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О *p*-СВЕРХРАЗРЕШИМОСТИ ОДНОГО КЛАССА КОНЕЧНЫХ ГРУПП

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ON *p*-SUPERSOLUBILITY OF ONE CLASS FINITE GROUPS

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Доказано следующее: конечная группа G p-сверхразрешима тогда и только тогда, когда она имеет нормальную подгруппу N с p-сверхразрешимой фактор-группой G / N такой, что либо $N - p'$ -группа, либо p делит $|N|$, и $|G : N_G(L)|$ является степенью числа p для любой циклической p-подгруппы L из N порядка p или порядка 4 (если $p = 2$ и в N силовская 2-подгруппа является неабелевой).

Ключевые слова*: конечная группа*, *p-нильпотентная группа*, *p-сверхразрешимая группа*.

The following is proved: A finite group G is p-supersoluble if and only if it has a normal subgroup N with p-supersoluble quotient G / N such that either N is p' -group or p divides $|N|$ and $|G : N_G(L)|$ equals to a power of p for any cyclic p-subgroup L of N of order p or order 4 (if $p = 2$ and a Sylow 2-subgroup of N is non-abelian).

Keywords: *finite group*, *p-nilpotent group*, *p-supersoluble group*.

Mathematics Subject Classification (**2000**): 20D10, 20D15.

Introduction

Throughout this paper, all groups are finite. A group G is said to be p -supersoluble if every chief factor of G is either cyclic or is a p' -group. There are a large number of criteria supersolubility groups (see [1]). And at the same time p -supersolvable groups remain relatively poorly understood. In this note we prove the following result in this trend.

Theorem. *Let G be a group and p a prime. Then G is p-supersoluble if and only if it has a normal subgroup N with p-supersoluble quotient* G / N *such that either N is* p' *-group or p divides* $|N|$ *and* $|G : N_G(L)|$ *equals to a power of p for any cyclic p-subgroup L of N of order p or order* 4 (*if p* = 2 *and a Sylow* 2*-subgroup of N is non-abelian*).

Bercovich and Kazarin proved [2, Theorem 1] that if for any cyclic p-subgroup L of G of prime order or order 4, $|G : C_G(L)|$ equals to a power of p, then G is p-nilpotent. From our theorem we obtain the following similar result for the p-supersoluble groups.

Corollary **0.1** *Let G be a group and p a prime divisor of* $|G|$. *If for any cyclic p-subgroup L of G of order p or order* 4, $|G : N_G(L)|$ *equals to a power of p*, *then G is p-supersoluble*.

Corollary **0.2** (Ito, Gaschütz [3, IV, Theorem 5.7]). *If every minimal subgroup of a group G is normal in G*, *then the commutator subgroup* G' *of G is 2-closed.*

Corollary **0.3** (Buckley [4]). *Let G be a group of odd order. If every minimal subgroup of G is normal in G*, *then G is supersoluble.*

All unexplained notations and terminology are standard. The reader is referred to [5] or [6] if necessary.

1 Some lemmas

In order to prove the theorem we need the following lemmas.

Lemma **1.1** *Let* $L \le G$ *and p be a prime divisor of L. Suppose that L is p-closed and* $|G : N_G(L)|$ *is a power of p. Then the Sylow p-subgroup of L is contained in* $O_p(G)$.

Proof. Let L_p be the Sylow p-subgroup of L, P a Sylow subgroup of G containing L_p. Since L_p is characteristic in L, $N_G(L) \le N_G(L_p)$. Hence

$$(L_p)^G = (L_p)^{N_G(L_p)P} = (L_p)^P \le O_p(G). \qquad \square$$

Lemma **1.2** [7, Theorem 2.4]. *Let P be a p-group*, α *a* p' *-automorphism of P.*

(1) *If* $[\alpha, \Omega_2(P)] = 1$, *then* $\alpha = 1$.

(2) *If* $[\alpha, \Omega_1(P)] = 1$ *and either p is odd or P is abelian*, *then* $\alpha = 1$.

Proof. We use, following [5], $\mathcal{A}(p-1)$ to denote the formation of all abelian groups of exponent dividing $p-1$. The symbol $Z_{\mathcal{U}}(G)$ denotes the largest normal subgroup of a group G such that every chief factor of G below $Z_{\mathcal{U}}(G)$ is cyclic. $\square$

Lemma **1.3** [8, Lemma 2.2]. *Let E be a normal* p *-subgroup of a group G*. *If* $E \le Z_{\mathcal{U}}(G)$, *then*

$$(G / C_G(E))^{\mathcal{A}(p-1)} \le O_p(G / C_G(E)).$$

***Lemma* 1.4** *Let P be a normal p -subgroup of a group G. Let D be a characteristic subgroup of a p-subgroup P such that every non-trivial p'-automorphism of P induces a non-trivial automorphism of D. Suppose that $D \le Z_{\mathcal{U}}(G)$. Then $P \le Z_{\mathcal{U}}(G)$.*

Proof. Let $C = C_G(P)$, H/K any chief factor of G below P. Then

$$O_p(G/C_G(H/K)) = 1$$

by [1, Appendix C, Corollary 6.4]. Since $D \le Z_{\mathcal{U}}(G)$, then $(G/C_G(D))^{\mathcal{A}(p-1)}$ is a p-group by Lemma 1.3. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group. Thus

$$G/C_G(H/K) \in \mathcal{A}(p-1)$$

and so $|H/K| = p$ by [1, Chapter 1, Theorem 1.4]. Therefore $P \le Z_{\mathcal{U}}(G)$. □

***Lemma* 1.5** *Let G be a p-group of class at most 2. Suppose that $\exp(G/Z(G))$ divides p.*

(1) *If $p > 2$, then $\exp(\Sigma(G)) = p$.*

(2) *If G is a non-abelian 2-group, then $\exp(\Sigma(G)) = p$.*

Proof. See page 3 in [2].

***Lemma* 1.6** *Let P be a normal p-subgroup of a group G. Suppose that $|G : N_G(L)$ equals to a power of p for any cyclic p-subgroup L of P of prime order or order 4 (if P is a non-abelian 2-group). Then $P \le Z_{\mathcal{U}}(G)$.*

Proof. Suppose that this lemma is false and let G be a counterexample with $|G||P|$ minimal. Let $Z = Z_{\mathcal{U}}(G)$. If P is not a non-abelian 2-group we use Ω to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega = \Omega_2(P)$. Let D be a Thompson critical subgroup of P.

(1) *G has a normal subgroup $R \le P$ such that P/R is a non-cyclic chief factor of G, $R \le Z$ and $V \le R$ for any normal subgroup $V \ne P$ of G contained in P.*

Let P/R be a chief factor of G. Then the hypothesis holds for (G, R). Therefore $R \le Z$ by the choice of (G, P) and so P/R is not cyclic. Now let $V \ne P$ be any normal subgroup of G contained in P. Then $V \le Z$. If $V \not\le R$, then from the G – isomorphism

$$P/R = VR/R \simeq V/V \cap R$$

we deduce $P \le Z$, contrary to the choice of (G, P). Hence $V \le R$.

(2) $\Omega = P = D$.

Indeed, suppose that $\Omega < P$. Then, in view of (1), $\Omega \le Z$. Hence $P \le Z$ by Lemmas 1.3 and 1.4, which contradicts the choice of (G, P). Hence $\Omega = P$. In view of Theorem 3.11 in [9, Chapter 5] we obtain similarly that $P = D$.

(3) *The final contradiction.*

Let L/R be any minimal subgroup of $P/R \cap Z(G_p/R)$, where G_p is a Sylow p-subgroup of G. Let $x \in P \setminus R$. Then $|\langle x \rangle|$ is ether prime or 4 by Lemma 1.5. Hence $|G : N_G(\langle x \rangle)|$ is a power of p. Hence $L/R = \langle x \rangle R/R$ is normal in G/R, which contradicts (1). □

***Lemma* 1.7** *Let $L \le N$ be subgroups of a group G, where N is normal in G. Then $|N : N_N(L)|$ divides $|G : N_G(L)|$.*

Proof. This follows from

$$|N : N_N(L)| = |N : N_G(L) \cap N| = \\ = |NN_G(L) : N_G(L)|. \quad □$$

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and such that each group G has the smallest normal subgroup (denoted by $G^{\mathfrak{F}}$) whose quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ for any group G with $G/\Phi(G) \in \mathfrak{F}$. A group G is said to be a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but $H \in \mathfrak{F}$ for every proper subgroup H of G. In what follows we shall need the following result.

***Lemma* 1.8** [5, Chapter VI, Theorem 25.4]. *Let $\mathfrak{F}$ be a saturated formation. Let G be a minimal non-$\mathfrak{F}$-group such that $G^{\mathfrak{F}}$ is soluble.*

(a) *$P = G^{\mathfrak{F}}$ is a p-group for some prime p and P is of exponent p or of exponent 4 (if P is a non-abelian 2-group).*

(b) *$P/\Phi(P)$ is a chief factor of G and*

$$(P/\Phi(P)) \leftthreetimes (G/C_G(P/\Phi(P))) \notin \mathfrak{F}.$$

***Lemma* 1.9** *Let p be a prime and G a p-soluble group. Assume that $O_{p'}(G) = 1$. Then the following statements are equivalent.*

(i) *G is p-supersoluble*;

(ii) *G is supersoluble*;

(iii) *$G/O_p(G)$ is an abelian group of exponent dividing $p-1$.*

Proof. (i) $\Rightarrow$ (ii). Since G is p-supersoluble, for every chief p-factor H/K of G, we have $|H/K| = p$ and so $G/C_G(H/K)$ is an abelian group of exponent dividing $p-1$ (see [1, Chapter 1, Theorem 1.4]. Since $O_{p'}(G) = 1$, the intersection of the centralizers of all chief factors H/K of $|H/K| = p$ is $O_{p',p}(G) = O_p(G)$. Hence G is supersoluble by [1]. By using the same arguments, we also see that (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). □

2 Proof of Theorem

We have only to prove that if a group G has a normal subgroup N with p-supersoluble quotient such that either N is p'-group or p divides $|N|$ and $|G : N_G(L)|$ equals to a power of p for any cyclic

p-subgroup L of N of order p or order 4 (if $p=2$ and a Sylow 2-subgroup of N is non-abelian), then G is p-supersoluble. Suppose that this is false and let G be a counterexample with $|G|+|N|$ minimal. Then p divides $|N|$. Let P be Sylow p-subgroup of N.

(1) $O_{p'}(G)=1$.

Let $D=O_{p'}(G)$ and H/D a cyclic p-subgroup of ND/D of prime order p or order 4. Then in view of Schur-Zassenhaus Theorem, there is a cyclic subgroup H_0 of G such that $|H_0|=|H/D|$ and $H_0D=H$. Since $(|H_0|,|D|)=1$ and $H_0\le ND$ we have $H_0\le N$. Hence $|G:N_G(H_0)|$ equals to a power of p by the hypothesis and so $|G/D:N_{G/D}(H/D)|$ equals to a power of p. Therefore the hypothesis holds for $(G/D, ND/D)$. If $D\ne 1$, then G/D is p-supersoluble by the choice of $|G|+|N|$, which implies the p-supersolubility of G. Thus $D=1$.

(2) *N is p-supersoluble.*

Suppose that N is not p-supersoluble. Then p divides $|N|$. Moreover, $N=G$, otherwise N is p-supersoluble by Lemma 1.7 and the choice of G.

(a) *For any subgroup H of G we have* $O_p(H)\le Z_{\mathcal{U}}(H)$.

By Lemma 1.6, $O_p(G)\le Z_{\mathcal{U}}(G)$. Let $P=O_p(H)$ and H be a cyclic p-subgroup of H of prime order or order 4 (if $p=2$ and a Sylow 2-subgroup of N is non-abelian). If P is not a non-abelian 2-group we use Ω to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega=\Omega_2(P)$. By Lemma 1.1, $H\le O_p(G)$. Hence

$$\Omega\le Z_{\mathcal{U}}(G)\cap H\le Z_{\mathcal{U}}(H)$$

and so $P\le Z_{\mathcal{U}}(H)$ by Lemmas 1.2 and 1.4.

(b) *G is not soluble.*

Assume that G is soluble and let H be a minimal non-p-supersoluble subgroup of G. Let $\mathfrak{F}$ be the class of all p-supersoluble groups. Then $\mathfrak{F}$ is a saturated formation by [3, Chapter VI, Theorem 8.6]. Hence by Lemma 1.8, $P=H^{\mathfrak{F}}$ is a p-group and P is of exponent p or of exponent 4 (if P is a non-abelian 2-group). Moreover, $P/\Phi(P)$ is a non-cyclic chief factor of H. But from Claim (a) we deduce that $P\le Z_{\mathcal{U}}(H)$, so $|P/\Phi(P)|=p$. This contradiction shows that we have (b).

(c) $O^q(G)=G$ *for all primes* $q\ne p$.

Suppose that for some prime $q\ne p$ we have $O^q(G)\ne G$. The hypothesis is true for $(O^q(G), O^q(G))$ by Lemma 1.7, so $O^q(G)$ is p-supersoluble by the choice of $(G,N)=(G,G)$. Since $O_{p'}(O^q(G))$ a characteristic subgroup of $O^q(G)$, it is normal in G. Hence in view of (1), $O_{p'}(O^q(G))=1$. Therefore $C_G(H/K)$ is supersoluble and $C_G(O^q(G))$ is an abelian group of exponent dividing $p-1$ by Lemma 1.9. Therefore by Claim (a),

$$O_p(O^q(G))=O_p(G)\le Z_{\mathcal{U}}(G)$$

and so G is p-supersoluble. This contradiction shows that we have (c).

(d) $O_p(G)\le Z_\infty(G)$.

In view of Lemma 1.1, $O_p(G)\ne 1$. Let H/K be any chief factor of G below $O_p(G)$. By Claim (a), $|H/K|=p$. Hence $G/C_G(H/K)$ a cyclic group of exponent dividing $p-1$. Suppose that $C_G(H/K)\ne G$. The for some prime $q\ne p$ we have $O^q(G)\ne G$, which contradicts (c). Hence $C_G(H/K)=G$ for each chief factor H/K of G below $O_p(G)$.

The final contradiction for (2).

Let H be a minimal non-p-nilpotent subgroup of G. Then, by [3, Chapter IV, Theorem 5.4], $H=P\rtimes Q$ is a Schmidt subgroup of G, where P is a Sylow p-subgroup of H and Q is a Sylow q-subgroup of H for some prime Moreover, P is of exponent p or of exponent 4 (if P is a non-abelian 2-group). By Claim (a), $P\le Z_{\mathcal{U}}(H)$, so $|P|=p$. But then by Lemma 1.1 and (d),

$$P\le Z_\infty(G)\cap H\le Z_\infty(H),$$

so H is nilpotent. This contradiction completes the proof of (2).

(3) $O_p(N)$ *is a Sylow p-subgroup of N and* $O_p(N)\le Z_{\mathcal{U}}(G)$.

By (1), $O_{p'}(N)=1$, so N is supersoluble and $O_p(N)$ is a Sylow p-subgroup of N by Lemma 1.9. On the other hand, $O_p(N)$ is characteristic in N and hence it is normal in G. Finally, $O_p(N)\le Z_{\mathcal{U}}(G)$ by Lemma 1.6.

Final contradiction.

Every chief factor of G below of N either is cyclic or is a p'-group by Claim (3). Hence G is p-supersoluble since G/N is p-super-soluble by hypothesis, contrary to the choice of G. □

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