A note on the $\mathfrak{F}$-hypercenter of a finite group

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All groups considered here are finite. In [**1**] R. Baer showed that on the one hand the hypercenter $\mathrm{Z}_\infty(G)$ of a group G coincides with the intersection of all maximal nilpotent subgroups of G and on the other hand $\mathrm{Z}_\infty(G)$ coincides with the intersection of normalizers of all Sylow subgroups of G.

Let $\mathfrak{X}$ be a class of groups. A chief factor H/K of a group G is called $\mathfrak{X}$-central if $(H/K)\rtimes G/C_G(H/K)\in\mathfrak{X}$. A normal subgroup N of G is said to be $\mathfrak{X}$-hypercentral in G if $N=1$ or $N\neq 1$ and every chief factor of G below N is $\mathfrak{X}$-central. The $\mathfrak{X}$-hypercenter $\mathrm{Z}_\mathfrak{X}(G)$ is the product of all normal $\mathfrak{X}$-hypercentral subgroups of G (see [**2**, 1, Definition 2.2]). So if $\mathfrak{X}=\mathfrak{N}$ is the class of all nilpotent groups then $\mathrm{Z}_\mathfrak{N}(G)$ is just the hypercenter $\mathrm{Z}_\infty(G)$ of a group G.

Recall that $\mathrm{Int}_\mathfrak{F}(G)$ is the intersection of all $\mathfrak{F}$-maximal subgroups of a group G and $\underset{i\in I}{\times}\mathfrak{F}_{\pi_i}=(G=\underset{i\in I}{\times}\mathrm{O}_{\pi_i}(G)|\mathrm{O}_{\pi_i}(G)\in\mathfrak{F}_{\pi_i})$ is a hereditary saturated formation where $\sigma=\{\pi_i|i\in I\}$ is a partition of $\mathbb{P}$ into mutually disjoint subsets and $\mathfrak{F}_{\pi_i}$ is a hereditary saturated formation with $\pi(\mathfrak{F}_{\pi_i})=\pi_i$ for all $i\in I$. Denote the intersection of all normalizers of $\mathfrak{F}$-maximal subgroups of G by $\mathrm{NI}_\mathfrak{F}(G)$.

Theorem. *Let $\sigma=\{\pi_i|i\in I\}$ be a partition of $\mathbb{P}$ into mutually disjoint subsets, $\mathfrak{F}_{\pi_i}$ be a hereditary saturated formation with $\pi(\mathfrak{F}_{\pi_i})=\pi_i$ for all $i\in I$ and $\mathfrak{F}=\underset{i\in I}{\times}\mathfrak{F}_{\pi_i}$. The following statements are equivalent*:

(1) $\mathrm{Int}_\mathfrak{F}(G)=\mathrm{Z}_\mathfrak{F}(G)$ *for every group G.*

(2) $\mathrm{Int}_{\mathfrak{F}_{\pi_i}}(G)=\mathrm{Z}_{\mathfrak{F}_{\pi_i}}(G)$ *for every π_i-group G and every $i\in I$.*

(3) $\bigcap\limits_{i\in I}\mathrm{NI}_{\mathfrak{F}_{\pi_i}}(G)=\mathrm{Z}_\mathfrak{F}(G)$ *for every group G.*

Corollary (Baer [**1**]). *Let G be a group. Then*

(1) *The hypercenter of G is the intersection of all normalizers of all Sylow subgroups of G.*

(2) *The hypercenter of G is the intersection of all maximal nilpotent subgroups of G.*

Remark. A. N. Skiba [**3**] described all hereditary saturated formations $\mathfrak{F}$ with $\mathrm{Int}_\mathfrak{F}(G)=\mathrm{Z}_\mathfrak{F}(G)$ in terms of the canonical local definition of $\mathfrak{F}$. In the proof of Theorem we don't use his description.

References

[1] Baer R., Group Elements of Prime Power Index // Trans. Amer. Math Soc., V.75, 1, 1953, 20–47.

[2] Guo W., Structure theory for canonical classes of finite groups // Springer 2015.

[3] Skiba A. N., On the $\mathfrak{F}$-hypercentre and the intersection of all $\mathfrak{F}$-maximal subgroups of a finite group // Journal of Pure and Applied Algebra, V.216, 4, 2012, 789–799.

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