FINITE GROUPS WITH RESTRICTIONS ON TWO MAXIMAL SUBGROUPS

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Introduction

Throughout this paper, all groups are finite and \( G \) always denotes a finite group.

A subgroup \( A \) of a group \( G \) is called seminormal in \( G \), if there exists a subgroup \( B \) such that \( G = AB \) and \( AB = BA \neq G \) for every proper subgroup \( B \) of \( B \).

Groups with some seminormal subgroups were investigated in works of many authors, see, for example, [1]–[10]. In particular, the supersolubility of a group with seminormal Sylow subgroups was obtained in [7], [9]. In [6] the supersolubility of a group with seminormal 2-maximal subgroups was proved. In [10] first two authors obtained the sufficient conditions for the supersolubility of \( G \) under the condition that all Sylow subgroups or all maximal subgroups of two non-conjugate maximal subgroups of \( G \) are seminormal in \( G \). We introduce the new concept that unites subnormality and seminormality.

Definition. A subgroup \( A \) of a group \( G \) is called semisubnormal in \( G \), if either \( A \) is subnormal in \( G \), or is seminormal in \( G \).

Let \( M \) and \( H \) be non-conjugate maximal subgroups of \( G \). In the present paper we proved the supersolubility of a group \( G \) under the condition that all Sylow subgroups of \( M \) and \( H \) are semisubnormal in \( G \). We also obtained the nilpotency of the second derived subgroup \((G')'\) of a group \( G \) under the condition that all maximal subgroups of \( M \) and \( H \) are semisubnormal in \( G \).

1 Preliminary results

We use the standard terminology of [11], [12]. Recall that \( A^G = \{ A^g \mid g \in G \} \) is the subgroup generated by all subgroups of \( G \) that are conjugate to \( A \). Denote by \( \pi(G) \) the set of all prime divisors of order of \( G \) and by \( |G: A| \) the index of subgroup \( A \) in \( G \). We use \( N < G \) to denote a normal subgroup \( N \) of \( G \). For maximal subgroup \( M \) of \( G \) we will use the following notation: \( M \triangleleft < G \). We write \( O_p(G) \) to denote the greatest normal \( p \)-subgroups of \( G \). The semidirect product of a normal subgroup \( A \) and a subgroup \( B \) is written as follows: \( A \rtimes B \). A subgroup \( U \) is called subnormal in \( G \), if there exist the subgroups \( U_0, U_1, \ldots, U_s \) such that

\[
U = U_0 \triangleleft U_1 \triangleleft \ldots \triangleleft U_s \triangleleft G.
\]

Let \( \mathfrak{F} \) be a non-empty formation. If \( G \) is a group then \( G^\mathfrak{F} \) denotes the \( \mathfrak{F} \)-residual of \( G \), that is the intersection of all those normal subgroups \( N \) of \( G \) for which \( G/N \in \mathfrak{F} \). We define \( \mathfrak{F} \circ \mathfrak{F} = \{ G \mid G^\mathfrak{F} \in \mathfrak{F} \} \) and call \( \mathfrak{F} \circ \mathfrak{F} \) the formation product of \( \mathfrak{F} \) and \( \mathfrak{F} \), see [13, IV, 1.7]. As usually, \( \mathfrak{F}^2 = \mathfrak{F} \circ \mathfrak{F} \). A formation
is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper $\mathcal{N}$, $\mathcal{U}$ and $\mathcal{A}$ denote the formations of all nilpotent, all supersoluble and all abelian groups respectively. The other definitions and terminology about formations could be referred to [11], [13], [14].

**Lemma 1.1.** (1) If $H$ is a semisubnormal subgroup of $G$ and $H \leq X \leq G$, then $H$ is semisubnormal in $X$. (2) If $H$ is a semisubnormal subgroup of $G$ and $N$ is normal in $G$, then $HN/\mathcal{N}$ is semisubnormal in $G/\mathcal{N}$. (3) If $H$ is a semisubnormal subgroup of $G$ and $Y$ is a non-empty set of elements from $G$, then $H^\triangledown = \{H^y \mid y \in Y\}$ is semisubnormal in $G$. In particular, $H^\triangledown$ is semisubnormal in $G$ for any $g \in G$.

**Proof.** If $H$ is subnormal in $G$, then the statements (1)–(3) are true, see [11, Lemma 2.41, Theorem 2.43]. If $H$ is seminormal, then this statements was proved in [8, Lemma 2]. Thus the statements (1)–(3) are true.

**Lemma 1.2.** (1) Let $p$ be the greatest in $\pi(G)$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ is semisubnormal in $G$, then $P$ is normal in $G$. (2) If any Sylow subgroup of $G$ is semisubnormal in $G$, then $G$ is supersoluble. (3) Let $H$ be a maximal subgroup of $G$. If $H$ is semisubnormal in $G$, then the index of $H$ in $G$ is a prime. (4) If every maximal subgroup of $G$ is semisubnormal in $G$, then $G$ is supersoluble. (5) If the index of $H$ in $G$ is a prime, then $H$ is semisubnormal in $G$.

**Proof.** (1) It is clear that if $P$ is subnormal in $G$, then $P$ is normal in $G$. If $P$ is seminormal in $G$ and $p$ is greatest in $\pi(G)$, then by [7, Lemma 4], $P$ is normal in $G$.

(2) Suppose that $G$ has at least one subnormal Sylow subgroup $P$. Then $P$ is normal in $G$ and therefore is seminormal in $G$. Hence any Sylow subgroup of $G$ is seminormal in $G$. By [7, Corollary 6], $G$ is supersoluble.

(3) If $H$ is subnormal in $G$, then $H$ is normal in $G$ and by [11, Lemma 3.17 (6)], $|G: H|$ is prime. Let $H$ be a seminormal subgroup in $G$ and $K$ be a subgroup of $G$ such that $HK = G$ and $HK_r$ is a proper subgroup of $G$ for every proper subgroup $K_r$ of $K$. Let prime $r$ divides the index $|G: H|$ and $R$ be a Sylow $r$-subgroup of $K$. Then $HR = G$ and $G = H \langle x \rangle$ for $x \in R \setminus H$. We choose an element $x$ such that its order is the smallest. Then $H \langle x' \rangle = \langle x' \rangle H = H$ and $|G: H| = r$.

(4) Let $M$ be a maximal subgroup of $G$. By (3), the index of $M$ in $G$ is a prime. By [12, VI.9.2 (2)], $G$ is supersoluble.

(5) Let $|G: H| = r$ and $R$ be a Sylow $r$-subgroup of $G$. Then $R$ is not contained in $H$ and there exists an element $x \in R \setminus H$. Let $|x| = r^a$ and $\langle x \rangle \cap H = r^a$. It is obvious that $a > a_r$, hence

$$\langle x \rangle H = \left(\frac{|\langle x \rangle|}{|\langle x \rangle \cap H|}\right)r^a \geq |G|/|\langle x \rangle H| = G.$$

Now $x'$ belongs to $H$ and $H$ is seminormal in $G$, and therefore is semisubnormal in $G$.

**Lemma 1.3.** (1) If $A$ is a semisubnormal $2$-nilpotent subgroup of $G$, then $A^G$ is soluble. (2) Let $p$ be the smallest prime divisor of order of $G$. If $A$ is semisubnormal in $G$ and $p$ does not divide the order of $A$, then $p$ does not divide the order of $A^G$.

**Proof.** (1) If $A$ is subnormal in $G$, then by [11, Theorem 5.31], $A^G$ is soluble. If $A$ is seminormal in $G$, then $A^G$ is soluble by [8, Lemma 10].

(2) If $A$ is a subnormal $p'$-subgroup of $G$, then by [11], $A^G$ is a $p'$-subgroup. If $A$ is a seminormal $p'$-subgroup of $G$, then $A^G$ is a $p'$-subgroup by [8, Lemma 11].

**Lemma 1.4** [15, Lemma 6]. Let $G$ be a soluble group. Assume that $G \notin \mathcal{U}$, but $G/K \in \mathcal{U}$ for every non-trivial normal subgroup $K$ of $G$. Then:

(1) $G$ contains a unique minimal normal subgroup $N$, $N = F(G) = \mathcal{O}_p(G) = C_\mathcal{O}(N)$ for some $p \in \pi(G)$;

(2) $Z(G) = \mathcal{O}_p(G) = \Phi(G) = 1$;

(3) $G$ is primitive, $G = N \rtimes M$, where $M$ is maximal in $G$ with trivial core;

(4) $N$ is an elementary abelian subgroup of order $p^n$, $n > 1$;

(5) if $V$ is a subgroup $G$ and $G = VN$, then $V = M'$ for some $x \in G$.

**Lemma 1.5.** Let $\mathcal{F}$ be a formation. Then $\mathcal{N} \circ \mathcal{F}$ is a saturated formation.

**Proof.** According to [14], the product $\mathcal{N} \circ \mathcal{F}$ is a local formation. Since saturated formation and local formation are equivalent concepts, $\mathcal{N} \circ \mathcal{F}$ is a saturated formation.

**Lemma 1.6.** Let $\mathcal{F}$ be a saturated formation and $G$ be a group. Assume that $G \notin \mathcal{F}$, but $G/N \in \mathcal{F}$ for all non-trivial normal subgroups $N$ of $G$. Then $G$ is a primitive group.

**Proof.** Since $\mathcal{F}$ is a saturated formation, it follows that $\Phi(G) = 1$ and $G$ contains a unique minimal normal subgroup $N$. For some maximal subgroup $M$ of $G$, we have $G = NM$, because $\Phi(G) = 1$. It is obvious that the core $M_G = 1$. Hence $G$ is a primitive group.
Lemma 1.7 [11, Theorem 4.40–4.42]. Let $G$ be a soluble primitive group and $M$ is a primitivator of $G$. Then the following statements hold:

1. $\Phi(G) = 1$;
2. $F(G) = C_{G}(F(G)) = O_{p}(G)$ and $F(G)$ is an elementary abelian subgroup of order $p^{n}$ for some prime $p$ and some positive integer $n$;
3. $G$ contains a unique minimal normal subgroup $N$ and moreover, $N = F(G)$;
4. $G = F(G) \times M$ and $O_{p}(M) = 1$.

Lemma 1.8 [18]. Let $G$ be a minimal nonsupersoluble group. Then the following holds:

1. $G$ is soluble;
2. $G$ contains a unique normal Sylow subgroup $P$ and $P = G^{u}$;
3. $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ such that $|P/\Phi(P)| > p$.

2 Supersolubility of a group with semisubnormal Sylow subgroups of two maximal subgroups

Lemma 2.1. Let $M$ be a maximal subgroup of $G$. If all Sylow subgroups of $M$ is semisnormal in $G$, then $[G : M]$ is a prime, $M$ and $G/M_{G}$ is supersoluble. In particular, $G$ is soluble.

Proof. First we prove that $G$ is soluble. We use induction on the order of $M$. Let $R$ be an arbitrary Sylow subgroup of $M$. By Lemma 1.1, $R$ is semisnormal in $M$. Because it is true for any Sylow subgroup of $M$, it follows that $M$ is supersoluble by Lemma 1.2 (2). In particular, $M$ is 2-nilpotent. Hence every subgroup of $M$ is also 2-nilpotent. By Lemma 1.3 (1), $R^{G}$ is soluble. If $MR^{G} = G$, then $G$ is soluble, since $G/R^{G} = M/R^{G} \subseteq M/MR^{G}$ is supersoluble. Let $R^{G} \leq M$. Hence $G/R^{G}$ has a maximal subgroup $M/R^{G}$. Let $S/R^{G}$ be a Sylow $t$-subgroup of $M/R^{G}$ and $T$ be a Sylow $t$-subgroup of $S$. By Lemma 1.1.65, $TR^{G}$ is a Sylow $t$-subgroup of $S/R^{G}$. Then $S = TR^{G}$ and $T$ is a Sylow $t$-subgroup of $M$. By hypothesis, $T$ is semisnormal in $G$ and by Lemma 1.1, $TR^{G} = S/R^{G}$ is semisnormal in $G/R^{G}$. Then by induction, $G/R^{G}$ is soluble, consequently $G$ is soluble. So the supersolubility of $G$ is proved.

We use induction on the order of $G$ and prove that $G/M_{G}$ is supersoluble. If $M_{G} \neq 1$, then $M/M_{G}$ is a maximal subgroup of $G/M_{G}$. As in the previous indent it is easy to verify that the quotient $G/M_{G}$ with maximal subgroup $M/M_{G}$ satisfy all conditions of the lemma. By induction, $(G/M_{G})/(M/M_{G})_{G/M_{G}}$ is supersoluble. Since $(M/M_{G})_{G/M_{G}} = 1$, it follows that $G/M_{G}$ is supersoluble and $[G/M_{G} : M/M_{G}] = [G : M]$ is a prime.

Therefore we consider that $M_{G} = 1$. Now $G$ is primitive and $G = N \times M_{G}$, where $N$ is a $r$-subgroup. Since $M$ is supersoluble, it follows that $M = P \times T$, where $P = M_{P}$ is a Sylow $p$-subgroup for the greatest $p \in \pi(M)$. Let $p = r$. Then $O_{p}(M) = 1$, a contradiction. Hence $p \neq r$ and $P$ is a Sylow $p$-subgroup of $G$. Suppose that $P$ is subnormal in $G$. Then $P \triangleleft G$, a contradiction. Consequently $P$ is seminormal in $G$. Now $G$ has a subgroup $U$ such that $G = PU$. It is clear that $N \subseteq U$. Let $x$ be an element of prime order that lies in $N$. Then $P(x) \leq G$. If $p > r$, then $P \triangleleft P(x)$. Thus $P \triangleleft \langle M, x \rangle = G$, a contradiction. If $p < r$, then $N$ is a Sylow $r$-subgroup of $G$, since $p$ is the greatest in $\pi(M)$. Now all Sylow subgroups of $G$ is semisnormal in $G$. By Lemma 1.2 (2), $G$ is supersoluble. Hence $[G : M]$ is a prime. The lemma is proved. □

Remark 2.1. Soluble groups containing a supersoluble subgroup of prime index were studied in [16], [17].

Theorem 2.1. Suppose that $G$ has two nonconjugate maximal subgroups $H$ and $K$. If all Sylow subgroups of $H$ and of $K$ are semisnormal in $G$, then $G$ is supersoluble.

Proof. We use induction on the order of $G$. By Lemma 2.1, $G$ is soluble, $H$ and $K$ are supersoluble. Besides, quotients $G/H_{G}$ and $G/K_{G}$ are supersoluble. In particular, indices of subgroups $H$ and $K$ in $G$ are primes. By Lemma 1.2 (5), subgroups $H$ and $K$ are semisnormal in $G$.

Let $N$ be an arbitrary non-trivial normal subgroup in $G$. If $N$ is not contained in $H \cap K$, then $N$ is either not contained in $H$, or $N$ not contained in $K$. If $N$ is not contained in $H$, then $HN = G$ and $G/N = HN/N \cong H/H \cap N$ is supersoluble. Similarly, if $N$ is not contained in $K$, then $KN = G$ and $G/N$ is supersoluble. Let $N \subseteq H \cap K$. Then $G/N = (H/N)(K/N)$. Let $\overline{R}$ be a Sylow $r$-subgroup of $H/N$. Then $H$ has a Sylow $r$-subgroup $R$ such that $\overline{R} = RN/N$. By hypothesis, $R$ is semisnormal in $G$. By Lemma 1.1 (2), $\overline{R} = RN/N$ is semisnormal in $G/N$. Similarly, every Sylow subgroup of $K/N$ is semisnormal in $G/N$. By induction, $G/N$ is supersoluble.

So, in any case $G/N$ is supersoluble. By Lemma 1.6, $G$ is primitive and statements (1)–(5) of the Lemma 1.4 are true. In particular, $|N| = p^{n} > p$.

If $N \not\subseteq H$, then $G = N \times H$. Since $H$ is semisnormal in $G$, then by Lemma 1.2 (5), $|N| = |G : H|$ is a prime, a contradiction. Similarly, in the case when $N \not\subseteq K$. Hence we consider that $N \subseteq H \cap K$.

Because $H$ and $K$ are supersoluble and $N = C_{G}(N)$,
we have \( p \) is the greatest in \( \pi(H) \) and in \( \pi(K) \), hence \( p \) is the greatest in \( \pi(G) \). Since \( O_p(G/N) = 1 \) and \( G/N \) is supersoluble, \( p \) does not divide the order of \( G/N \) and \( N \) is a Sylow \( p \)-subgroup of \( G \).

Let \( N_i \leq N \), \( |N_i| = p \) and \( R \) be a Sylow \( r \)-subgroup of \( M \). Since \( M = G_r = H_rK_r \), it follows that \( R = H_rK_r \), for some Sylow \( r \)-subgroups \( H_r \) and \( K_r \) of \( H \) and of \( K \) respectively. By hypothesis, subgroups \( H_r \) and \( K_r \) are semisubnormal in \( G \). If \( H_r \) is subnormal in \( G \), then by [14, Corollary 7.72 (1)], \( H_r \leq O_p(G) \leq O_p(G) = 1 \). Similarly, if \( K_r \) is subnormal in \( G \), then \( K_r \leq O_p(G) = 1 \). Consequently \( H_r \) and \( K_r \) are semisubnormal in \( G \). Hence there exists a subgroup \( U \) such that \( G = H_rU \) and \( H_r \) is permutable with any subgroup of \( U \). Since \( N \leq U \), we have \( H_r \) is permutable with \( N_r \). Similarly, \( K_r \) is permutable with \( N_i \). Hence \( R \) is permutable with \( N_i \). It is true for any \( r \in \pi(M) \). Therefore \( M \) is permutable with \( N_i \). Now \( MN_i \) is a subgroup of \( G \) and \( N_i \) is normal in \( MN_i \). Since \( N \) is abelian, \( N_i \) is normal in \( NM = G \), a contradiction with \( |N| > p \).

The theorem is proved. \( \Box \)

**Example 2.1.** The group \( G = \text{PSL}(2,5) \) has maximal subgroups \( H = Z_4 \times Z_2 \) and \( K = Z_3 \times Z_2 \). Maximal subgroups of Sylow subgroups of \( H \) and \( K \) are trivial, hence are semisubnormal in \( G \), but \( G \) is not soluble. Therefore the semisubnormality of maximal subgroups of Sylow subgroups of \( H \) and \( K \) under the conditions of Theorem 2.1 is not sufficient condition for the solubility of \( G \).

**Corollary 2.1.** [10, Theorem E]. Suppose that \( G \) has two non-conjugate maximal subgroups \( H \) and \( K \). If all Sylow subgroups of \( H \) and of \( K \) are semisubnormal in \( G \), then \( G \) is supersoluble.

3 On a group with semisubnormal maximal subgroups of two maximal subgroups

**Lemma 3.1.** Let \( M \) be a maximal subgroup of \( G \). If all maximal subgroups of \( M \) are semisubnormal in \( G \), then \( G \) is soluble.

*Proof.* We use induction on the order of \( G \). Let \( K \) be a maximal subgroup of \( M \). By hypothesis, \( K \) is semisubnormal in \( G \) and by Lemma 1.1 (1), \( K \) is semisubnormal in \( M \). By Lemma 1.2 (4), \( M \) is supersoluble and consequently is 2-nilpotent. Then \( K \) is also 2-nilpotent and by Lemma 1.3, \( K^G \) is soluble. Since \( M \) is a maximal subgroup of \( G \), then either \( MK^G = G \) or \( K^G \leq M \). If \( MK^G = G \), then \( G \) is soluble. Let \( K^G \leq M \). Then \( M/K^G \) is a maximal subgroup of \( G/K^G \). Let \( S \) be a maximal subgroup of \( M/K^G \). Then \( M \) has a maximal subgroup \( S \) such that \( K^G \leq S \) and \( S = S/K^G \). By hypothesis, \( S \) is semisubnormal in \( G \). By Lemma 1.1, \( SK^G/K^G \) is semisubnormal in \( G/K^G \). Since \( K^G \leq S \), we have \( S = SK^G \) and \( S/K^G \) is semisubnormal in \( G/K^G \). By induction, \( G/K^G \) is soluble. Then \( G \) is soluble. The lemma is proved. \( \Box \)

**Example 3.1.** In the condition of the Lemma 3.1, the index \( |G:M| \) may not be a prime. For example, the group \( G = A_5 = A \times B \). The subgroup \( B \) has the order 3. Besides, \( B \) is maximal in \( G \) and all maximal subgroups of \( B \) are semisubnormal in \( G \), but \( |G:B| = 4 \) is not a prime.

**Example 3.2.** The alternating group \( G = A_4 \) of degree 4 has two non-conjugate maximal subgroups \( A = Z_3 \) and \( B = Z_3 \times Z_2 \). It is clear that all maximal subgroups of \( A \) and of \( B \) are semisubnormal in \( G \). But \( G \) is non-supersoluble.

**Theorem 3.1.** Let \( H \) and \( K \) are non-conjugate maximal subgroups of \( G \). If all maximal subgroups of \( H \) and of \( K \) are semisubnormal in \( G \), then the second derived subgroup \( (G')'' \) is nilpotent.

*Proof.* Note that the nilpotency of the second derived subgroup \( (G')'' \) is equivalent to \( G \in \mathfrak{N} \rtimes \mathfrak{A}^2 \).

Assume that the claim is false and let \( G \) be a minimal counterexample. By Lemma 3.1, \( G \) is soluble. By Lemma 1.1 (1), every maximal subgroup of \( H \) is semisubnormal in \( H \) and by Lemma 1.2 (4), \( H \) is supersoluble. Similarly, \( K \) is supersoluble.

Let \( N \) be an arbitrary non-trivial normal subgroup of \( G \). Then either \( HN = G \), or \( HN = H \). If \( HN = G \), then

\[
G/N = HN/N \cong H/H \cap N \in \mathfrak{N} \rtimes \mathfrak{A} \subseteq \mathfrak{N} \rtimes \mathfrak{A}^2.
\]

If \( HN = H \), then \( N \subseteq H \). Similarly either \( KN = G \) and \( G/N \in \mathfrak{N} \rtimes \mathfrak{A}^2 \), or \( N \leq K \). Let \( N \leq H \cap K \). Then \( G/N \) has non-conjugate maximal subgroups \( H/N \) and \( K/N \). If \( \overline{S} \) is a maximal subgroup of \( H/N \), then \( H \) has a maximal subgroup \( S \) such that \( \overline{S} = S/N \). By hypothesis, \( S \) is semisubnormal in \( G \) and by Lemma 1.1 (2), \( \overline{S} = S/N \) is semisubnormal in \( G/N \). Similarly, if \( \overline{T} \) is a maximal subgroup of \( K/N \), then it is semisubnormal in \( G/N \). Therefore for \( G/N \) with non-conjugate maximal subgroups \( H/N \) and \( K/N \) the conditions of the theorem are satisfied. By induction, \( G/N \in \mathfrak{N} \rtimes \mathfrak{A}^2 \).

By Lemmas 1.5 and 1.6, \( G \) is primitive. Then for \( G \) we have Lemma 1.7. Hence \( \Phi(G) = 1 \) if \( G \) contains a unique minimal normal subgroup \( N \) such that \( N = C_G(N) \).

Suppose that at least one of the subgroups \( H \) or \( K \) is normal in \( G \). For example, let \( H \) be normal in \( G \). Then \( |G:H| = q \) and by [16, Theorem 1], \( G = N \rtimes T \), where \( T \) has abelian subgroup of index \( q \). Since \( T \in \mathfrak{A}^2 \), it follows that \( G \in \mathfrak{N} \rtimes \mathfrak{A}^2 \), a contradiction.
Therefore in the future we assume that the subgroups $H$ and $K$ are non-normal. By [16, Theorem 2], $G = N \times T$, where

$$T / C_T(N) \cong T / C_T(N)$$

and $< z > := Z(T)$. Since $N = C_N(T)$, we have $C_T(N) = 1$ and $T = \langle < y > \times < z > \rangle$. Hence $< z > := Z(T)$. Thus $T \in \mathfrak{A}$ and $G \in \mathfrak{N} \times \mathfrak{A}$, a contradiction. The theorem is proved.

**Corollary 3.1.1.** If all 2-maximal subgroups of $G$ are semisubnormal in $G$, then the derived subgroup $G'$ is nilpotent.

**Proof.** Note that the nilpotency of the derived subgroup $G'$ is equivalent to $G \in \mathfrak{N} \mathfrak{G}$.

Assume that the claim is false and let $G$ be a minimal counterexample. It is easy to show that $G / N$ satisfies the hypothesis of the corollary, where $N$ is an arbitrary non-trivial normal subgroup of $G$. By induction, $G / N \in \mathfrak{N} \mathfrak{G}$. Hence by Lemmas 1.5 and 1.6, $G$ is primitive.

Let $M$ be an arbitrary maximal subgroup of $G$. Then by Lemmas 1.1 (1) and 1.2 (4), $M$ is supersolvable. Hence either $G$ is supersoluble, or $G$ is a minimal non-supersolvable group.

If $G$ is supersoluble, then $G \in \mathfrak{N} \mathfrak{G}$ by [11, Theorem 4.52], a contradiction.

Let $G$ be a minimal non-supersolvable group. By Lemmas 1.7 and 1.8, $G$ is soluble, $P$ is a unique minimal normal subgroup of $G$, $|P| > p$ and $P$ is a Sylow $p'$-subgroup of $G$ such that $G = P \times M$, where $M$ is a maximal subgroup of $G$. Besides, $M$ is a Hall $p'$-subgroup of $G$. Let $P_1$ be a subgroup of prime order $p$.

If $M$ is abelian, then $G \in \mathfrak{N} \mathfrak{G}$, a contradiction. Therefore we assume that $M$ is non-abelian. Hence $M$ has maximal subgroups $M_1$ and $M_2$ such that $M = \langle M_1, M_2 \rangle$. If at least one of the subgroups $M_1$ or $M_2$ is normal in $G$, then $O_p(G) \neq 1$, a contradiction. Thus $M_1$ and $M_2$ are seminormal in $G$. Hence there are the subgroups $V_1$ and $V_2$ such that

$$M_1 V_1 = M_2 V_1 = G, M_1 P = P M_1, M_2 P = P M_2,$$

because $P \leq V_1 \cap V_2$. Then $M_1 \leq N_G(P)$ and $M_2 \leq N_G(P)$. Therefore $P_1 \leq G = P \{ M_1, M_2 \}$, a contradiction. The corollary is proved.

**REFERENCES**


