A test for a local formation of finite groups to be a formation of soluble groups with the Shemetkov property[1]

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Abstract. L.A. Shemetkov posed a Problem 9.74 in Kourovka Notebook to find all local formations $\mathfrak{F}$ of finite groups such that every finite minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order. All known solutions to this problem are obtained under the assumption that every minimal non-$\mathfrak{F}$-group is soluble. Using the above mentioned solutions we present a polynomial in n time check for a local formation $\mathfrak{F}$ with bounded $\pi(\mathfrak{F})$ to be a formation of soluble groups with the Shemtkov property where $n = \max \pi(\mathfrak{F})$.

Keywords. Finite group; Schmidt group; soluble group; local formation; formation with the Shemetkov property; N-critical graph of a group.

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Introduction and the results

All groups considered here are finite. Recall that a non-nilpotent group all whose proper subgroups are nilpotent is called a Schmidt group in the honor of O. Yu. Schmidt who described the structure of such groups [14] in 1924. In 1951 N. Ito [8, Proposition 2] proved that a non-p-nilpotent group all whose proper subgroups are p-nilpotent is a Schmidt group. Recall that for a class of groups $\mathfrak{X}$ a group $G \notin \mathfrak{X}$ is called a minimal non-$\mathfrak{X}$-group if all its proper subgroups belong $\mathfrak{X}$. V.N. Semenchuk and A.F. Vasil'ev [15] described all hereditary local formations of soluble groups $\mathfrak{F}$ such that every soluble minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order. In 1984 L.A. Shemetkov asked [12, Problem 9.74] in Kourovka Notebook to find all local formations $\mathfrak{F}$ of finite groups such that every finite minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order. The solutions to this problem are presented in [10, Corollary 1] and [4, Theorem 2]. Recall that a formation $\mathfrak{F}$ is said to have the Shemetkov property if every minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order.

Theorem 1 ([9, Corollary 2.4.23])**.** *Let $\mathfrak{F}$ be a hereditary local formation. Then $\mathfrak{F}$ has the Shemetkov property if and only if it satisfies the following conditions:*

1) *Every minimal non-$\mathfrak{F}$-group is soluble;*

2) *$\mathfrak{F}$ is locally defined by f where $f(p_i) = \mathfrak{G}_{\pi_i}$ for all $p_i \in \pi(\mathfrak{F})$ where π_i is a subset of $\pi(\mathfrak{F})$ with $p_i \in \pi_i$.*

It is natural to ask if condition 1) of Theorem 1 can be deduced from condition 2) of this theorem. The aim of this paper is to solve a particular case of this question: can one deduce from condition 2) of Theorem 1 that $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property?

Theorem 2. *Let $\pi = \{p_1, p_2, \dots, p_k\}$ be a set of primes not greater than n, π_i be a subset of π with $p_i \in \pi_i$. Assume that $\mathfrak{F}$ is a local formation with $\pi(\mathfrak{F}) = \pi$ locally defined by f where $f(p_i) = \mathfrak{G}_{\pi_i}$. In $O(n^2)$ operations one can check whether $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property.*

In the section "Proof of Corollary 1" we will show how this algorithm works on a simple example.

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Corollary 1. *Let* $\pi = \{2, 3, 5, 7\}$ *and* $\mathfrak{F}$ *be a local formation with* $\pi(\mathfrak{F}) = \pi$ *locally defined by* f *where* $f(2) = f(3) = \mathfrak{G}_{\{2,3,5,7\}}$*,* $f(5) = \mathfrak{G}_{\{3,5,7\}}$ *and* $f(7) = \mathfrak{G}_{\{5,7\}}$*. Then* $\mathfrak{F}$ *is a formation of soluble groups with the Shemetkov property.*

The proof of Theorem 2 is based on the concept of N-critical graph. Recall that a Schmidt (p, q)-group is a Schmidt group with the normal Sylow p-subgroup. An *N-critical graph* $\Gamma_{Nc}(G)$ of a group G [19, Definition 1.3] is a directed graph on the vertex set $\pi(G)$ and (p, q) is an edge of $\Gamma_{Nc}(G)$ iff G has a Schmidt (p, q)-subgroup. We can modify the proof of Theorem 2 to show

Corollary 2. *Let* Γ *be a directed graph such that* $V(\Gamma)$ *is a finite set of primes and* $n = \max V(\Gamma)$*. One can check if every group* G *with* $\Gamma_{Nc}(G) = \Gamma$ *is soluble in a polynomial in* n *time.*

From the proves of Corollaries 2 and 3 follows

Corollary 3. *If a* $\{2, 3, 5, 7\}$*-group* G *does not contain Schmidt* $(5, 2)$*-subgroups,* $(7, 2)$*-subgroups and* $(7, 3)$*-subgroups, then* G *is soluble.*

1 Preliminaries

All unexplained notations and terminologies are standard. The reader is referred to [5, 6] if necessary. Here Z_n is the cyclic group of order n; $\pi(G)$ is the set of all prime divisors of $|G|$; $\pi(\mathfrak{X}) = \bigcup_{G\in\mathfrak{X}} \pi(G)$; $\Phi(G)$ is the Frattini subgroup of G; $\mathcal{M}(\mathfrak{X})$ is the class of all minimal non-$\mathfrak{X}$-groups. Recall that $\mathfrak{G}_\pi$ is the class of all π-groups and $\mathbb{F}_p$ denotes a field with p elements. Whenever V is a G-module over $\mathbb{F}_p$, $V \rtimes G$ denotes the semidirect product of V with G corresponding to the action of G on V as G-module.

1.1 Formations

Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images (i.e. from $G \in \mathfrak{F}$ and $N \trianglelefteq G$ it follows that $G/N \in \mathfrak{F}$) and subdirect products (i.e. from $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$ it follows that $G/(N_1 \cap N_2) \in \mathfrak{F}$).

A formation $\mathfrak{F}$ is called

(a) *hereditary* if from $G \in \mathfrak{F}$ and $H \leq G$ it follows that $H \in \mathfrak{F}$.

(b) *saturated* if from $G/N \in \mathfrak{F}$ where $N \trianglelefteq G$ and $N \leq \Phi(G)$ it follows that $G \in \mathfrak{F}$.

A function of the form $f : \mathbb{P} \to \{formations\}$ is called a *formation function*. Recall [6, IV, Definitions 3.1] that a formation $\mathfrak{F}$ is called *local* if

$$\mathfrak{F} = (G \mid G/C_G(H/K) \in f(p) \text{ for every } p \in \pi(H/K) \text{ and every chief factor } H/K \text{ of } G)$$

for some formation function f. In this case f is called a *local definition* of $\mathfrak{F}$.

1.2 N-critical graph

Here a (directed) graph Γ is a pair of sets $V(\Gamma)$ and $E(\Gamma)$ where $V(\Gamma)$ is a set of vertices of Γ and $E(\Gamma)$ is a set of edges of Γ, i.e. the set of ordered pairs of elements from $V(\Gamma)$. Two graphs Γ_1 and Γ_2 are called equal (denoted by $\Gamma_1 = \Gamma_2$) if $V(\Gamma_1) = V(\Gamma_2)$ and $E(\Gamma_1) = E(\Gamma_2)$. Graph Γ_1 is called a subgraph of Γ_2 (denoted by $\Gamma_1 \subseteq \Gamma_2$) if $V(\Gamma_1) \subseteq V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$. Graph Γ is called a union of graphs Γ_1 and Γ_2 (denoted by $\Gamma = \Gamma_1 \cup \Gamma_2$) if $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2)$.

The N-critical graph of a class of groups $\mathfrak{X}$ [19, Definition 3.1] is defined by

$$\Gamma_{Nc}(\mathfrak{X}) = \bigcup_{G\in\mathfrak{X}} \Gamma_{Nc}(G).$$

Lemma 1 ([19, Theorem 2.7(2)]). *Let H be a subgroup of G and G_i be groups where $1 \leq i \leq n$. Then $\Gamma_{Nc}(H) \subseteq \Gamma_{Nc}(G)$ and*

$$\Gamma_{Nc}\left(\times_{i=1}^{n} G_i\right) = \bigcup_{i=1}^{n} \Gamma_{Nc}(G_i).$$

Proposition 1 ([19, Proposition 6.1]). *The following statements hold:*

(*a*) *If p is a prime, then $V(\Gamma_{Nc}(PSL(2,2^p))) = \pi(2(2^{2p}-1))$ and $E(\Gamma_{Nc}(PSL(2,2^p))) = \{(2,q) \mid q \in \pi(2^p-1)\} \cup \{(q,2) \mid q \in \pi(2^{2p}-1)\}$.*

(*b*) *If p is an odd prime, then $V(\Gamma_{Nc}(PSL(2,3^p))) = \pi(3(3^{2p}-1))$ and $E(\Gamma_{Nc}(PSL(2,3^p))) = \{(3,q) \mid q \in \pi(3^p-1) \setminus \{2\}\} \cup \{(2,3)\} \cup \{(q,2) \mid q \in \pi(3^{2p}-1) \setminus \{2\}\}$.*

(*c*) *If $p > 5$ is a prime with $p^2+1 \equiv 0 \mod 5$, then $V(\Gamma_{Nc}(PSL(2,p))) = \pi(p(p^2-1))$ and $E(\Gamma_{Nc}(PSL(2,p))) = \{(p,q) \mid q \in \pi\left(\frac{p-1}{2}\right)\} \cup \{(2,3)\} \cup \{(q,2) \mid q \in \pi(p^2-1) \setminus \{2\}\}$.*

(*d*) *If p is an odd prime, then $V(\Gamma_{Nc}(Sz(2^p))) = \pi(2(2^{2p}+1)(2^p-1))$ and $E(\Gamma_{Nc}(Sz(2^p))) = \{(2,q) \mid q \in \pi(2^p-1)\} \cup \{(q,2) \mid q \in \pi((2^p-1)(2^{2p}+1))\}$.*

(*e*) *$V(\Gamma_{Nc}(PSL(3,3))) = \{2,3,13\}$ and $E(\Gamma_{Nc}(PSL(3,3))) = \{(2,3),(3,2),(13,3)\}$.*

1.3 Algorithms

We assume that basic arithmetic operations and comparison are done in the same amount of time (equal 1). Recall that $O(f(n))$ means a function $g(n)$ such that there exist $C > 0$ and a natural number n_0 such that for all $n \geq n_0$ holds $|g(n)| < C|f(n)|$. It is well known that the decomposition of a natural number n into the product of primes can be done in $O(\sqrt{n})$ operations.

2 Proof of Theorem 2

(*a*) *$\mathfrak{F}$ is hereditary formation.*

Note that $f(p)$ is a hereditary formation for all $p \in \pi = \pi(\mathfrak{F})$. Hence $\mathfrak{F}$ is a hereditary formation by [6, IV, Proposition 3.14].

(*b*) *$\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property if and only if $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$ for every minimal simple non-abelian group G.*

Suppose that $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$ for every minimal simple non-abelian group G. Assume that $\mathfrak{F}$ contains a non-soluble group G_1. Since $\mathfrak{F}$ is a hereditary, it contains a minimal non-soluble group G_2. Note that $G_3 \simeq G_2/\Phi(G_2) \in \mathfrak{F}$ is a minimal simple non-abelian group. Hence $\Gamma_{Nc}(G_3) \subseteq \Gamma_{Nc}(\mathfrak{F})$, a contradiction.

Thus every group in $\mathfrak{F}$ is soluble. Assume now that $\mathcal{M}(\mathfrak{F})$ contains a non-soluble group G_1. Note that G_1 is a minimal non-soluble group. By Gaschütz-Lubeseder-Schmid Theorem $\mathfrak{F}$ is a saturated formation. Hence a minimal simple non-abelian group $G_2 \simeq G_1/\Phi(G_1) \in \mathcal{M}(\mathfrak{F})$. Since G_2 is not a Schmidt group or a group of prime order, $\Gamma_{Nc}(G_2)$ is the join of $\Gamma_{Nc}(M)$ where M runs through all maximal subgroups of G_2. From $M \in \mathfrak{F}$ for every maximal subgroup M of G_2 it follows that $\Gamma_{Nc}(G_2) \subseteq \Gamma_{Nc}(\mathfrak{F})$, a contradiction. Thus every group in $\mathcal{M}(\mathfrak{F})$ is soluble. Therefore $\mathfrak{F}$ has the Shemetkov property by Theorem 1 or [5, Theorem 6.4.12].

Suppose now that $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property. Assume that there is a minimal simple non-abelian group G with $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F})$. Since G is non-soluble, it contains a minimal non-$\mathfrak{F}$-group H as a subgroup. Since $\pi(G) = V(\Gamma_{Nc}(G)) \subseteq V(\Gamma_{Nc}(\mathfrak{F}))$ and $\mathfrak{F}$ has the Shemetkov property, we see that H is a Schmidt (p,q)-group for some $(p,q) \notin E(\Gamma_{Nc}(\mathfrak{F}))$. From $\Gamma_{Nc}(H) \subseteq \Gamma_{Nc}(G)$ by Lemma 1 it follows that $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$, a contradiction. Thus $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$ for every minimal simple non-abelian group G.

(*c*) *$E(\Gamma_{Nc}(\mathfrak{F})) = \{(p_i,p_j) \mid p_i \in \pi, p_j \in \pi_i\}$.*

Let Γ be a graph with $V(\Gamma) = \pi$ and $E(\Gamma) = \{(p_i,p_j) \mid p_i \in \pi, p_j \in \pi_i\}$. Assume that $(p,q) \in E(\Gamma_{Nc}(\mathfrak{F}))$. Note that $p = p_i$ and $q = p_j$ for some $p_i, p_j \in \pi$ with $p_i \neq p_j$. Then $\mathfrak{F}$

contains a Schmidt (p_i, p_j)-group G_1. Hence $G_2 \simeq G_1/\Phi(G_1) \in \mathfrak{F}$ is also a Schmidt (p_i, p_j)-group. In this case G has the unique minimal normal subgroup N, N is a p_i-group and $G/C_G(N) \simeq Z_{p_j}$. Therefore $Z_{p_j} \in f(p_i) = \mathfrak{G}_{\pi_i}$. Hence $p_j \in \pi_i$. Thus $E(\Gamma_{Nc}(\mathfrak{F})) \subseteq E(\Gamma)$.

Let $p_i \in \pi$ and $p_j \in \pi_i$. According to [6, B, Theorem 10.3] a group Z_{p_j} has a faithful irreducible module V over a field $\mathbb{F}_{p_i}$. Let $G = V \rtimes Z_{p_j}$. Note that V is the unique minimal normal subgroup of G and every maximal subgroup of G is either a p_i-group or a p_j-group. Hence G is a Schmidt (p_i, p_j)-group. From $G/C_G(V) \simeq Z_{p_j} \in f(p_i)$ and $f(p_j) \neq \emptyset$ it follows that $G \in \mathfrak{F}$. Hence $(p_i, p_j) \in E(\Gamma_{Nc}(\mathfrak{F}))$. Therefore $E(\Gamma) \subseteq E(\Gamma_{Nc}(\mathfrak{F}))$. Thus $E(\Gamma) = E(\Gamma_{Nc}(\mathfrak{F}))$.

(*d*) *Let p be a prime and $p \leq n$. If G is isomorphic to any group from $PSL(2, 2^p)$, $PSL(2, 3^p)$ and $Sz(2^p)$ for odd p, $PSL(2, p)$ for $p > 5$ with $5 \in \pi(p^2 + 1)$, then we can check if $\Gamma_{Nc}(G) \subseteq \Gamma$ in $O(n)$ operations.*

First we need to check that $\pi(G) \subseteq \pi$. Note that

$$|G| \in \{(2^p + 1)(2^{2p} - 2^p), (3^p + 1)(3^{2p} - 3^p)/2, 2^{2p}(2^{2p} + 1)(2^p - 1), (p + 1)(p^2 - p)/2\}.$$

Since $p \leq n$, the rude estimates shows that $|G| \leq 2^{6n}$. Hence $|G|$ has no more than $6n$ not necessary different primes divisors. Note that if $\Gamma_{Nc}(G) \subseteq \Gamma$, then all these divisors belong to π. Since $|\pi| < n$, we see that in no more than $7n$ divisions we can check the if $\pi(G) \subseteq \pi = V(\Gamma)$. Suppose now that $\pi(G) \subseteq \pi$.

Assume that $G \simeq PSL(2, 2^p)$. Since $\pi(G) \subseteq \pi$, in $O(n)$ operations we can compute $\pi(2^p - 1)$ and $\pi(2^p + 1)$. Note that $|\pi(2^p - 1) \cup \pi(2^p + 1)| < n$ in our case. According to Proposition 1(a) taking vertex 2, we need to check that from it starts an arrow to every vertex from $\pi(2^p - 1)$; taking every vertex from $\pi(2^p - 1) \cup \pi(2^p + 1)$, we need to check that from it starts an arrow to vertex 2. It is clear that all these can be done in $O(n)$ operations.

Assume that $G \simeq Sz(2^p)$. Since $\pi(G) \subseteq \pi$, in $O(n)$ operations we can compute $\pi(2^p - 1)$ and $\pi(2^{2p} + 1)$. Note that $|\pi(2^p - 1) \cup \pi(2^{2p} + 1)| < n$ in our case. According to Proposition 1(d) taking vertex 2, we need to check that from it starts an arrow to every vertex from $\pi(2^p - 1)$; taking every vertex from $\pi(2^p - 1) \cup \pi(2^{2p} + 1)$, we need to check that from it starts an arrow to vertex 2. It is clear that all these can be done in $O(n)$ operations.

Assume that $G \simeq PSL(2, 3^p)$. Since $\pi(G) \subseteq \pi$, in $O(n)$ operations we can compute $\pi(3^p - 1)$ and $\pi(3^p + 1)$. Note that $|\pi(3^p - 1) \cup \pi(3^p + 1)| < n$ in our case. According to Proposition 1(b) taking vertex 3, we need to check that from it starts an arrow to every vertex from $\pi(3^p - 1) \setminus \{2\}$; taking every vertex from $\pi(3^p - 1) \cup \pi(3^p + 1) \setminus \{2\}$, we need to check that from it starts an arrow to vertex 2; also we need to check, if $(2, 3) \in E(\Gamma)$. It is clear that all these can be done in $O(n)$ operations.

Assume that $G \simeq PSL(2, p)$. Since $\pi(G) \subseteq \pi$, in $O(n)$ operations we can compute $\pi(p - 1)$ and $\pi(p + 1)$. Note that $|\pi(p - 1) \cup \pi(p + 1)| < n$ in our case. According to Proposition 1(c) taking vertex p, we need to check that from it starts an arrow to every vertex from $\pi(\frac{p-1}{2})$; taking every vertex from $\pi(p - 1) \cup \pi(p + 1) \setminus \{2\}$, we need to check that from it starts an arrow to vertex 2; also we need to check, if $(2, 3) \in E(\Gamma)$. It is clear that all these can be done in $O(n)$ operations.

(*e*) *In $O(n^{3/2})$ operations we can show that there are up to isomorphism no more than $4n$ minimal simple non-abelian groups G for which $\Gamma_{Nc}(G) \subseteq \Gamma$ is possible and list this groups.*

Recall [7, II, Bemerkung 7.5] that minimal simple non-abelian groups up to isomorphism are $PSL(2, 2^p)$ for a prime p, $PSL(2, 3^p)$ and $Sz(2^p)$ for an odd prime p, $PSL(2, p)$ where $p > 5$ is a prime with $5 \in \pi(p^2 + 1)$ and $PSL(3, 3)$.

Assume that $G \simeq PSL(2, 2^p)$ or $G \simeq Sz(2^p)$. Then $\pi(2^p - 1) \subseteq \pi$. Hence if $q \in \pi(2^p - 1)$, then $2^p \equiv 1 \mod q$. Note that $2^{q-1} \equiv 1 \mod q$ by Fermat's Little Theorem. Hence $2^{(q-1,p)} \equiv 1 \mod q$. Since p is a prime, $p \in \pi(q - 1)$.

Assume that $G \simeq PSL(2, 3^p)$ where p is an odd prime. Then $\pi(3^p - 1) \setminus \{2\} \subseteq \pi$. So there is $q \in \pi(3^p - 1) \setminus \{2, 3\}$. By analogy $p \in \pi(q - 1)$.

Let $\rho = \cup_{q\in\pi}(\pi(q-1))$. Note $\pi(q-1)$ can be computed in $O(n^{1/2})$ divisions for every $q \in \pi$ and $\max\rho < n$. From $|\pi| < n$ it follows that ρ can be computed in $O(n^{3/2})$ operations. Hence there are up to isomorphism no more than $3|\rho| < 3n$ minimal simple groups G from

$$\{PSL(2,2^p), PSL(2,3^p), Sz(2^p) \mid p \in \mathbb{P}\}$$

for which $\Gamma_{Nc}(G) \subseteq \Gamma$ is possible. Note that if $\Gamma(PSL(2,p)) \subseteq \Gamma$, then $p \in \pi$. Since $|\pi| \leq n$, in $O(n^{3/2})$ operations we can show that there are up to isomorphism no more than $4n$ minimal simple non-abelian groups G for which $\Gamma_{Nc}(G) \subseteq \Gamma$ is possible and list this groups.

(f) *The final step.*

According to (b) we need only to check that $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$ for every minimal simple non-abelian group G. Using (e) we can list (up to isomorphism) no more than $4n$ such groups G for which $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F})$ is possible in $O(n^{3/2})$ operations. According to (d) for every listed group G in $O(n)$ operations we can check if $\Gamma_{Nc}(G) \subseteq \Gamma_{Nc}(\mathfrak{F})$ (note that we can check if $\Gamma_{Nc}(PSL(3,3)) \subseteq \Gamma_{Nc}(\mathfrak{F})$ in $O(1)$ operations). Thus in $O(n^2)$ operations we can check if $\Gamma_{Nc}(G) \not\subseteq \Gamma_{Nc}(\mathfrak{F})$ for every minimal simple non-abelian group G.

Algorithm 1: IsSolubleSFormation($\mathfrak{F}$)

Result: True, if $\mathfrak{F}$ is a formation of soluble groups with the Shemtkov property and false otherwise.

Data: $\pi = \{p_1, p_2, \dots, p_k\}$ is a set of primes not greater than n, π_i is a subset of π with $p_i \in \pi_i$ for $i \in \{1, \dots, k\}$

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Γ ← graph with V(Γ) = π and (p_i, p_j) ∈ E(Γ) iff p_j ∈ π_i;
if {(2,3),(3,2),(13,3)} ⊆ E(Γ) or {(2,3),(3,2),(5,2)} ⊆ E(Γ) then
    return false;
end
for p ∈ π with 5 ∈ π(p^2 + 1) and p > 5 do
    if π((p^3 − p)/2) ⊆ π then
        if {(p,q) | q ∈ π((p−1)/2)} ∪ {(2,3)} ∪ {(q,2) | q ∈ π(p^2 − 1) \ {2}} ⊆ E(Γ) then
            return false;
        end
    end
end
ρ ← ∪_{p∈π}(π(p − 1) \ {2});
for p ∈ ρ do
    if π(2(2^{2p} − 1)) ⊆ π then
        if {(2,q) | q ∈ π(2^p − 1)} ∪ {(q,2) | q ∈ π(2^{2p} − 1)} ⊆ E(Γ) then
            return false;
        end
    end
    if π(2(2^p − 1)(2^{2p} + 1)) ⊆ π then
        if {(2,q) | q ∈ π(2^p − 1)} ∪ {(q,2) | q ∈ π((2^p − 1)(2^{2p} + 1))} ⊆ E(Γ) then
            return false;
        end
    end
    if π(3(3^{2p} − 1)/2) ⊆ π then
        if {(3,q) | q ∈ π(3^p − 1) \ {2}} ∪ {(2,3)} ∪ {(q,2) | q ∈ π(3^{2p} − 1) \ {2}} ⊆ E(Γ)
         then
            return false;
        end
    end
end
return true;
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3 Proof of Corollary 1

Let do the steps according to "IsSolubleSFormation":
1. $V(\Gamma_{Nc}(\mathfrak{F})) = \{2,3,5,7\}$ and

$$E(\Gamma_{Nc}(\mathfrak{F})) = \{(2,3),(2,5),(2,7),(3,2),(3,5),(3,7),(5,3),(5,7),(7,5)\}.$$

2. $\{(2,3),(3,2),(13,3)\} \not\subseteq E(\Gamma)$ and $\{(2,3),(3,2),(5,2)\} \not\subseteq E(\Gamma)$.
3. The only prime $p > 5$ in π with $5 \in \pi(p^2+1)$ is 7. Note that $\{(7,3),(2,3),(3,2)\} \not\subseteq E(\Gamma)$.
4. $\rho = (\pi(2-1) \cup \pi(3-1) \cup \pi(5-1) \cup \pi(7-1)) \setminus \{2\} = \{3\}$.
5. $\pi(2(2^6-1)) \subseteq \pi$ but $\{(2,7),(7,2),(3,2)\} \not\subseteq E(\Gamma)$.
6. $\pi(2(2^3-1)(2^6+1)) \not\subseteq \pi$.
7. $\pi(3(3^6-1)/2) \not\subseteq \pi$
8. Thus $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property.

4 Proof of Corollary 2

Let $\pi = \{p_1, \dots, p_k\} = V(\Gamma)$, $\pi_i = \{p_i\} \cup \{p_j \mid (p_i, p_j) \in E(\Gamma)\}$ and $\mathfrak{F}$ be a local formation with $\pi(\mathfrak{F}) = \pi$ locally defined by f where $f(p_i) = \mathfrak{G}_{\pi_i}$ for all $p_i \in \pi$. Since $n = \max \pi$, in a polynomial in n time we can check does $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property by Theorem 2.

Let prove that every group G with $\Gamma_{Nc}(G) = \Gamma$ is soluble if and only if $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property. Note that $\Gamma_{Nc}(\mathfrak{F}) = \Gamma$ by step (c) of the proof of Theorem 2.

Suppose that every group G with $\Gamma_{Nc}(G) = \Gamma$ is soluble. Assume that there is a minimal simple non-abelian group G_1 with $\Gamma_{Nc}(G_1) \subseteq \Gamma$. Let $H(p_i, p_j)$ be a Schmidt (p_i, p_j)-group and

$$G_2 = G_1 \times (\times_{(p_i,p_j)\in E(\Gamma)} H(p_i, p_j)) \times (\times_{p_i \in V(\Gamma)} Z_{p_i}).$$

Since $E(\Gamma) \cup V(\Gamma)$ is finite, by Lemma 1

$$\Gamma_{Nc}(G_2) = \Gamma_{Nc}(G_1) \cup \left(\bigcup_{(p_i,p_j)\in E(\Gamma)} \Gamma_{Nc}(H(p_i,p_j)) \right) \cup \left(\bigcup_{p_i \in V(\Gamma)} \Gamma_{Nc}(Z_{p_i}) \right) = \Gamma.$$

Therefore there exists a non-soluble group G_2 with $\Gamma_{Nc}(G_2) = \Gamma$, a contradiction. Thus $\Gamma_{Nc}(G_1) \not\subseteq \Gamma$ for every minimal simple non-abelian group G_1. Thus $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property by step (b) of the proof of Theorem 2.

Suppose that $\mathfrak{F}$ is a formation of soluble groups with the Shemetkov property. Assume that there exists a non-soluble group G with $\Gamma_{Nc}(G) = \Gamma$. Then G contains a minimal non-$\mathfrak{F}$-group H as a subgroup. Since $\pi(G) = \pi = \pi(\mathfrak{F})$, H is a Schmidt (p,q)-group for some $(p,q) \notin E(\Gamma)$. Hence $\Gamma_{Nc}(G) \neq \Gamma$, a contradiction. Thus every group G with $\Gamma_{Nc}(G) = \Gamma$ is soluble.

5 Final remarks

Let $\mathfrak{F}, \pi, \pi_i$ be the same as in Theorem 2. Then if "IsSolubleSFormation($\mathfrak{F}$)" returns true, then we automatically prove a lot of results about $\mathfrak{F}$. Lets list some of them.

From [11] and [17] or [2] it follows that $\mathfrak{F}$ has *the Kegel property*, i.e.

If $A, B, C \in \mathfrak{F}$ and $G = AB = BC = CA$, then $G \in \mathfrak{F}$.

From [1, Lemma 2.2] it follows that $\mathfrak{F}$ has *the property* $\mathcal{P}_2$, i.e.

If $G = A_1 \dots A_k$ where $A_i A_j \in \mathfrak{F}$ for $i \neq j$, then $G \in \mathfrak{F}$.

From [16] (see also [18, Theorem 1]) it follows that $\mathfrak{F}$ has *the Belonogov property* in the class of all soluble groups

If A, B, C are non-conjugate maximal $\mathfrak{F}$- subgroups of a soluble group G, then $G \in \mathfrak{F}$.

It is well known that the class of all nilpotent groups can be characterized as a class of groups whose every Sylow (or cyclic primary) subgroup is subnormal. Recall [5, Definition 6.1.2] that a subgroup H of a group G is said to be $\mathfrak{F}$-subnormal in G if either $H = G$ or there exists a chain of subgroups $H = H_0 < \dots < H_n = G$ such that H_{i-1} is a maximal subgroup of H_i and $H_i/Core_{H_i}(H_{i-1}) \in \mathfrak{F}$ for $i = 1, \dots, n$. From [13, Corollaries 3.9 and 3.10] it follows that for $\mathfrak{F}$ hold the analogues of this characterization.

If every Sylow (or every cyclic primary) subgroup of G is $\mathfrak{F}$-subnormal in G, then $G \in \mathfrak{F}$.

From [19, Theorem 4.4] or [3, Theorem 1] it follows that for $\mathfrak{F}$ holds an analogue of Frobenious p-nilpotency criterion.

A π-group G belongs $\mathfrak{F}$ iff $N_G(P)/C_G(P)$ is a π_i-group for every p_i-subgroup P of G and $p_i \in \pi$.

We want to note that the following problems seems interesting and remains open.

Problem 1. Let $\pi = \{p_1, p_2, \dots, p_k\}$ be a set of primes not greater than n, π_i be a subset of π with $p_i \in \pi_i$. Assume that $\mathfrak{F}$ is a local formation with $\pi(\mathfrak{F}) = \pi$ locally defined by f where $f(p_i) = \mathfrak{G}_{\pi_i}$. Can one check that $\mathfrak{F}$ has the Shemetkov property in polynomial in n time?

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