# Arithmetic graphs and the products of finite groups*

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**Abstract**

The Hawkes graph $\Gamma_H(G)$ of $G$ is the directed graph whose vertex set coincides with $\pi(G)$ and it has the edge $(p,q)$ whenever $q \in \pi(G/O_{p',p}(G))$. The Sylow graph $\Gamma_s(G)$ of $G$ is the directed graph with vertex set $\pi(G)$ and $(p,q)$ is an edge of $\Gamma_s(G)$ whenever $q \in \pi(N_G(P)/PC_G(P))$ for some Sylow $p$-subgroup $P$ of $G$. The $N$-critical graph $\Gamma_{Nc}(G)$ of a group $G$ the directed graph whose vertex set coincides with $\pi(G)$ such that $(p,q)$ is an edge of $\Gamma_{Nc}(G)$ whenever $G$ contains a Schmidt $(p,q)$-subgroup, i.e. a Schmidt $\{p,q\}$-subgroup with a normal Sylow $p$-subgroup. In the paper the Hawkes, the Sylow and the $N$-critical graphs of the products of totally permutable, mutually permutable and $\mathfrak{N}$-connected subgroups are studied.

**Keywords**: finite group; Hawkes graph; Sylow graph; $N$-critical graph; totally permutable product; mutually permutable product; $\mathfrak{N}$-connected subgroups.
**MSC2020**: 20D40.

## 1 Introduction

All groups considered are **finite**, $G$ always denotes a group and $\pi(G)$ is the set of all prime divisors of $|G|$. If a graph has no isolated vertices, then we will define it just by its edges.

There were many papers in which with every group a certain graph is assigned and the connection of the geometry of the graph with the properties of the group is studied since 1878 (for example, see [1, 7, 11, 12, 13, 14, 23, 24] and etc.). Among such graphs there is an interesting family of arithmetic graphs, i.e. graphs whose vertices are prime divisors of the group's order.

In 1968 Hawkes [11] introduced the directed graph $\Gamma_H(G)$ of $G$ whose vertex set coincides with $\pi(G)$ and it has the edge $(p,q)$ whenever $q \in \pi(G/O_{p',p}(G))$. This graph has many interesting properties [11, 21, 23]. For example [11], if it does not have a loop $(p,p)$, then the $p$-length of $G$ is at most 1.

Recall [8, 12] that the Sylow graph $\Gamma_s(G)$ of $G$ is the directed graph with vertex set $\pi(G)$ and $(p,q)$ is an edge of $\Gamma_s(G)$ whenever $q \in \pi(N_G(P)/PC_G(P))$ for some Sylow $p$-subgroup $P$ of $G$. For applications and properties of this graph see [8, 12, 17, 23]. In particular [17], every connected component of $\Gamma_s(G)$ corresponds to a normal Hall subgroup of $G$.

Recall that a Schmidt $(p,q)$-group is a Schmidt group (i.e. non-nilpotent group, all whose proper subgroups are nilpotent) $G$ with $\pi(G) = \{p,q\}$ and a normal Sylow $p$-subgroup. The $N$-critical [23] graph $\Gamma_{Nc}(G)$ of a group $G$ is the directed graph whose vertex set coincides with $\pi(G)$ such that $(p,q)$ is an edge of $\Gamma_{Nc}(G)$ whenever $G$ contains a Schmidt $(p,q)$-subgroup. For its properties and applications see [16, 18, 23].

Recall that $\mathfrak{N}$ denotes the class of all nilpotent groups. Following Carocca [6] subgroups $H$ and $K$ are called $\mathfrak{N}$-connected if $\langle x,y\rangle \in \mathfrak{N}$ for every $x \in H$ and $y \in K$. Products of $\mathfrak{N}$-connected subgroups were studied in [6, 9, 10] and other. Here we prove

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**Theorem 1.** *Let a group $G$ be the product of pairwise permutable and $\mathfrak{N}$-connected subgroups $G_1, \ldots, G_n$. Then*

$$\Gamma_s(G) = \bigcup_{i=1}^{n} \Gamma_s(G_i), \Gamma_H(G) = \bigcup_{i=1}^{n} \Gamma_H(G_i), \Gamma_{Nc}(G) = \bigcup_{i=1}^{n} \Gamma_{Nc}(G_i).$$

Recall that $G = AB$ is called a totally permutable product of subgroups $A$ and $B$ if every subgroup of $A$ permutes with every subgroup of $B$. Asaad and Shaalan [4] proved that a totally permutable product of two supersoluble groups is also supersoluble. This result started a study of totally permutable products in connection with the theory of group's classes (for example, see [3, Chapter 4]).

**Theorem 2.** *Let a group $G$ be the product of pairwise totally permutable subgroups $G_1, \ldots, G_n$ and $\Gamma(G) = \{(p, q) \mid p, q \in \pi(G), q \in \pi(p-1)\}$. Then*

$$\Gamma_s(G) \subseteq \bigcup_{i=1}^{n} \Gamma_s(G_i) \cup \Gamma(G), \Gamma_H(G) \subseteq \bigcup_{i=1}^{n} \Gamma_H(G_i) \cup \Gamma(G), \Gamma_{Nc}(G) \subseteq \bigcup_{i=1}^{n} \Gamma_{Nc}(G_i) \cup \Gamma(G).$$

**Example 1.** The symmetric group $S_3$ of degree 3 is a totally permutable product of cyclic groups $Z_3$ and $Z_2$ of orders 3 and 2 respectively. Note that $(3, 2)$ is the unique edge of the Sylow graph, the Hawkes graph and the $N$-critical graph of $S_3$ and the Sylow graphs, the Hawkes graphs and the $N$-critical graphs of $Z_3$ and $Z_2$ have no edges. Thus $\Gamma(S_3) \not\subseteq \Gamma(Z_3) \cup \Gamma(Z_2)$ for $\Gamma \in \{\Gamma_s, \Gamma_H, \Gamma_{Nc}\}$.

**Remark 1.** Theorems 1 and 2 follow from a more general result (see Theorem 5).

Recall [3, Definition 4.1.1] that a group $G$ is called a mutually permutable product of its subgroups $A$ and $B$ if $G = AB$, $A$ permutes with every subgroup of $B$ and $B$ permutes with every subgroup of $A$. The products of mutually permutable subgroups are widely studied (see [3, Chapter 4]).

**Theorem 3.** *Let $G = AB$ be a mutually permutable product of its subgroups $A$ and $B$ and $\Gamma(A, B) = \{(p, q) \mid p \in \pi(A), q \in \pi(B) \cap \pi(p-1)$ or $p \in \pi(B), q \in \pi(A) \cap \pi(p-1)\}$. Then*

$$\Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \subseteq \Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B) \text{ and}$$

$$\Gamma_H(A) \cup \Gamma_H(B) \subseteq \Gamma_H(G) \subseteq \Gamma_H(A) \cup \Gamma_H(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\}.$$

**Example 2.** Note that the symmetric group $S_4$ of degree 4 is a mutually permutable product of its Sylow 2-subgroup $P$ and the alternating group $A_4$ of degree 4. Now $E(\Gamma_H(P)) = \emptyset, E(\Gamma_H(A_4)) = \{(2, 3)\}, E(\Gamma(P, A_4)) = \{(3, 2)\}$ and $E(\Gamma_H(S_4)) = \{(2, 2), (2, 3), (3, 2)\}$. Hence $\Gamma_H(S_4) \not\subseteq \Gamma_H(P) \cup \Gamma_H(A_4) \cup \Gamma(P, A_4)$.

Recall [20] that a formation $\mathfrak{F}$ has the Shemetkov property if every $s$-critical for $\mathfrak{F}$ group is a Schmidt group or a group of prime order. For various properties and applications of such formations see [4, Chapter 6.4].

**Corollary 3.1.** *A hereditary formation $\mathfrak{F}$ with the Shemetkov property is closed under taking products of mutually permutable $\mathfrak{F}$-subgroups if and only if it contains all supersoluble Schmidt $\pi(\mathfrak{F})$-groups.*

**Corollary 3.2** ([5, Theorem 2])**.** *Let $p$ be a prime and $\pi$ be a $p$-special set of primes (i.e. $q \notin \pi$ whenever $p$ divides $q(q-1)$). If $G$ is the mutually permutable product of two subgroups $A$ and $B$ which are normal extensions of $p$-groups by $\pi$-groups, the same is true for $G$.*

**Theorem 4.** *Let a group $G$ be a product of pairwise mutually permutable soluble subgroups $G_1, \dots, G_n$ and $\Gamma(G_i, G_j)$ be defined the same way as in Theorem 3. Then*

$$\Gamma_H(G) \subseteq \bigcup_{1\leq i\leq n} \Gamma_H(G_i) \cup \bigcup_{1\leq i,j\leq n, i\neq j} \Gamma(G_i, G_j) \cup \{(p,p) \mid p \in \pi(G)\} \text{ and}$$

$$\Gamma_{Nc}(G) \subseteq \bigcup_{1\leq i\leq n} \Gamma_{Nc}(G_i) \cup \bigcup_{1\leq i,j\leq n, i\neq j} \Gamma(G_i, G_j).$$

The following result for $n = 2$ was proved in [22, Corollary 7].

**Corollary 4.1.** *Let a group $G$ be a product of pairwise mutually permutable subgroups $G_1, \dots, G_n$ If every Schmidt subgroup of $G_1, \dots, G_n$ is supersoluble, then every Schmidt subgroup of $G$ is supersoluble.*

# 2 Preliminaries

Here $\pi(n)$ is the set of prime divisors of $n$; $\pi(\mathfrak{F}) = \cup_{G\in\mathfrak{F}}\pi(G)$; $S_n$ and $A_n$ are the symmetric and the alternating group of degree $n$ respectively; $Z_n$ is the cyclic group of order $n$; $\Phi(G)$ is the Frattini subgroup of $G$; $\mathrm{O}_\pi(G)$ is the greatest normal $\pi$-subgroup of a group $G$ for a set of primes $\pi$. If $\pi = \{p\}$, then $\mathrm{O}_\pi(G)$ is denoted by $\mathrm{O}_p(G)$. If $\pi = \mathbb{P}\setminus\{p\}$, then $\mathrm{O}_\pi(G)$ is denoted by $\mathrm{O}_{p'}(G)$; $\mathrm{O}_{p',p}(G)$ is the greatest normal $p$-nilpotent subgroup of a group $G$. It can be defined by $\mathrm{O}_{p',p}(G)/\mathrm{O}_{p'}(G) = \mathrm{O}_p(G/\mathrm{O}_{p'}(G))$.

Recall that here a (directed) graph $\Gamma$ is a pair of sets $V(\Gamma)$ and $E(\Gamma)$ where $V(\Gamma)$ is a set of vertices of $\Gamma$ and $E(\Gamma)$ is a set of edges of $\Gamma$, i.e. the set of ordered pairs of elements from $V(\Gamma)$. An edge $(v, v)$ is called a loop. Two graphs $\Gamma_1$ and $\Gamma_2$ are called equal (denoted by $\Gamma_1 = \Gamma_2$) if $V(\Gamma_1) = V(\Gamma_2)$ and $E(\Gamma_1) = E(\Gamma_2)$. Graph $\Gamma_1$ is called subgraph of $\Gamma_2$ (denoted by $\Gamma_1 \subseteq \Gamma_2$) if $V(\Gamma_1) \subseteq V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$. Graph $\Gamma$ is called a union of graphs $\Gamma_1$ and $\Gamma_2$ (denoted by $\Gamma = \Gamma_1 \cup \Gamma_2$) if $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.

Let $\Gamma \in \{\Gamma_s, \Gamma_H, \Gamma_{Nc}\}$ and $\mathfrak{X}$ be a class of groups. Recall [23, Definition 3.1] that

$$\Gamma(\mathfrak{X}) = \bigcup_{G\in\mathfrak{X}} \Gamma(G).$$

**Lemma 1** ([23, Theorem 2.7])**.** *Let $G$ be a group. Then*

1. *If $\Gamma \in \{\Gamma_H, \Gamma_{Nc}\}$, then $\Gamma(H) \subseteq \Gamma(G)$ for any $H \leq G$.*

2. *If $\Gamma \in \{\Gamma_s, \Gamma_H, \Gamma_{Nc}\}$, then $\Gamma(G/N) \subseteq \Gamma(G)$ for any $N \trianglelefteq G$.*

3. *If $\Gamma \in \{\Gamma_s, \Gamma_H, \Gamma_{Nc}\}$, then $\Gamma(G/N_1) \cup \Gamma(G/N_2) = \Gamma(G)$ for any $N_1, N_2 \trianglelefteq G$ with $N_1 \cap N_2 = 1$.*

4. *If $\Gamma \in \{\Gamma_H, \Gamma_{Nc}\}$, then $\Gamma(N_1) \cup \Gamma(N_2) = \Gamma(G)$ for any $N_1, N_2 \trianglelefteq G$.*

5. *If $\Gamma \in \{\Gamma_s, \Gamma_H, \Gamma_{Nc}\}$, then $\Gamma(G_1 \times \dots \times G_n) = \Gamma(G_1) \cup \dots \cup \Gamma(G_n)$ for any groups $G_1, \dots, G_n$.*

Let $\mathfrak{X}$ be a class of groups. Recall that a chief factor $H/K$ of $G$ is called $\mathfrak{X}$-*central* (see [19, p. 127–128]) in $G$, for a class of groups $\mathfrak{X}$, provided that the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ of $H/K$ with $G/C_G(H/K)$ corresponding to the action by conjugation of $G$ on $H/K$ belongs $\mathfrak{X}$. The $\mathfrak{X}$-*hypercenter* $\mathrm{Z}_\mathfrak{X}(G)$ of $G$ is the greatest normal subgroup of $G$ such that every chief factor of $G$ below it is $\mathfrak{X}$-central (it exists according to [19, Lemma 14.1]). If $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, then $\mathrm{Z}_\mathfrak{N}(G)$ is the hypercenter $\mathrm{Z}_\infty(G)$ of $G$.

## 3 The proof of Theorems 1 and 2

Recall [10, Proposition 1(8)] that if $G = G_1 \dots G_n$ is the product of pairwise permutable and $\mathfrak{N}$-connected subgroups, then $[G_i, \prod_{j=1,j\neq i}^n G_j] \leq \mathrm{Z}_\infty(G)$ for any $i \in \{1, \dots, n\}$. According to [3, Lemma 4.2.12] if $G = G_1 \dots G_n$ is the product of totally permutable subgroups, then $[G_i, \prod_{j=1,j\neq i}^n G_j] \leq \mathrm{Z}_\mathfrak{U}(G)$ for any $i \in \{1, \dots, n\}$ where $\mathfrak{U}$ stands for the class of all supersoluble groups. These observations lead us to the following definition.

**Definition 1.** We say that $G$ is the product of subgroups $G_1, G_2 \dots, G_n$ with $\mathfrak{F}$-hypercentral condition for commutators if $G = G_1 \dots G_n$, $G_iG_j$ is a subgroup of $G$ for every $i, j \in \{1, \dots, n\}$ and $[G_i, \prod_{j=1,j\neq i}^n G_j] \leq \mathrm{Z}_\mathfrak{F}(G)$ for any $i \in \{1, \dots, n\}$.

The main property of products with $\mathfrak{F}$-hypercentral condition for commutators is

**Lemma 2.** *Let $\mathfrak{F}$ be a hereditary formation with $\mathfrak{N} \subseteq \mathfrak{F}$. If a group $G$ is the product of subgroups $G_1, \dots, G_n$ with $\mathfrak{F}$-hypercentral condition for commutators, then*

$$G/\mathrm{Z}_\mathfrak{F}(G) \simeq G_1/\mathrm{Z}_\mathfrak{F}(G_1) \times \dots \times G_n/\mathrm{Z}_\mathfrak{F}(G_n).$$

*Proof.* (*a*) $\overline{H}_i = G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \cap (\prod_{j=1,j\neq i}^n G_j)\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \simeq 1$ for any $i \in \{1, \dots, n\}$.

Since $G$ satisfies $\mathfrak{F}$-hypercentral condition for commutators, we see that every element of $\overline{H}_i$ commutes with every element of $G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$ and $(\prod_{j=1,j\neq i}^n G_j)\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$. Hence it commutes with every element of $G/\mathrm{Z}_\mathfrak{F}(G)$. Therefore $\overline{H}_i \leq \mathrm{Z}(G/\mathrm{Z}_\mathfrak{F}(G))$. From $\mathfrak{N} \subseteq \mathfrak{F}$ it follows that $\mathrm{Z}(G/\mathrm{Z}_\mathfrak{F}(G)) \leq \mathrm{Z}_\mathfrak{F}(G/\mathrm{Z}_\mathfrak{F}(G)) \simeq 1$. Thus $\overline{H}_i \simeq 1$.

(*b*) $G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \trianglelefteq G/\mathrm{Z}_\mathfrak{F}(G)$ for any $i \in \{1, \dots, n\}$.

Since $G$ satisfies $\mathfrak{F}$-hypercentral condition for commutators, we see that every element of $G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$ commutes with every element of $(\prod_{j=1,j\neq i}^n G_j)\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$. Now from $G = G_1 \dots G_n$ it follows that $G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \trianglelefteq G/\mathrm{Z}_\mathfrak{F}(G)$.

(*c*) $G/\mathrm{Z}_\mathfrak{F}(G) \simeq G_1/\mathrm{Z}_\mathfrak{F}(G_1) \times \dots \times G_n/\mathrm{Z}_\mathfrak{F}(G_n)$.

Now from (*a*) and (*b*) it follows that

$$G/\mathrm{Z}_\mathfrak{F}(G) = G_1\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \times \dots \times G_n\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G).$$

Note that every $\mathfrak{F}$-central chief factor of $G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$ is an $\mathfrak{F}$-central chief factor of $G/\mathrm{Z}_\mathfrak{F}(G)$. From $\mathrm{Z}_\mathfrak{F}(G/\mathrm{Z}_\mathfrak{F}(G)) \simeq 1$ it follows that

$$\mathrm{Z}_\mathfrak{F}(G_i\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)) \simeq \mathrm{Z}_\mathfrak{F}(G_i/(G_i \cap \mathrm{Z}_\mathfrak{F}(G))) \simeq 1.$$

Since $\mathfrak{F}$ is hereditary, we see that $G_i \cap \mathrm{Z}_\mathfrak{F}(G) \leq \mathrm{Z}_\mathfrak{F}(G_i)$ by [2, Lemma 2.4(iii)]. Now from $\mathrm{Z}_\mathfrak{F}(G_i/(G_i \cap \mathrm{Z}_\mathfrak{F}(G))) \simeq 1$ it follows that $G_i \cap \mathrm{Z}_\mathfrak{F}(G) = \mathrm{Z}_\mathfrak{F}(G_i)$. Thus

$$G/\mathrm{Z}_\mathfrak{F}(G) \simeq G_1/(G_1 \cap \mathrm{Z}_\mathfrak{F}(G)) \times \dots \times G_n/(G_n \cap \mathrm{Z}_\mathfrak{F}(G_n)) \simeq G_1/\mathrm{Z}_\mathfrak{F}(G_1) \times \dots \times G_n/\mathrm{Z}_\mathfrak{F}(G_n).$$

Lemma is proved. □

Denote by $\Gamma(\mathfrak{F})_{|G}$ the induced subgraph of $\Gamma(\mathfrak{F})$ on $\pi(G)$.

**Lemma 3.** *Let $\mathfrak{F}$ be a hereditary formation with $\mathfrak{N} \subseteq \mathfrak{F}$, $\Gamma \in \{\Gamma_s, \Gamma_{Nc}, \Gamma_H\}$ and $G$ be a group. Then*

$$\Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \subseteq \Gamma(G) \subseteq \Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma(\mathfrak{F})_{|G}.$$

*Proof.* From 2 of Lemma 1 it follows that $\Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \subseteq \Gamma(G)$. Assume that there is a group $G$ with $\Gamma(G) \not\subseteq \Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma(\mathfrak{F})_{|G}$. Note that $V(\Gamma(G)) = V(\Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma(\mathfrak{F})_{|G})$. Hence there is $(p, q) \in E(\Gamma(G)) \setminus E(\Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma(\mathfrak{F})_{|G})$.

Let $\Gamma = \Gamma_{Nc}$. It means there is a Schmidt $(p,q)$-subgroup $H$ of $G$ with $H \notin \mathfrak{F}$. From $\Gamma_{Nc}(G/\mathrm{Z}_\mathfrak{F}(G)) \subseteq \Gamma_{Nc}(G)$ it follows that $H\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G) \simeq H/H \cap \mathrm{Z}_\mathfrak{F}(G)$ is nilpotent. Since $\mathfrak{F}$ is hereditary, $H \cap \mathrm{Z}_\mathfrak{F}(G) \leq \mathrm{Z}_\mathfrak{F}(H)$ by [2, Lemma 2.4(iii)]. From $\mathfrak{N} \subseteq \mathfrak{F}$ it follows that $H\mathrm{Z}_\mathfrak{F}(G) \in \mathfrak{F}$. Therefore $H \in \mathfrak{F}$, a contradiction. Thus $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma_{Nc}(\mathfrak{F})_{|G}$.

Let $\Gamma = \Gamma_s$. Then there is an element $x$ of $G$ which induces an automorphisms of order $q^\alpha$ on a Sylow $p$-subgroup $P$ of $G$. WLOG we may assume that $x$ is a $q$-element of $G$. Note that $x\mathrm{Z}_\mathfrak{F}(G)$ acts trivially on a Sylow $p$-subgroup $P\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$ of $G/\mathrm{Z}_\mathfrak{F}(G)$. Hence $P\langle x\rangle\mathrm{Z}_\mathfrak{F}(G)/\mathrm{Z}_\mathfrak{F}(G)$ is a nilpotent group. By analogy with the previous paragraph, $P\langle x\rangle\mathrm{Z}_\mathfrak{F}(G) \in \mathfrak{F}$. Hence $(p,q) \in \Gamma_s(\mathfrak{F})$, a contradiction.

Let $\Gamma = \Gamma_H$. From [23, Proposition 2.3(1)] it follows that there is a chief factor $H/K$ of $G$ below $\mathrm{Z}_\mathfrak{F}(G)$ with $p \in \pi(H/K)$ and $q \in \pi(G/C_G(H/K))$. From $(H/K) \rtimes G/C_G(H/K) \in \mathfrak{F}$ it follows that $(p,q) \in \Gamma_H(\mathfrak{F})$, a contradiction. Thus $\Gamma_H(G) \subseteq \Gamma_H(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma_H(\mathfrak{F})_{|G}$. □

The main result of this section is

**Theorem 5.** *Let $\mathfrak{F}$ be a hereditary formation with $\mathfrak{N} \subseteq \mathfrak{F}$ and $\Gamma \in \{\Gamma_s, \Gamma_{Nc}, \Gamma_H\}$. If a group $G$ is the product of subgroups $G_1, \dots, G_n$ with $\mathfrak{F}$-hypercentral condition for commutators, then*

$$\Gamma(G) \subseteq \bigcup_{i=1}^{n} \Gamma(G_i) \cup \Gamma(\mathfrak{F})_{|G}.$$

*Proof.* From Lemma 2 it follows that

$$G/\mathrm{Z}_\mathfrak{F}(G) \simeq G_1/\mathrm{Z}_\mathfrak{F}(G_1) \times \cdots \times G_n/\mathrm{Z}_\mathfrak{F}(G_n).$$

Now $\Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) = \cup_{i=1}^{n}\Gamma(G_i/\mathrm{Z}_\mathfrak{F}(G_i))$ by 5 of Lemma 1. Note that $\Gamma(\mathfrak{F})_{|G_i} \subseteq \Gamma(\mathfrak{F})_{|G}$. Therefore by Lemma 3

$$\begin{aligned}\Gamma(G) \cup \Gamma(\mathfrak{F})_{|G} &= \Gamma(G/\mathrm{Z}_\mathfrak{F}(G)) \cup \Gamma(\mathfrak{F})_{|G} = \\ &\bigcup_{i=1}^{n} \Gamma(G_i/\mathrm{Z}_\mathfrak{F}(G_i)) \cup \Gamma(\mathfrak{F})_{|G} = \bigcup_{i=1}^{n} (\Gamma(G_i/\mathrm{Z}_\mathfrak{F}(G_i)) \cup \Gamma(\mathfrak{F})_{|G_i}) \cup \Gamma(\mathfrak{F})_{|G} \\ &= \bigcup_{i=1}^{n} (\Gamma(G_i) \cup \Gamma(\mathfrak{F})_{|G_i}) \cup \Gamma(\mathfrak{F})_{|G} = \bigcup_{i=1}^{n} \Gamma(G_i) \cup \Gamma(\mathfrak{F})_{|G}\end{aligned}$$

Thus $\Gamma(G) \subseteq \cup_{i=1}^{n}\Gamma(G_i) \cup \Gamma(\mathfrak{F})_{|G}$. □

**Lemma 4.** *If a group $G$ has a Sylow tower, then $\Gamma_s(G) = \Gamma_{Nc}(G) = \Gamma_H(G)$.*

*Proof.* According to [23, Proposition 2.4] $\Gamma_s(G) \subseteq \Gamma_{Nc}(G) \subseteq \Gamma_H(G)$ for any group $G$. Hence we need to prove only that $\Gamma_s(G) = \Gamma_H(G)$ for a Sylow tower group $G$. Assume the contrary, let a Sylow tower group $G$ be a minimal order counterexample. Since $V(\Gamma_H(G)) = V(\Gamma_s(G)) = \pi(G)$, we see that there is $(p,q) \in E(\Gamma_H(G)) \setminus E(\Gamma_s(G))$. Since $G$ has a Sylow tower, we see that $\mathrm{O}_{p',p}(G)$ contains all Sylow $p$-subgroups of $G$. Therefore $p \neq q$.

Let $N$ be a minimal normal subgroup of $G$. Recall that the class of Sylow tower groups is closed under taking epimorphic images. Therefore $G/N$ is a Sylow tower group. From $|G/N| < |G|$ and our assumption it follows that $\Gamma_s(G/N) = \Gamma_H(G/N)$, in particular $(p,q) \notin E(\Gamma_H(G))$. If $G$ has two minimal normal subgroups $N_1$ and $N_2$, then $\Gamma_s(G) = \Gamma_s(G/N_1) \cup \Gamma_s(G/N_2) = \Gamma_H(G/N_1) \cup \Gamma_H(G/N_2) = \Gamma_H(G)$ by Lemma 1, a contradiction. Thus $G$ has the unique minimal normal subgroup $N$. Since $G$ is soluble, $N$ is an $r$-group for some prime $r$. If $r \neq p$, then $\mathrm{O}_{p',p}(G/N) = \mathrm{O}_{p',p}(G)/N$. Hence $(p,q) \in E(\Gamma_H(G/N))$, a contradiction. Now $r = p$. Since $G$ is a Sylow tower group with the unique minimal normal subgroup $N$, we see that a Sylow

$p$-subgroup $P$ of $G$ is normal in $G$. Let $Q$ be a Sylow $q$-subgroup of $G$. Then $Q \leq N_G(P)$. From $(p, q) \notin E(\Gamma_s(G))$ it follows that $Q \leq C_G(P)$. Now $Q \leq C_G(H/K)$ where $H/K$ is a chief $p$-factor of $G$. Recall that $\mathrm{O}_{p',p}(G)$ is the intersection of centralizers of all chief $p$-factors of $G$. So $Q \leq \mathrm{O}_{p',p}(G)$. Thus $(p, q) \notin E(\Gamma_H(G))$, the final contradiction. □

*Proof of Theorem 1.* From Lemma 4 it follows that $\Gamma_s(\mathfrak{N}) = \Gamma_{Nc}(\mathfrak{N}) = \Gamma_H(\mathfrak{N})$. Note that $V(\Gamma_H(\mathfrak{N})) = \mathbb{P}$ and $E(\Gamma_H(\mathfrak{N})) = \emptyset$. Let $\Gamma \in \{\Gamma_s, \Gamma_{Nc}, \Gamma_H\}$. Now from the proof of Theorem 5 it follows that

$$\Gamma(G) = \Gamma(G) \cup \Gamma(\mathfrak{N})_{|G} = \bigcup_{i=1}^{n} \Gamma(G_i) \cup \Gamma(\mathfrak{N})_{|G} = \bigcup_{i=1}^{n} \Gamma(G_i).$$

Theorem 1 is proved. □

*Proof of Theorem 2.* From Lemma 4 it follows that $\Gamma_s(\mathfrak{U}) = \Gamma_{Nc}(\mathfrak{U}) = \Gamma_H(\mathfrak{U})$. Note that $V(\Gamma_H(\mathfrak{U})) = \mathbb{P}$. It is well known that a group $G$ is supersoluble iff $G/\mathrm{O}_{p',p}(G)$ is abelian of exponent dividing $p-1$. Hence $E(\Gamma_H(\mathfrak{U})) \subseteq \{(p, q) \mid q \in \pi(p-1)\}$. On the other hand if $q \in \pi(p-1)$, then a cyclic group $Z_p$ of order $p$ has a power automorphism of order $q$. Now $H = Z_p \rtimes Z_q$ is supersoluble and $q \in \pi(H/\mathrm{O}_{p',p}(H))$. Thus $E(\Gamma_H(\mathfrak{U})) = \{(p, q) \mid q \in \pi(p-1)\}$. Now Theorem 2 directly follows from Theorem 5. □

# 4 The proof of Theorems 3 and 4

We need the following lemma in the proof of Theorem 3.

**Lemma 5.** *Let $P$ be a Sylow $p$-subgroup of a Schmidt $(p, q)$-subgroup $S$ of a group $G$. If $A$ is a subgroup of $G$ with $P \leq A$ and $G = AC_G(P)$, then $A$ contains a Schmidt $(p, q)$-subgroup.*

*Proof.* Let $Q$ be a Sylow $q$-subgroup of $S$, then $Q = \langle x \rangle$ is cyclic. Since $G = AC_G(P) = C_G(P)A$, there exist $y \in A$ and $z \in C_G(P)$ with $x = zy$. Now $P = P^x = P^y$. It means that $P \trianglelefteq P\langle y \rangle \leq A$. Assume that $A$ does not contain a Schmidt $(p, q)$-group. Now $P\mathrm{O}_q(\langle y \rangle)$ is a $p$-closed $\{p, q\}$-group without Schmidt $(p, q)$-subgroups. It means that $P\mathrm{O}_q(\langle y \rangle)$ is nilpotent. Hence $\langle y_1 \rangle = \mathrm{O}_q(\langle y \rangle) \leq C_G(P)$. Let $y_2 = \mathrm{O}_{q'}(\langle y \rangle)$. So $y = y_1 y_2$ It is well known that $C_G(P) \trianglelefteq N_G(P)$. Note that $x, y \in N_G(P)$. Now $\langle x \rangle C_G(P)/C_G(P)$ is a non-trivial $q$-group. From the other hand $\langle x \rangle C_G(P)/C_G(P) = \langle z y_1 y_2 \rangle C_G(P)/C_G(P) = \langle y_2 \rangle C_G(P)/C_G(P)$ is a $q'$-group, a contradiction. □

*Proof of Theorem 3.* ***Let prove that*** $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B)$. Note that $\Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \subseteq \Gamma_{Nc}(G)$ by Lemma 1.

Assume that $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B)$ is false. Let chose a minimal order group $G$ such that $G$ is a mutually permutable product of subgroups $A$ and $B$ and $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B)$. It means that there is $(p, q) \notin E(\Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B))$ such that $(p, q) \in E(\Gamma_{Nc}(G))$. Therefore $G$ has a Schmidt $(p, q)$-subgroup $S$.

Since $A_G B_G \neq 1$ by [3, Theorem 4.3.11], WLOG we may assume that $A_G$ contains a minimal normal subgroup $N$ of $G$.

Now $G/N = (A/N)(BN/N)$ is a mutually permutable product of subgroups $A/N$ and $BN/N$ by [3, Lemma 4.1.10]. Hence $\Gamma_{Nc}(G/N) \subseteq \Gamma_{Nc}(A/N) \cup \Gamma_{Nc}(BN/N) \cup \Gamma(A/N, BN/N)$. Note that $BN/N \simeq B/(B \cap N)$. It means that $\Gamma_{Nc}(A/N) \subseteq \Gamma_{Nc}(A)$ and $\Gamma_{Nc}(BN/N) = \Gamma_{Nc}(B/(B \cap N)) \subseteq \Gamma_{Nc}(B)$ by Lemma 1. By the definition of $\Gamma(A, B)$ we see that

$$\Gamma(A/N, BN/N) = \Gamma(A/N, B/(B \cap N)) \subseteq \Gamma(A, B).$$

Now $\Gamma_{Nc}(G/N) \subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B)$. So $(p, q) \notin E(\Gamma_{Nc}(G/N))$. Hence $SN/N$ is not a Schmidt group. Therefore $S \cap N$ contains a Sylow $p$-subgroup $P_0$ of $S$. Denote a Sylow $q$-subgroup of $S$ by $Q_0$.

Assume that $N \leq A \cap B$. There exist Sylow $q$-subgroups $Q, Q_1$ and $Q_2$ of $G, A$ and $B$ respectively such that $Q = Q_1Q_2$. Note that there is $x \in G$ with $Q_0 \leq Q^x$. Now $S \leq NQ^x = (NQ_1)^x(NQ_2)^x$. Let $T = NQ^x$, $H = (NQ_1)^x$ and $K = (NQ_2)^x$. From $\Gamma_{Nc}(H) = \Gamma_{Nc}(NQ_1) \subseteq \Gamma_{Nc}(A)$ and $\Gamma_{Nc}(K) = \Gamma_{Nc}(NQ_2) \subseteq \Gamma_{Nc}(B)$ it follows that $(p, q) \notin E(\Gamma_{Nc}(H) \cup \Gamma_{Nc}(K))$. Let $P$ be a Sylow $p$-subgroup of $N$ with $P_0 \leq P$. By Frattini's Argument $H = NN_H(P)$. So there is a $q$-subgroup $Q_3$ of $N_H(P)$ with $H = NQ_3$. Note that $PQ_3$ is a $p$-closed $\{p, q\}$-group without Schmidt $(p, q)$-subgroups. It means that $PQ_3$ is nilpotent. Hence $Q_3 \leq C_H(P)$. Therefore $H = NC_H(P)$. Similar arguments show that $K = NC_K(P)$. So $T = NC_T(P)$. Now $N$ contains a Schmidt $(p, q)$-group by Lemma 5, a contradiction.

Assume now that $N \not\leq A \cap B$. It means that $N \cap B = 1$ by [3, Lemma 4.3.3(4)]. Suppose that $B \leq C_G(N)$. Now $A$ has a Schmidt $(p, q)$-group by Lemma 5, a contradiction. Thus $B \not\leq C_G(N)$. In this case $N$ is cyclic and $A \leq C_G(N)$ by [3, Lemma 4.3.3(5)]. Since $G = AB = C_G(N)(NB)$, we see that $NB$ contains a Schmidt $(p, q)$-group by Lemma 5. Hence $q \in \pi(NB/C_{NB}(N)) \subseteq \pi(B)$. Since $N$ is a cyclic $p$-group, we see that $NB/C_{NB}(N) \simeq G/C_G(N)$ is abelian of exponent dividing $p - 1$. Therefore $(p, q) \in E(\Gamma(A, B))$, the final contradiction.

***Let prove that*** $\Gamma_H(G) \subseteq \Gamma_H(A) \cup \Gamma_H(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\}$. Note that $\Gamma_H(A) \cup \Gamma_H(B) \subseteq \Gamma_H(G)$ by Lemma 1.

Assume that $\Gamma_H(G) \subseteq \Gamma_H(A) \cup \Gamma_H(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\}$ is false. Let chose a minimal order group $G$ such that $G$ is a mutually permutable product of subgroups $A$ and $B$ and $\Gamma_H(G) \not\subseteq \Gamma_{Nc}(A) \cup \Gamma_{Nc}(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\}$. It means that there is $(p, q) \notin E(\Gamma_H(A) \cup \Gamma_H(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\})$ such that $(p, q) \in E(\Gamma_H(G))$. In particular, $p \neq q$.

WLOG we may assume that $A$ contains a minimal normal subgroup $N$ of $G$ by [3, Theorem 4.3.11]. Now $G/N = (A/N)(BN/N)$ is a mutually permutable product of subgroups $A/N$ and $BN/N$ by [3, Lemma 4.1.10]. Hence $\Gamma_H(G/N) \subseteq \Gamma_H(A/N) \cup \Gamma_H(BN/N) \cup \Gamma(A/N, BN/N) \cup \{(p, p) \mid p \in \pi(G/N)\}$. Note that $\Gamma_H(A/N) \subseteq \Gamma_H(A)$, $\Gamma_H(BN/N) \subseteq \Gamma_H(B)$ by Lemma 1, and $\Gamma(A/N, BN/N) \subseteq \Gamma(A, B)$. Now $\Gamma_H(G/N) \subseteq \Gamma_H(A) \cup \Gamma_H(B) \cup \Gamma(A, B) \cup \{(p, p) \mid p \in \pi(G)\}$. So $(p, q) \notin E(\Gamma_H(G/N))$. From 3 of Lemma 1 and our assumption it follows that $N$ must be the unique minimal normal subgroup of $G$. If $\Phi(G) \not\simeq 1$, then similar arguments show that $(p, q) \notin E(\Gamma_H(G/\Phi(G)))$. Note that $\Gamma_H(G/\Phi(G)) = \Gamma_H(G)$ by [23, Theorem 2.7], a contradiction. So $\Phi(G) = 1$. Thus $G$ is a primitive group with $C_G(N) \leq N$.

If $N$ is a $p'$-group, then $\mathrm{O}_{p',p}(G/N) = \mathrm{O}_{p',p}(G)/N$ and $G/\mathrm{O}_{p',p}(G) \simeq (G/N)/\mathrm{O}_{p',p}(G/N)$. Hence $(p, q) \in E(\Gamma_H(G/N))$, a contradiction. Now $p \in \pi(N)$. Therefore $\mathrm{O}_{p'}(G) = 1$. Assume that $N \leq A \cap B$. Hence $\mathrm{O}_{p'}(A) = \mathrm{O}_{p'}(B) = 1$. Now $(\pi(A) \setminus \{p\}) \subseteq \pi(A/\mathrm{O}_{p',p}(A))$ and $(\pi(B) \setminus \{p\}) \subseteq \pi(B/\mathrm{O}_{p',p}(B))$. It means that $(p, q) \in E(\Gamma_H(A) \cup \Gamma_H(B))$, a contradiction. Therefore $N \not\leq A \cap B$. It means that $N \cap B = 1$ by [3, Lemma 4.3.3(4)].

Now either $A \leq C_G(N)$ or $B \leq C_G(N)$ by [3, Lemma 4.3.3(5)]. If $B \leq C_G(N)$, then from $C_G(N) \leq N \leq A$ it follows that $A = G$, and $(p, q) \in E(\Gamma_H(A))$, a contradiction. Thus $B \not\leq C_G(N)$. In this case $N$ is cyclic and $A \leq C_G(N)$ by [3, Lemma 4.3.3(5)]. Hence $N \leq A \leq C_G(N) \leq N$. Thus $N = C_G(N) = A$ is a cyclic group of order $p$. In this case $G/N$ is an abelian group of exponent dividing $p-1$. From $\mathrm{O}_{p'}(G) = 1$ it follows that is a $\mathrm{O}_{p',p}(G) = N$. Therefore $\pi(G/\mathrm{O}_{p',p}(G)) \subseteq \pi(p - 1)$. Hence $q \in \pi(p - 1)$. Thus $(p, q) \in E(\Gamma(A, B))$, the final contradiction. □

*Proof of Corollary 3.1.* Let $\mathfrak{F}$ be a hereditary formation with the Shemetkov property. Then $\mathfrak{F} = (G \mid \Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F}))$ by [23, Theorem 4.4]. Assume that $\mathfrak{F}$ is closed under taking mutually permutable products. Let $G$ be a supersoluble Schmidt $\pi(\mathfrak{F})$-group. Then $G/\Phi(G)$ is a mutually permutable product of groups $Z_p$ and $Z_q$ of orders $p$ and $q$ for some $p, q \in \pi(\mathfrak{F})$ with $q \in \pi(p-1)$. Hence $G/\Phi(G) \in \mathfrak{F}$. Since the class of soluble $\mathfrak{F}$-groups is saturated [4, Corollary 6.4.5], $G \in \mathfrak{F}$. Thus $\mathfrak{F}$ contains every supersoluble Schmidt $\pi(\mathfrak{F})$-group.

Assume now that $\mathfrak{F}$ contains every supersoluble Schmidt $\pi(\mathfrak{F})$-group. Hence $(p, q) \in E(\Gamma_{Nc}(\mathfrak{F}))$ for every $p, q \in \pi(\mathfrak{F})$ with $q \in \pi(p-1)$. Now if $G = AB$ is a mutually permutable product of $\mathfrak{F}$-subgroups $A$ and $B$, then $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F})$ by Theorem 3. Hence $G \in \mathfrak{F}$. Thus $G$ is closed under taking mutually permutable products. ☐

*Proof of Corollary 3.2.* Let $\pi$ be a $p$-special set of primes and $\mathfrak{F}$ be the class of normal extensions of $p$-groups by $\pi$-groups. Then $p \not\in \pi$. Hence $\mathfrak{F}$ is the formation of all $p$-closed $\pi(\mathfrak{F})$-groups. Recall that an $s$-critical group for the class of all $p$-closed groups is a Schmidt $(q, p)$-group for some prime $q$. Hence $\mathfrak{F}$ is the formation with the Shemetkov property. Note that $\mathfrak{F}$ contains every supersoluble $\pi(\mathfrak{F})$-group. Thus $\mathfrak{F}$ is closed under taking mutually permutable products by Corollary 3.1. ☐

We need the following Lemma in the prove of Theorem 4.

**Lemma 6.** *Let $G$ be a soluble group and $n \geq 3$. If $G = G_1 \dots G_n$ is a product of pairwise permutable subgroups $G_1, \dots, G_n$, then*

$$\Gamma_{Nc}(G) = \bigcup_{1 \leq i < j \leq n} \Gamma_{Nc}(G_i G_j).$$

*Proof.* Assume that $n = 3$, then $G = (G_1G_2)(G_1G_3) = (G_1G_2)(G_2G_3) = (G_1G_3)(G_2G_3)$. Now $\Gamma_{Nc}(G) = \Gamma_{Nc}(G_1G_2) \cup \Gamma_{Nc}(G_1G_3) \cup \Gamma_{Nc}(G_2G_3)$ by [23, Theorem 7.1(1)]. Assume that we prove Lemma 6 for all $n$ with $3 \leq n \leq k$, let prove it for $n = k+1$. Let $H_l = \prod_{j=1, j \neq l}^{n}$. Then by our assumption $\Gamma_{Nc}(H_l) = \bigcup_{1 \leq i < j \leq n, i, j \neq l} \Gamma_{Nc}(G_i G_j)$. From $G = H_1H_2 = H_1H_3 = H_2H_3$ and [23, Theorem 7.1(1)] it follows that

$$\Gamma_{Nc}(G) = \Gamma_{Nc}(H_1) \cup \Gamma_{Nc}(H_2) \cup \Gamma_{Nc}(H_3) = \bigcup_{1 \leq i < j \leq k+1} \Gamma_{Nc}(G_i G_j).$$

Now Lemma 6 follows from the mathematical induction principle. ☐

*Proof of Theorem 4.* Let a group $G$ be a product of pairwise mutually permutable soluble subgroups $G_1, \dots, G_n$. From [3, Theorem 4.1.14] it follows that $G$ is soluble. Now

$$\Gamma_{Nc}(G) = \bigcup_{1 \leq i < j \leq n} \Gamma_{Nc}(G_i G_j).$$

According to Theorem 3 $\Gamma_{Nc}(G_iG_j) \subseteq \Gamma_{Nc}(G_i) \cup \Gamma_{Nc}(G_j) \cup \Gamma(G_i, G_j)$. Thus

$$\Gamma_{Nc}(G) \subseteq \bigcup_{1 \leq i \leq n} \Gamma_{Nc}(G_i) \cup \bigcup_{1 \leq i, j \leq n, i \neq j} \Gamma(G_i, G_j).$$

From $\Gamma_{Nc}(G) \subseteq \Gamma_H(G)$ for every group $G$ and [15, Lemma 3] it follows that if $(p, q) \in E(\Gamma_H(G)) \setminus E(\Gamma_{Nc}(G))$ for a soluble group $G$, then $p = q$. Thus

$$\Gamma_H(G) \subseteq \bigcup_{1 \leq i \leq n} \Gamma_H(G_i) \cup \bigcup_{1 \leq i, j \leq n, i \neq j} \Gamma(G_i, G_j) \cup \{(p, p) \mid p \in \pi(G)\}.$$

Theorem 4 is proved. ☐

*Proof of Corollary 4.1.* It is clear that the class $\mathfrak{F}$ of all groups whose Schmidt subgroups are supersoluble is a hereditary formation with the Shemetkov property and $\Gamma_{Nc}(\mathfrak{F}) = \Gamma_{Nc}(\mathfrak{U}) = \{(p, q) \mid q \in \pi(p-1)\}$. Note that $\Gamma_{Nc}(\mathfrak{F})$ does not have cycles. Hence every $\mathfrak{F}$-group has a Sylow tower by [23, Theorem 6.2(b)]. In particular, $G_1, \dots, G_n$ are soluble. Now $G = G_1 \dots G_n$ is a product of mutually permutable soluble $\mathfrak{F}$-subgroups $G_1, \dots, G_n$. Therefore $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F})$ by Theorem 4. Thus $G \in \mathfrak{F}$ by [23, Theorem 4.4]. ☐

## 5 Final remarks and open questions

From 4 of Lemma 1 it follows that if a group $G = AB$ is the product of its normal subgroups $A$ and $B$, then $\Gamma(G) = \Gamma(A) \cup \Gamma(B)$ where $\Gamma \in \{\Gamma_H, \Gamma_{Nc}\}$. Note that $S_4$ is the product of its normal subgroups $S_4$ and $A_4$ and $\Gamma_s(S_4) \cup \Gamma_s(A_4) \neq \Gamma_s(S_4)$. Nevertheless the following question seems interesting.

**Question 1.** If a group $G = AB$ is the product of its normal subgroups $A$ and $B$. Is $\Gamma_s(G) \subseteq \Gamma_s(A) \cup \Gamma_s(B)$?

Through $\overline{\Gamma}$ here we denote an undirected graph on the same vertex set as $\Gamma$ in which two vertices are connected by the edge if they are connected in $\Gamma$. In the proves of [8, 12] the Sylow graph was considered as a directed one but in [12] it was defined as an undirected one. Therefore the graph $\overline{\Gamma}_s$ seems interesting. Moreover

**Proposition 1.** *If a soluble group $G = AB$ is the product of its normal subgroups $A$ and $B$, then $\overline{\Gamma}_s(G) = \overline{\Gamma}_s(A) \cup \overline{\Gamma}_s(B)$.*

*Proof.* From [16, Theorem 4.2(2)] it follows that $\overline{\Gamma}_s(H) = \overline{\Gamma}_{Nc}(H)$ for any soluble group $H$. Now $\overline{\Gamma}_s(G) = \overline{\Gamma}_s(A) \cup \overline{\Gamma}_s(B)$ follows from 4 of Lemma 1. □

Note [16, Proof of Theorem 4.2] that there are groups $H$ with $\overline{\Gamma}_s(H) \neq \overline{\Gamma}_{Nc}(H)$. That is why we ask

**Question 2.** If a group $G = AB$ is the product of its normal subgroups $A$ and $B$. Is $\overline{\Gamma}_s(G) = \overline{\Gamma}_s(A) \cup \overline{\Gamma}_s(B)$?

In Theorem 3 only $N$-critical and Hawkes graph of mutually permutable product was described. What can be said about the Sylow graph of mutually permutable product? For example

**Question 3.** If a group $G = AB$ is the product of mutually permutable subgroups $A$ and $B$. Is $\Gamma_s(G) \subseteq \Gamma_s(A) \cup \Gamma_s(B) \cup \Gamma(A, B)$?

The proof of Theorem 3 is based on the properties of mutually permutable products of 2 subgroups. The analogues of these properties for products of more than 2 subgroups are not known now. That is why we use some properties of $N$-critical graph of a soluble group (see Lemma 6) to prove Theorem 4. Hence we have the following 2 questions.

**Question 4.** Does the conclusion of Theorem 4 hold for the product of pairwise mutually permutable subgroups $G_1, \dots, G_n$?

**Question 5.** Let $G$ be a group and $n \geq 3$. If $G = G_1 \dots G_n$ is the product of pairwise permutable subgroups $G_1, \dots, G_n$, then is

$$\Gamma_{Nc}(G) = \bigcup_{1 \leq i < j \leq n} \Gamma_{Nc}(G_i G_j)?$$

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