak{F}$. Hence $k \leq n - 1$.

Note that $H^\mathfrak{F} = (H^\mathfrak{F})_{\mathfrak{F}(k)} \leq H_{\mathfrak{F}(k)}$. It means that $H/H_{\mathfrak{F}(k)} \in \mathfrak{F}$. Hence $k \geq n - 1$. Thus $k = n - 1$. \square

If $\mathfrak{F}, \mathfrak{H}, \mathfrak{K} \neq \emptyset$ are formations, then $(\mathfrak{F}\mathfrak{H})\mathfrak{K} = \mathfrak{F}(\mathfrak{H}\mathfrak{K})$ by [3, IV, Theorem 1.8]. That is why the class $\mathfrak{F}^n = \underbrace{\mathfrak{F} \dots \mathfrak{F}}_n$ is a well defined formation.

Lemma 3.4. *For a natural number n and a Fitting formation \mathfrak{F} holds $\mathfrak{F}^n = (G \mid h_{\mathfrak{F}}(G) \leq n)$.*

Proof. From Lemma 3.3 it follows that if $G \in (G \mid h_{\mathfrak{F}}(G) \leq n)$, then $G^{\mathfrak{F}^n} = 1$. It means that $(G \mid h_{\mathfrak{F}}(G) \leq n) \subseteq \mathfrak{F}^n$. Assume that there is a group $G \in \mathfrak{F}^n$ with $h_{\mathfrak{F}}(G) > n$. Note that $\overline{G^{\mathfrak{F}}} \neq \overline{G}$ for every quotient group $\overline{G} \neq 1$ of G . Then $h_{\mathfrak{F}}(G^{\mathfrak{F}^n}) > 0$ by Lemma 3.3. It means that $G^{\mathfrak{F}^n} \neq 1$, a contradiction. Therefore $\mathfrak{F}^n \subseteq (G \mid h_{\mathfrak{F}}(G) \leq n)$. Thus $\mathfrak{F}^n = (G \mid h_{\mathfrak{F}}(G) \leq n)$. \square

In the next lemma we recall the key properties of mutually permutable products

Lemma 3.5. *Let a group $G = AB$ be a mutually permutable product of subgroups A and B . Then*

(1) [1, Lemma 4.1.10] $G/N = (AN/N)(BN/N)$ is a mutually permutable product of subgroups AN/N and BN/N for every normal subgroup N of G .

(2) [1, Lemma 4.3.3(4)] If N is a minimal normal subgroup of a group G , then $\{N \cap A, N \cap B\} \subseteq \{1, N\}$.

(3) [1, Lemma 4.3.3(5)] If N is a minimal normal subgroup of G contained in A and $B \cap N = 1$, then $N \leq C_G(A)$ or $N \leq C_G(B)$. If furthermore N is not cyclic, then $N \leq C_G(B)$.

(4) [1, Theorem 4.3.11] $A_G B_G \neq 1$.

(5) [1, Corollary 4.1.26] A' and B' are subnormal in G .

Recall that $\pi(G)$ is the set of all prime divisors of $|G|$, $\pi(\mathfrak{F}) = \bigcup_{G \in \mathfrak{F}} \pi(G)$ and \mathfrak{N}_{π} denote the class of all nilpotent π -groups.

Lemma 3.6. *Let \mathfrak{F} be a Fitting formation. Assume that $h_{\mathfrak{F}}(G) \leq h + 1$ for every mutually permutable product G of two \mathfrak{F} -subgroups. Then*

$$\max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} - 1 \leq h_{\mathfrak{F}}(G) \leq \max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} + h$$

for every mutually permutable product G of two subgroups A and B with $h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B) < \infty$.

Proof. If $A = 1$ or $B = 1$, then there is nothing to prove. Assume that $A, B \neq 1$. Let a group $G = AB$ be the product of mutually permutable subgroups A and B . From $h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B) < \infty$ it follows that $\pi(G) \subseteq \pi(\mathfrak{F})$. According to [3, IX, Lemma 1.8] $\mathfrak{N}_{\pi(\mathfrak{F})} \subseteq \mathfrak{F}$. Note that A' and B' are subnormal in G by (5) of Lemma 3.5. Since $H^{\mathfrak{F}} \trianglelefteq H^{\mathfrak{N}_{\pi(\mathfrak{F})}} \trianglelefteq H'$ holds for every $\pi(\mathfrak{F})$ -group H , subgroups $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal in G . Let $C = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle^G = \langle \{(A^{\mathfrak{F}})^x \mid x \in G\} \cup \{(B^{\mathfrak{F}})^x \mid x \in G\} \rangle$. Then by (2) of Theorem 2.6 and by Lemma 3.3

$$\begin{aligned} h_{\mathfrak{F}}(C) &= \max \{ \{(h_{\mathfrak{F}}(A^{\mathfrak{F}})^x) \mid x \in G\} \cup \{(h_{\mathfrak{F}}(B^{\mathfrak{F}})^x) \mid x \in G\} \} \\ &= \max\{h_{\mathfrak{F}}(A^{\mathfrak{F}}), h_{\mathfrak{F}}(B^{\mathfrak{F}})\} = \max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} - 1. \end{aligned}$$

Now $G/C = (AC/C)(BC/C)$ is a mutually permutable product of \mathfrak{F} -subgroups AC/C and BC/C by (1) of Lemma 3.5. It means that $h_{\mathfrak{F}}(G/C) \leq h + 1$ by our assumption. With the help of Lemma 2.4 we see that

$$h_{\mathfrak{F}}(G) \leq h_{\mathfrak{F}}(C) + h_{\mathfrak{F}}(G/C) \leq \max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} - 1 + 1 + h = \max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} + h.$$

From the other hand, $h_{\mathfrak{F}}(G) \geq h_{\mathfrak{F}}(C) = \max\{h_{\mathfrak{F}}(A), h_{\mathfrak{F}}(B)\} - 1$ by (2) of Theorem 2.6. \square

Lemma 3.7. *Let \mathfrak{F} be a Fitting formation. Assume that a group G is the least order group with*

- (1) G is a mutually permutable product of two subgroups A and B with $h_{\mathfrak{F}}(A) \geq h_{\mathfrak{F}}(B)$;
- (2) $h_{\mathfrak{F}}(G) = h_{\mathfrak{F}}(A) - 1$.

Then G has the unique minimal normal subgroup N , $N \leq A$ and $h_{\mathfrak{F}}(A/N) = h_{\mathfrak{F}}(A) - 1$.

Proof. Let N be a minimal normal subgroup of G . Then $N \cap A \in \{N, 1\}$ by (2) of Lemma 3.5.

Assume that $N \cap A = 1$. Now $G/N = (AN/N)(BN/N)$ is a mutually permutable product of groups AN/N and BN/N by (1) of Lemma 3.5. By our assumption and $h_{\mathfrak{F}}(G) \geq h_{\mathfrak{F}}(G/N) \geq h_{\mathfrak{F}}(AN/N) = h_{\mathfrak{F}}(A)$, a contradiction. Hence $N \cap A = N$ for every minimal normal subgroup N of G .

Now $h_{\mathfrak{F}}(G) + 1 = h_{\mathfrak{F}}(A) > h_{\mathfrak{F}}(G) \geq h_{\mathfrak{F}}(G/N) \geq h_{\mathfrak{F}}(A/N) \geq h_{\mathfrak{F}}(A) - 1$. It means that $h_{\mathfrak{F}}(G) = h_{\mathfrak{F}}(A/N) = h_{\mathfrak{F}}(A) - 1$.

If G has two minimal normal subgroups N_1 and N_2 , then $h_{\mathfrak{F}}(A/N_1) = h_{\mathfrak{F}}(A/N_2) = h_{\mathfrak{F}}(A) - 1$. It means $h_{\mathfrak{F}}(A) < h_{\mathfrak{F}}(A) - 1$ by Lemma 3.4, a contradiction. Hence G has a unique minimal normal subgroup N . \square

4 Proof of Theorem 1.1(1)

Our proof relies on the notion of the \mathfrak{X} -hypercenter. A chief factor H/K of G is called \mathfrak{X} -central in G provided

$$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{X}$$

(see [18, p. 127–128] or [7, 1, Definition 2.2]). A normal subgroup N of G is said to be \mathfrak{X} -hypercentral in G if $N = 1$ or $N \neq 1$ and every chief factor of G below N is \mathfrak{X} -central. The symbol $Z_{\mathfrak{X}}(G)$ denotes the \mathfrak{X} -hypercenter of G , that is, the product of all normal \mathfrak{X} -hypercentral in G subgroups. According to [18, Lemma 14.1] or [7, 1, Theorem 2.6] $Z_{\mathfrak{X}}(G)$ is the largest normal \mathfrak{X} -hypercentral subgroup of G . If $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, then $Z_{\mathfrak{N}}(G) = Z_{\infty}(G)$ is the hypercenter of G .

Lemma 4.1. *Let n be a natural number. Then $(\mathfrak{N}^*)^n = (G \mid h^*(G) \leq n) = (G \mid G = Z_{(\mathfrak{N}^*)^n}(G))$.*

Proof. First part follows from Lemma 3.4. It is well known that the class of all quasinilpotent groups is a composition (or Baer-local, or solubly saturated) formation (see [2, Example 2.2.17]). According to [18, Theorem 7.9] $(\mathfrak{N}^*)^n$ is a composition formation. Now $(\mathfrak{N}^*)^n = (G \mid G = Z_{(\mathfrak{N}^*)^n}(G))$ by [7, 1, Theorem 2.6]. \square

For a normal section H/K of G the subgroup $C_G^*(H/K) = HC_G(H/K)$ is called an *inneriser* (see [2, Definition 1.2.2]). It is the set of all elements of G that induce inner automorphisms on H/K . From the definition of the generalized Fitting subgroup it follows that it is the intersection of innerisers of all chief factors.

Lemma 4.2. *Let N be a normal subgroup of a group G . If N is a direct product of isomorphic simple groups and $h^*(G/C_G^*(N)) \leq k - 1$, then $F_{(k)}^*(G/N) = F_{(k)}^*(G)/N$.*

Proof. Assume that $h^*(G/C_G^*(N)) \leq k - 1$. Let $F/N = F_{(k)}^*(G/N)$. Then $F_{(k)}^*(G) \subseteq F$. Now $F/C_F^*(N) \simeq FC_G^*(N)/C_G^*(N) \trianglelefteq G/C_G^*(N)$. Therefore $h^*(F/C_F^*(N)) \leq k - 1$. It means that $h^*(F/C_F^*(H/K)) \leq k - 1$ for every chief factor H/K of F below N . Hence $(H/K) \rtimes (F/C_F^*(H/K)) \in (\mathfrak{N}^*)^k$ for every chief factor H/K of F below N . It means that $N \leq Z_{(\mathfrak{N}^*)^k}(F)$. Thus $F \in (\mathfrak{N}^*)^k$ by Lemma 4.1. So $F \subseteq F_{(k)}^*(G)$. Thus $F_{(k)}^*(G) = F$. \square

Lemma 4.3. *If a group $G = AB$ is a product of mutually permutable quasinilpotent subgroups A and B , then $h^*(G) \leq 2$.*

Proof. To prove this lemma we need only to prove that if a group $G = AB$ is a product of mutually permutable quasinilpotent subgroups A and B , then $G \in (\mathfrak{N}^*)^2$ by Lemma 4.1. Assume the contrary. Let G be a minimal order counterexample.

(1) G has a unique minimal normal subgroup N and $G/N \in (\mathfrak{N}^*)^2$.

Note that G/N is a mutually permutable product of quasinilpotent subgroups (AN/N) and (BN/N) by (1) of Lemma 3.5. Hence $G/N \in (\mathfrak{N}^*)^2$ by our assumption. Since $(\mathfrak{N}^*)^2$ is a formation, we see that G has a unique minimal normal subgroup. According to (4) of Lemma 3.5 $A_G B_G \neq 1$. WLOG we may assume that G has a minimal normal subgroup $N \leq A$.

(2) $N \leq A \cap B$.

Suppose that $N \cap B = 1$. Then $A \leq C_G(N)$ or $B \leq C_G(N)$ by (3) of Lemma 3.5. If $A \leq C_G(N)$, then $N \rtimes G/C_G(N) \simeq N \rtimes B/C_B(N) \in (\mathfrak{N}^*)^2$. If $B \leq C_G(N)$, then $N \rtimes G/C_G(N) \simeq N \rtimes A/C_A(N) \in (\mathfrak{N}^*) \subseteq (\mathfrak{N}^*)^2$ by [2, Corollary 2.2.5]. In both cases $N \leq Z_{(\mathfrak{N}^*)^2}(G)$. It means that $G \in (\mathfrak{N}^*)^2$, a contradiction. Now $N \cap B \neq 1$. Hence $N \leq A \cap B$ by (2) of Lemma 3.5.

(3) N is non-abelian.

Assume that N is abelian. Since A is quasinilpotent, we see that $A/C_A(N)$ is a p -group. By analogy $B/C_B(N)$ is a p -group. Note that $A/C_A(N) \simeq AC_G(N)/C_G(N)$ and $B/C_B(N) \simeq BC_G(N)/C_G(N)$. From $G = AB$ it follows that $G/C_G(N)$ is a p -group. Since N is a chief factor of G , we see that $G/C_G(N) \simeq 1$. So $N \leq Z_\infty(G) \leq Z_{(\mathfrak{N}^*)^2}(G)$. Thus $G \in (\mathfrak{N}^*)^2$, a contradiction. It means that N is non-abelian.

(4) *The final contradiction.*

Now N is a direct product of minimal normal subgroups of A . Since A is quasinilpotent, we see that every element of A induces an inner automorphism on every minimal normal subgroup of A . Hence every element of A induces an inner automorphism on N . By analogy every element of B induces an inner automorphism on N . From $G = AB$ it follows that every element of G induces an inner automorphism on N . So $NC_G(N) = G$ or $G/C_G(N) \simeq N$. Now $N \rtimes (G/C_G(N)) \in (\mathfrak{N}^*)^2$. It means that $N \leq Z_{(\mathfrak{N}^*)^2}(G)$. Thus $G \in (\mathfrak{N}^*)^2$ and $h^*(G) \leq 2$, the final contradiction. \square

Proof of Theorem 1.1(1). Let a group G be a mutually permutable product of subgroups A and B . From Theorem 2.6 and Lemma 4.3 it follows that

$$\max\{h^*(A), h^*(B)\} - 1 \leq h^*(G) \leq \max\{h^*(A), h^*(B)\} + 1.$$

Assume that $\max\{h^*(A), h^*(B)\} - 1 = h^*(G)$. WLOG let $h^*(A) = h^*(G) - 1$. We may assume that a group G is the least order group with such properties. Then G has the unique minimal normal subgroup N , $N \leq A$ and $h^*(A/N) = h^*(A) - 1$ by Lemma 3.7.

Assume that $h^*(A/C_A^*(N)) < h^*(A) - 1$. Then

$$F_{(h^*(A)-1)}^*(A/N) = F_{(h^*(A)-1)}^*(A)/N < A/N$$

by Lemma 4.2. It means that $h^*(A) = h^*(A/N)$, a contradiction. Hence $h^*(A/C_A^*(N)) = h^*(A) - 1$.

Since $G/C_G^*(N) = (AC_G^*(N)/C_G^*(N))(BC_G^*(N)/C_G^*(N))$ is a mutually permutable products of subgroups $AC_G^*(N)/C_G^*(N)$ and $BC_G^*(N)/C_G^*(N)$ by (1) of Lemma 3.5 and $A/C_A^*(N) \simeq AC_G^*(N)/C_A^*(N)$, we see that $h^*(G/C_G^*(N)) \geq h^*(A/C_A^*(N)) = h^*(A) - 1$ by our assumptions. Note that $F^*(G) \leq C_G^*(N)$. Now $h^*(G) - 1 = h^*(G/F^*(G)) \geq h^*(G/C_G^*(N)) \geq h^*(A/C_A^*(N)) = h^*(A) - 1$. It means that $h^*(G) \geq h^*(A)$, the final contradiction. \square

5 Proof of Theorem 1.1(2)

Lemma 5.1. *Let p be a prime and $\mathfrak{H} = \mathfrak{H}_p$. If a group $G = AB$ is a product of mutually permutable \mathfrak{H} -subgroups A and B , then $G \in \mathfrak{H}$.*

Proof. Assume the contrary. Let G be a minimal order counterexample.

(1) G has a unique minimal normal subgroup N , $G/N \in \mathfrak{H}$ and N is not p -soluble.

Note that G/N is a mutually permutable product of \mathfrak{H} -subgroups (AN/N) and (BN/N) by (1) of Lemma 3.5. Hence $G/N \in \mathfrak{H}$ by our assumption. Since \mathfrak{H} is a formation, we see that G

has a unique minimal normal subgroup. According to (4) of Lemma 3.5 $A_G B_G \neq 1$. WLOG we may assume that G has a minimal normal subgroup $N \leq A$.

If N is p -soluble, then $\overline{F}_p(G)/N = \overline{F}_p(G/N) = G$, i.e. So $\overline{F}_p(G) = G$. Thus $G \in \mathfrak{H}$, a contradiction.

(2) $N \leq A \cap B$.

Suppose that $N \cap B = 1$. Note that N is not cyclic by (1). Then $B \leq C_G(N)$ by (3) of Lemma 3.5. Hence $N \rtimes G/C_G(N) \simeq N \rtimes A/C_A(N) \in \mathfrak{H}$ by [2, Corollary 2.2.5]. It means that $N \leq Z_{\mathfrak{H}}(G)$. Therefore $G \in \mathfrak{H}$, a contradiction. Now $N \cap B \neq 1$. Hence $N \leq A \cap B$ by (2) of Lemma 3.5.

(4) *The final contradiction.*

Since N is the unique minimal normal subgroup of G and non-abelian, we see that $C_G(N) = 1$. So $C_A(N) = C_B(N) = 1$. Hence $R_p(A) = R_p(B) = 1$. In particular $F(A) = F(B) = 1$. Note that all minimal normal subgroups of A are in N . For B is the same situation. Thus $N = F^*(A) = F^*(B)$. So G/N is a mutually permutable product of p -soluble groups. Since the class of all p -soluble groups is closed by extensions by p -soluble groups, G/N is p -soluble by (1) and (4) of Lemma 3.5. From $N \leq F^*(G)$ it follows that $G \in \mathfrak{H}$, the contradiction. \square

Proof of Theorem 1.1(2). Let $\mathfrak{H} = \mathfrak{H}_p$ and a group G be a mutually permutable product of subgroups A and B . First we are going to prove that $\max\{h_{\mathfrak{H}}(A), h_{\mathfrak{H}}(B)\} = h_{\mathfrak{H}}(G)$.

By Lemmas 3.6 and 4.3 we have

$$\max\{h_{\mathfrak{H}}(A), h_{\mathfrak{H}}(B)\} - 1 \leq h_{\mathfrak{H}}(G) \leq \max\{h_{\mathfrak{H}}(A), h_{\mathfrak{H}}(B)\}.$$

Assume that $\max\{h_{\mathfrak{H}}(A), h_{\mathfrak{H}}(B)\} - 1 = h_{\mathfrak{H}}(G)$ for some mutually permutable product G of A and B . Assume that G is a minimal order group with this property. WLOG let $h_{\mathfrak{H}}(A) = h_{\mathfrak{H}}(G) - 1$. Then G has the unique minimal normal subgroup N , $N \leq A$ and $h_{\mathfrak{H}}(A/N) = h_{\mathfrak{H}}(A) - 1$ by Lemma 3.7.

If N is p -soluble, then $R_p(A/N) = R_p(A)/N$. It means that $\overline{F}_p(A/N) = \overline{F}_p(A)/N$. Thus $h_{\mathfrak{H}}(A/N) = h_{\mathfrak{H}}(A)$, a contradiction.

It means that $R_p(G) = 1$. Note that now N is a simple non-abelian group. Since N is a unique minimal normal subgroup of G , we see that $N = F^*(G)$. Now $h_{\mathfrak{H}}(G/N) = h_{\mathfrak{H}}(G) - 1$. Therefore

$$h_{\mathfrak{H}}(G) - 1 = h_{\mathfrak{H}}(G/N) \geq h_{\mathfrak{H}}(A/N) = h_{\mathfrak{H}}(A) - 1.$$

Thus $h_{\mathfrak{H}}(G) \geq h_{\mathfrak{H}}(A)$, the contradiction.

We proved that $\max\{h_{\mathfrak{H}}(A), h_{\mathfrak{H}}(B)\} = h_{\mathfrak{H}}(G)$.

Let G be a mutually permutable product of groups A and B . If A, B are p -soluble, then G is p -soluble by (1) and (4) of Lemma 3.5. Hence $\lambda_p(G) = \lambda_p(A) = \lambda_p(B) = 0$. Now assume that at least one of subgroups A, B is not p -soluble. Then G is not p -soluble by (1) and (4) of Lemma 3.5. WLOG let $h_{\mathfrak{H}}(A) \geq h_{\mathfrak{H}}(B)$. Hence A is not p -soluble. We proved that $h_{\mathfrak{H}}(A) = h_{\mathfrak{H}}(G)$. Note that $h_{\mathfrak{H}}(G) = \lambda_p(G)$, $h_{\mathfrak{H}}(A) = \lambda_p(A)$, $h_{\mathfrak{H}}(B) = \lambda_p(B)$ if B is not p -soluble by Lemma 2.5 and $0 = \lambda_p(B) < 1 = h_{\mathfrak{H}}(B) \leq h_{\mathfrak{H}}(A) = \lambda_p(A)$ otherwise. Thus $\max\{\lambda_p(A), \lambda_p(B)\} = \lambda_p(G)$. \square

6 Non-Frattini length

The Frattini subgroup $\Phi(G)$ play an important role in the theory of classes of groups. One of the useful properties of the Fitting subgroup of a soluble group is that it is strictly greater than the Frattini subgroup of the same group. Note that the generalized Fitting subgroup is non-trivial in every group but there are groups in which it coincides with the Frattini subgroup. That is why the following length seems interesting.

Definition 6.1. Let $1 = G_0 \leq G_1 \leq \dots \leq G_{2h} = G$ be a shortest normal series in which for i even $G_{i+1}/G_i \leq \Phi(G/G_i)$, and for i odd the factor G_{i+1}/G_i is a (non-empty) direct product of simple groups. Then $h = \tilde{h}(G)$ will be called the non-Frattini length of a group G .

Note that if G is a soluble group, then $\tilde{h}(G) = h(G)$. Another reason that leads us to this length is the generalization of the Fitting subgroup $\tilde{F}(G)$ introduced by P. Schmid [16] and L.A. Shemetkov [17, Definition 7.5] and defined by

$$\Phi(G) \subseteq \tilde{F}(G) \text{ and } \tilde{F}(G)/\Phi(G) = \text{Soc}(G/\Phi(G)).$$

P. Förster [4] showed that $\tilde{F}(G)$ can be defined by

$$\Phi(G) \subseteq \tilde{F}(G) \text{ and } \tilde{F}(G)/\Phi(G) = F^*(G/\Phi(G)).$$

Let Φ and \tilde{F} be functorials that assign $\Phi(G)$ and $\tilde{F}(G)$ to every group G . Then $\tilde{F} = \Phi \star F^*$. It is well known that Φ satisfies (F1) and (F2). Hence \tilde{F} satisfies (F1) and (F2) by Proposition 2.2.

Note that $\Phi(G/\Phi(G)) \simeq 1$. By analogy with the proof of Lemma 2.5 one can show that the non-Frattini length $\tilde{h}(G)$ of a group G and $h_{\tilde{F}}(G)$ coincide for every group G . The following theorem shows connections between the non-Frattini length and the generalized Fitting height.

Theorem 6.1. For any group G holds $\tilde{h}(G) \leq h^*(G) \leq 2\tilde{h}(G)$. There exists a group H with $\tilde{h}(H) = n$ and $h^*(H) = 2n$ for any natural n .

Proof. Since $\Phi(G)$ and $\text{Soc}(G/\Phi(G))$ are quasinilpotent, we see that $F^*(G) \leq \tilde{F}(G) \leq F_{(2)}^*(G)$. Now $F_{(n)}^*(G) \leq \tilde{F}_{(n)}(G) \leq F_{(2n)}^*(G)$. Hence if $\tilde{F}_{(n)}(G) = G$, then $F_{(n)}^*(G) \leq G$ and $F_{(2n)}^*(G) = G$. It means $\tilde{h}(G) \leq h^*(G) \leq 2\tilde{h}(G)$.

Let K be a group, K_1 be isomorphic to the regular wreath product of \mathbb{A}_5 and K . Note that the base B of it is the unique minimal normal subgroup of K_1 and non-abelian. According to [6], there is a Frattini $\mathbb{F}_3 K_1$ -module A which is faithful for K_1 and a Frattini extension $A \twoheadrightarrow K_2 \twoheadrightarrow K_1$ such that $A \overset{K_1}{\simeq} \Phi(K_2)$ and $K_2/\Phi(K_2) \simeq K_1$.

Let denote K_2 by $\mathbf{f}(K)$. Now $\mathbf{f}(K)/\tilde{F}(\mathbf{f}(K)) \simeq K$. From the definition of $h_{\tilde{F}} = \tilde{h}$ it follows that $\tilde{h}(\mathbf{f}(K)) = \tilde{h}(K) + 1$.

Note that $\Phi(\mathbf{f}(K)) \subseteq F^*(\mathbf{f}(K))$. Assume that $\Phi(\mathbf{f}(K)) \neq F^*(\mathbf{f}(K))$. It means that $F^*(\mathbf{f}(K)) = \tilde{F}(\mathbf{f}(K))$ is quasinilpotent. By [9, X, Theorem 13.8] it follows that $\Phi(\mathbf{f}(K)) \subseteq Z(F^*(\mathbf{f}(K)))$. It means that $1 < B \leq C_{K_1}(A)$. Thus A is not faithful, a contradiction.

Thus $\Phi(\mathbf{f}(K)) = F^*(\mathbf{f}(K))$ and $\mathbf{f}(K)/F^*(\mathbf{f}(K)) \simeq K_1$. Since K_1 has a unique minimal normal subgroup B and it is non-abelian, we see that $F^*(K_1) = B$. It means that $\mathbf{f}(K)/F_{(2)}^*(\mathbf{f}(K)) \simeq K$. From the definition of h^* it follows that $h^*(\mathbf{f}(K)) = h^*(K) + 2$.

As usual, let $\mathbf{f}^{(1)}(K) = \mathbf{f}(K)$ and $\mathbf{f}^{(i+1)}(K) = \mathbf{f}(\mathbf{f}^{(i)}(K))$. Then $\tilde{h}(\mathbf{f}^{(n)}(1)) = n$ and $h^*(\mathbf{f}^{(n)}(1)) = 2n$ for any natural n . \square

The following proposition directly follows from Theorem 2.6.

Proposition 6.2. Let a group $G = \langle A_i \mid 1 \leq i \leq n \rangle$ be the join of its subnormal subgroups A_i . Then $\tilde{h}(G) \leq \max\{\tilde{h}(A_i) \mid 1 \leq i \leq n\}$.

One of the main differences between the non-Frattini length and the generalized Fitting height is that the non-Frattini length of a normal subgroup can be greater than the non-Frattini length of a group.

Example 6.1. Let $E \simeq \mathbb{A}_5$. There is an $\mathbb{F}_5 E$ -module V such that $R = \text{Rad}(V)$ is a faithful irreducible $\mathbb{F}_5 E$ -module and V/R is an irreducible trivial $\mathbb{F}_5 E$ -module (how to construct such module, for example, see [14]). Let $G = V \rtimes E$. Now $\Phi(G) = R$ by [3, B, Lemma 3.14]. Note

that $G/\Phi(G) = G/R \simeq Z_5 \times E$. So $\tilde{F}(G) = G$ and $\tilde{h}(G) = 1$. Note that $G = V(RE)$ where V and RE are normal subgroups of G . Since V is abelian, we see that $\tilde{h}(V) = 1$. Note that R is a unique minimal normal subgroup of RE and $\Phi(RE) = 1$. It means that $\tilde{F}(RE) = R$ and $\tilde{h}(RE) = 2$. Thus $\tilde{h}(G) < \max\{\tilde{h}(V), \tilde{h}(RE)\}$ and \tilde{F} does not satisfy (F3).

Recall [1, Definition 4.1.1] that a group G is called a totally permutable product of its subgroups A and B if $G = AB$ and every subgroup of A permutes with every subgroup of B .

Theorem 6.3. *Let a group $G = AB$ be a totally permutable product of subgroups A and B . Then*

$$\max\{\tilde{h}(A), \tilde{h}(B)\} - 1 \leq \tilde{h}(G) \leq \max\{\tilde{h}(A), \tilde{h}(B)\} + 1.$$

Proof. If $A = 1$ or $B = 1$, then $\max\{\tilde{h}(A), \tilde{h}(B)\} = \tilde{h}(G)$. Assume that $A, B \neq 1$.

According to [1, Proposition 4.1.16] $A \cap B \leq F(G)$. Hence $A \cap B \leq F^*(G)$. Now $\overline{G} = G/F^*(G)$ is a totally permutable product of $\overline{A} = AF^*(G)/F^*(G)$ and $\overline{B} = BF^*(G)/F^*(G)$ by [1, Corollary 4.1.11]. Note that $\overline{A} \cap \overline{B} \simeq 1$. According to [1, Lemma 4.2.2] $[\overline{A}, \overline{B}] \leq F(\overline{G})$. So $[\overline{A}, \overline{B}] \leq F^*(\overline{G})$. It means that

$$\overline{G}/F^*(\overline{G}) = (\overline{A}F^*(\overline{G})/F^*(\overline{G})) \times (\overline{B}F^*(\overline{G})/F^*(\overline{G})).$$

Note that for the formation \mathfrak{U} of all supersoluble groups we have $\mathfrak{U} \subset \mathfrak{N}^2 \subset (\mathfrak{N}^*)^2$. Hence if $H = H_1H_2$ is a product of totally permutable $(\mathfrak{N}^*)^2$ -subgroups H_1 and H_2 , then $H \in (\mathfrak{N}^*)^2$ by [1, Theorem 5.2.1]. Analyzing the proof of [1, Theorem 5.2.2] we see that this theorem is true not only for saturated formation, but for formations $\mathfrak{F} = (G \mid G = Z_{\mathfrak{F}}(G))$. In particular, it is true for $(\mathfrak{N}^*)^2$. Thus if $H = H_1H_2 \in (\mathfrak{N}^*)^2$ is a product of totally permutable subgroups H_1 and H_2 , then $H_1, H_2 \in (\mathfrak{N}^*)^2$. Now $(\mathfrak{N}^*)^2$ satisfies conditions of [1, Proposition 5.3.9].

Therefore $A \cap F_{(2)}^*(G) = F_{(2)}^*(A)$ and $B \cap F_{(2)}^*(G) = F_{(2)}^*(B)$. Note that

$$\overline{A}F^*(\overline{G})/F^*(\overline{G}) \simeq AF_{(2)}^*(G)/F_{(2)}^*(G) \simeq A/F_{(2)}^*(A).$$

By analogy $\overline{B}F^*(\overline{G})/F^*(\overline{G}) \simeq B/F_{(2)}^*(B)$. Hence

$$G/F_{(2)}^*(G) \simeq (A/F_{(2)}^*(A)) \times (B/F_{(2)}^*(B)).$$

By Theorem 2.6 and $\tilde{h} = h_{\tilde{F}}$ we have $\tilde{h}(G/F_{(2)}^*(G)) = \max\{\tilde{h}(A/F_{(2)}^*(A)), \tilde{h}(B/F_{(2)}^*(B))\}$.

From $\tilde{F}(H) \leq F_{(2)}^*(H) \leq \tilde{F}_{(2)}(H)$ and Lemma 2.4 it follows that for any group $H \neq 1$ holds

$$\tilde{h}(H) - 1 = \tilde{h}(H/\tilde{F}(H)) \geq \tilde{h}(H/F_{(2)}^*(H)) \geq \tilde{h}(H/\tilde{F}_{(2)}(H)) \geq \tilde{h}(H) - 2.$$

Therefore

$$\{\tilde{h}(G) - \tilde{h}(G/F_{(2)}^*(G)), \tilde{h}(A) - \tilde{h}(A/F_{(2)}^*(A)), \tilde{h}(B) - \tilde{h}(B/F_{(2)}^*(B))\} \subseteq \{1, 2\}.$$

Thus $\max\{\tilde{h}(A), \tilde{h}(B)\} - 1 \leq \tilde{h}(G) \leq \max\{\tilde{h}(A), \tilde{h}(B)\} + 1$. □

While proving Theorem 6.3 we were not able to answer the following question:

Question 6.1. *Let a group $G = AB$ be a totally permutable product of subgroups A and B . Is $\max\{\tilde{h}(A), \tilde{h}(B)\} \leq \tilde{h}(G)$?*

The following question seems interesting

Question 6.2. *Do there exists a constant h with $|\max\{\tilde{h}(A), \tilde{h}(B)\} - \tilde{h}(G)| \leq h$ for any mutually permutable product $G = AB$ of subgroups A and B ?*

D.A. Towers [19] defined and studied analogues of $F^*(G)$ and $\tilde{F}(G)$ for Lie algebras. Using these subgroups and the radical (of a Lie algebra) one can introduce the generalized Fitting height, the non-soluble length and the non-Frattini length of a (finite dimension) Lie algebra.

Question 6.3. *Estimate the generalized Fitting height, the non-soluble length and the non-Frattini length of a (finite dimension) Lie algebra that is the sum of its two subalgebras (ideals, subideals, mutually or totally permutable subalgebras).*

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