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КОНЕЧНЫЕ ГРУППЫ С ОГРАНИЧЕНИЯМИ НА ДВЕ МАКСИМАЛЬНЫЕ ПОДГРУППЫ

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FINITE GROUPS WITH RESTRICTIONS ON TWO MAXIMAL SUBGROUPS

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Подгруппа A называется *полунормальной* в группе G, если существует подгруппа B такая, что $G = AB$ и AB_1 – собственная в G подгруппа для каждой собственной подгруппы B_1 из B. Если подгруппа A либо субнормальна в G, либо полунормальна в G, то A называется *полусубнормальной* в группе G. В настоящей работе доказана сверхразрешимость группы G при условии, что все силовские подгруппы из двух несопряженных максимальных подгрупп полусубнормальны в группе G. Установлена нильпотентность второго коммутанта $(G')'$ группы G при условии, что все максимальные подгруппы из двух несопряженных максимальных подгрупп полусубнормальны в группе G.

Ключевые слова: *сверхразрешимая группа*, *полусубнормальная подгруппа*, *коммутант*, *силовская подгруппа*, *максимальная подгруппа.*

A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AB_1 is a proper subgroup of G for every proper subgroup B_1 of B. We introduce the new concept that unites subnormality and seminormality. A subgroup A of a group G is called *semisubnormal* in G, if either A is subnormal in G, or is seminormal in G. In this paper we proved the supersolubility of a group G under the condition that all Sylow subgroups of two non-conjugate maximal subgroups of G are semisubnormal in G. Also we obtained the nilpotency of the second derived subgroup $(G')'$ of a group G under the condition that all maximal subgroups of two non-conjugate maximal subgroups are semisubnormal in G.

Keywords: *supersoluble groups*, *semisubnormal subgroup*, *derived subgroup*, *Sylow subgroup*, *maximal subgroup.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group.

A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and $AB_1 = B_1A \neq G$ for every proper subgroup B_1 of B.

Groups with some seminormal subgroups were investigated in works of many authors, see, for example, [1]–[10]. In particular, the supersolubility of a group with seminormal Sylow subgroups was obtained in [7], [9]. In [6] the supersolubility of a group with seminormal 2-maximal subgroups was proved. In [10] first two authors obtained the sufficient conditions for the supersolubility of G under the condition that all Sylow subgroups or all maximal subgroups of two non-conjugate maximal subgroups of G are seminormal in G.

We introduce the new concept that unites subnormality and seminormality.

Definition. A subgroup A of a group G is called *semisubnormal* in G, if either A is subnormal in G, or is seminormal in G.

Let M and H be non-conjugate maximal subgroups of G. In the present paper we proved the supersolubility of a group G under the condition that all Sylow subgroups of M and H are semisubnormal in G. We also obtained the nilpotency of the second derived subgroup $(G')'$ of a group G under the condition that all maximal subgroups of M and H are semisubnormal in G.

1 Preliminary results

We use the standart terminology of [11], [12]. Recall that $A^G = \langle A^g \mid g \in G \rangle$ is the subgroup generated by all subgroups of G that are conjugate to A. Denote by $\pi(G)$ the set of all prime divisors of order of G and by $|G : A|$ the index of subgroup A in G. We use $N \lhd G$ to denote a normal subgroup N of G. For maximal subgroup M of G we will use the following notation: $M < \cdot G$. We write $O_p(G)$ to denote the greatest normal p-subgroups of G. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. A subgroup U is called subnormal in G, if there exist the subgroups $U_0, U_1, \ldots, U_s$ such that

$$U = U_0 \lhd U_1 \lhd \ldots \lhd U_{s-1} \lhd U_s = G.$$

Let $\mathfrak{F}$ be a non-empty formation. If G is a group then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. We define $\mathfrak{F} \circ \mathfrak{H} = \{G \mid G^{\mathfrak{H}} \in \mathfrak{F}\}$ and call $\mathfrak{F} \circ \mathfrak{H}$ the formation product of $\mathfrak{F}$ and $\mathfrak{H}$, see [13, IV, 1.7]. As usually, $\mathfrak{F}^2 = \mathfrak{F} \circ \mathfrak{F}$. A formation

$\mathfrak{F}$ is said to be saturated if $G/\Phi(G)\in\mathfrak{F}$ implies that $G\in\mathfrak{F}$. In this paper $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{A}$ denote the formations of all nilpotent, all supersoluble and all abelian groups respectively. The other definitions and terminology about formations could be referred to [11], [13], [14].

***Lemma* 1.1.** (1) *If H is a semisubnormal subgroup of G and $H\le X\le G$, then H is semisubnormal in X.*

(2) *If H is a semisubnormal subgroup of G and N is normal in G, then HN/N is semisubnormal in G/N.*

(3) *If H is a semisubnormal subgroup of G and Y is a non-empty set of elements from G, then*

$$H^Y=\langle H^y \mid y\in Y\rangle$$

is semisubnormal in G. In particular, H^g is semisubnormal in G for any $g\in G$.

Proof. If *H* is subnormal in *G*, then the statements (1)–(3) are true, see [11, Lemma 2.41, Theorem 2.43]. If *H* is seminormal, then this statements was proved in [8, Lemma 2]. Thus the statements (1)–(3) are true.

***Lemma* 1.2.** (1) *Let p be the greatest in $\pi(G)$ and P be a Sylow p-subgroup of G. If P is semisubnormal in G, then P is normal in G.*

(2) *If any Sylow subgroup of G is semisubnormal in G, then G is supersoluble.*

(3) *Let H be a maximal subgroup of G. If H is semisubnormal in G, then the index of H in G is a prime.*

(4) *If every maximal subgroup of G is semisubnormal in G, then G is supersoluble.*

(5) *If the index of H in G is a prime, then H is semisubnormal in G.*

Proof. (1) It is clear that if *P* is subnormal in *G*, then *P* is normal in *G*. If *P* is seminormal in *G* and *p* is greatest in $\pi(G)$, then by [7, Lemma 4], *P* is normal in *G*.

(2) Suppose that *G* has at least one subnormal Sylow subgroup *P*. Then *P* is normal in *G* and therefore is seminormal in *G*. Hence any Sylow subgroup of *G* is seminormal in *G*. By [7, Corollary 6], *G* is supersoluble.

(3) If *H* is subnormal in *G*, then *H* is normal in *G* and by [11, Lemma 3.17 (6)], $|G:H|$ is prime. Let *H* be a seminormal subgroup in *G* and *K* be a subgroup of *G* such that $HK=G$ and HK_1 is a proper subgroup of *G* for every proper subgroup K_1 of *K*. Let prime *r* divides the index $|G:H|$ and *R* be a Sylow *r*-subgroup of *K*. Then $HR=G$ and $G=H\langle x\rangle$ for $x\in R\setminus H$. We choose an element *x* such that its order is the smallest. Then $H\langle x^r\rangle=\langle x^r\rangle H=H$ and $|G:H|=r$.

(4) Let *M* be a maximal subgroup of *G*. By (3), the index of *M* in *G* is a prime. By [12, VI.9.2 (2)], *G* is supersoluble.

(5) Let $|G:H|=r$ and *R* be a Sylow *r*-subgroup of *G*. Then *R* is not contained in *H* and there exists an element $x\in R\setminus H$. Let $|x|=r^a$ and $|\langle x\rangle\cap H|=r^{a_1}$. It is obvious that $a>a_1$, hence

$$|\langle x\rangle H|=\frac{|\langle x\rangle||H|}{|\langle x\rangle\cap H|}=\frac{r^a\frac{|G|}{r}}{r^{a_1}}\ge|G|,\ \langle x\rangle H=G.$$

Now x^r belongs to *H* and *H* is seminormal in *G*, and therefore is semisubnormal in *G*.

***Lemma* 1.3.** (1) *If A is a semisubnormal 2-nilpotent subgroup of G, then A^G is soluble.*

(2) *Let p be the smallest prime divisor of order of G. If A is semisubnormal in G and p does not divide the order of A, then p does not divide the order of A^G.*

Proof. (1) If *A* is subnormal in *G*, then by [11, Theorem 5.31], A^G is soluble. If *A* is seminormal in *G*, then A^G is soluble by [8, Lemma 10].

(2) If *A* is a subnormal p'-subgroup of *G*, then by [11], A^G is a p'-subgroup. If *A* is a seminormal p'-subgroup of *G*, then A^G is a p'-subgroup by [8, Lemma 11].

***Lemma* 1.4** [15, Lemma 6]. *Let G be a soluble group. Assume that $G\notin\mathfrak{U}$, but $G/K\in\mathfrak{U}$ for every non-trivial normal subgroup K of G. Then*:

(1) *G contains a unique minimal normal subgroup N, $N=F(G)=O_p(G)=C_G(N)$ for some $p\in\pi(G)$*;

(2) $Z(G)=O_{p'}(G)=\Phi(G)=1$;

(3) *G is primitive*; $G=N\rtimes M$, *where M is maximal in G with trivial core*;

(4) *N is an elementary abelian subgroup of order p^n, $n>1$*;

(5) *if V is a subgroup G and $G=VN$, then $V=M^x$ for some $x\in G$.*

***Lemma* 1.5.** *Let $\mathfrak{F}$ be a formation. Then $\mathfrak{N}\circ\mathfrak{F}$ is a saturated formation.*

Proof. According to [14], the product $\mathfrak{N}\circ\mathfrak{F}$ is a local formation. Since saturated formation and local formation are equivalent concepts, $\mathfrak{N}\circ\mathfrak{F}$ is a saturated formation.

***Lemma* 1.6.** *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G\notin\mathfrak{F}$, but $G/N\in\mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Proof. Since $\mathfrak{F}$ is a saturated formation, it follows that $\Phi(G)=1$ and *G* contains a unique minimal normal subgroup *N*. For some maximal subgroup *M* of *G*, we have $G=NM$, because $\Phi(G)=1$. It is obvious that the core $M_G=1$. Hence *G* is a primitive group.

***Lemma* 1.7** [11, Theorem 4.40–4.42]. *Let G be a soluble primitive group and M is a primitivator of G. Then the following statements hold*:

(1) $\Phi(G)=1$;

(2) $F(G)=C_G(F(G))=O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime p and some positive integer n*;

(3) *G contains a unique minimal normal subgroup N and moreover*, $N=F(G)$;

(4) $G=F(G)\rtimes M$ *and* $O_p(M)=1$.

***Lemma* 1.8** [18]. *Let G be a minimal non-supersoluble group. Then the following holds*:

(1) *G is soluble*;

(2) *G contains a unique normal Sylow subgroup P and* $P=G^{\mathfrak{U}}$;

(3) $P/\Phi(P)$ *is a minimal normal subgroup of* $G/\Phi(P)$ *such that* $|P/\Phi(P)|>p$.

2 Supersolubility of a group with semisubnormal Sylow subgroups of two maximal subgroups

***Lemma* 2.1.** *Let M be a maximal subgroup of G. If all Sylow subgroups of M is semisubnormal in G, then* $|G:M|$ *is a prime*, *M and* G/M_G *is supersoluble. In particular*, *G is soluble.*

Proof. First we prove that *G* is soluble. We use induction on the order of *G*. Let *R* be an arbitrary Sylow subgroup of *M*. By Lemma 1.1, *R* is semisubnormal in *M*. Because it is true for any Sylow subgroup of *M*, it follows that *M* is supersoluble by Lemma 1.2 (2). In particular, *M* is 2-nilpotent. Hence every subgroup of *M* is also 2-nilpotent. By Lemma 1.3 (1), R^G is soluble. If $MR^G=G$, then *G* soluble, since $G/R^G=MR^G/R^G\cong M/M\cap R^G$ is supersoluble. Let $R^G\le M$. Hence G/R^G has a maximal subgroup M/R^G. Let S/R^G be a Sylow *t*-subgroup of M/R^G and *T* be a Sylow *t*-subgroup of *S*. By [11, Theorem 1.65], TR^G/R^G is a Sylow *t*-subgroup of S/R^G. Then $S=TR^G$ and *T* is a Sylow *t*-subgroup of *M*. By hypothesis, *T* is semisubnormal in *G* and by Lemma 1.1, $TR^G/R^G=S/R^G$ is semisubnormal in G/R^G. Then by induction, G/R^G is soluble, consequently *G* is soluble. So the solubility of *G* is proved.

We use induction on the order of *G* and prove that G/M_G is supersoluble. If $M_G\ne 1$, then M/M_G is a maximal subgroup of G/M_G. As in the previous indent it is easy to verify that the quotient G/M_G with maximal subgroup M/M_G satisfy all conditions of the lemma. By induction, $(G/M_G)/(M/M_G)_{G/M_G}$ is supersoluble. Since $(M/M_G)_{G/M_G}=1$, it follows that G/M_G is supersoluble and $|G/M_G:M/M_G|=|G:M|$ is a prime.

Therefore we consider that $M_G=1$. Now *G* is primitive and $G=N\rtimes M$, where *N* is a *r*-subgroup. Since *M* is supersoluble, it follows that $M=P\rtimes T$, where $P=M_p$ is a Sylow *p*-subgroup for the greatest $p\in\pi(M)$. Let $p=r$. Then $O_p(M)\ne 1$, a contradiction. Hence $p\ne r$ and *P* is a Sylow *p*-subgroup of *G*. Suppose that *P* is subnormal in *G*. Then $P\lhd G$, a contradiction. Consequently *P* is seminormal in *G*. Now *G* has a subgroup *U* such that $G=PU$. It is clear that $N\le U$. Let *x* be an elementof prime order that lies in *N*. Then $P\langle x\rangle\le G$. If $p>r$, then $P\lhd P\langle x\rangle$. Thus $P\lhd\langle M,x\rangle=G$, a contradiction. If $p<r$, then *N* is a Sylow *r*-subgroup of *G*, since *p* is the greatest in $\pi(M)$. Now all Sylow subgroups of *G* is semisubnormal in *G*. By Lemma 1.2 (2), *G* is supersoluble. Hence $|G:M|$ is a prime. The lemma is proved. □

***Remark* 2.1.** Soluble groups containing a supersoluble subgroup of prime index were studied in [16], [17].

***Theorem* 2.1.** *Suppose that G has two non-conjugate maximal subgroups H and K. If all Sylow subgroups of H and of K are semisubnormal in G, then G is supersoluble.*

Proof. We use induction on the order of *G*. By Lemma 2.1, *G* is soluble, *H* and *K* are supersoluble. Besides, quotients G/H_G and G/K_G are supersoluble. In particular, indices of subgroups *H* and *K* in *G* are primes. By Lemma 1.2 (5), subgroups *H* and *K* are semisubnormal in *G*.

Let *N* be an arbitrary non-trivial normal subgroup in *G*. If *N* is not contained in $H\cap K$, then *N* is either not contained in *H*, or *N* not contained in *K*. If *N* is not contained in *H*, then $HN=G$ and

$$G/N=HN/N\cong H/H\cap N$$

is supersoluble. Similarly, if *N* is not contained in *K*, then $KN=G$ and G/N is supersoluble. Let $N\le H\cap K$. Then $G/N=(H/N)(K/N)$. Let $\overline{R}$ be a Sylow *r*-subgroup of H/N. Then *H* has a Sylow *r*-subgroup *R* such that $\overline{R}=RN/N$. By hypothesis, *R* is semisubnormal in *G*. By Lemma 1.1 (2), $\overline{R}=RN/N$ is semisubnormal in G/N. Similarly, every Sylow subgroup of K/N is semisubnormal in G/N. By induction, G/N is supersoluble.

So, in any case G/N is supersoluble. By Lemma 1.6, *G* is primitive and statements (1)–(5) of the Lemma 1.4 are true. In particular, $|N|=p^n>p$. If $N\not\le H$, then $G=N\rtimes H$. Since *H* is semisubnormal in *G*, then by Lemma 1.2 (5), $|N|=|G:H|$ is a prime, a contradiction. Similarly, in the case when $N\not\le K$. Hence we consider that $N\le H\cap K$. Because *H* and *K* are supersoluble and $N=C_G(N)$, we have p is the greatest in $\pi(H)$ and in $\pi(K)$, hence p is the greatest in $\pi(G)$. Since $O_p(G/N)=1$ and G/N is supersoluble, p does not divide the order of G/N and N is a Sylow p-subgroup of G.

Let $N_1 \le N$, $|N_1|=p$ and R be a Sylow r-subgroup of M. Since $M = G_{p'} = H_{p'}K_{p'}$, it follows that $R = H_rK_r$ for some Sylow r-subgroups H_r and K_r of H and of K respectively. By hypothesis, subgroups H_r and K_r are semisubnormal in G. If H_r is subnormal in G, then by [14, Corollary 7.7.2 (1)], $H_r \le O_r(G) \le O_{p'}(G)=1$. Similarly, if K_r is subnormal in G, then $K_r \le O_{p'}(G)=1$. Consequently H_r and K_r are semisubnormal in G. Hence there exists a subgroup U such that $G=H_rU$ and H_r is permutable with any subgroup of U. Since $N \le U$, we have H_r is permutable with N_1. Similarly, K_r is permutable with N_1. Hence R is permutable with N_1. It is true for any $r \in \pi(M)$. Therefore M is permutable with N_1. Now MN_1 is a subgroup of G and N_1 is normal in MN_1. Since N is abelian, N_1 is normal in $NM=G$, a contradiction with $|N|>p$. The theorem is proved. □

***Example* 2.1.** The group $G = PSL(2,5)$ has maximal subgroups $H = Z_3 \rtimes Z_2$ and $K = Z_5 \rtimes Z_2$. Maximal subgroups of Sylow subgroups of H and K are trivial, hence are semisubnormal in G, but G is not soluble. Therefore the semisubnormality of maximal subgroups of Sylow subgroups of H and K under the conditions of Theorem 2.1 is not sufficient condition for the solubility of G.

***Corollary* 2.1.1** [10, Theorem E]. *Suppose that G has two non-conjugate maximal subgroups H and K. If all Sylow subgroups of H and of K are seminormal in G, then G is supersoluble.*

3 On a group with semisubnormal maximal subgroups of two maximal subgroups

***Lemma* 3.1.** *Let M be a maximal subgroup of G. If all maximal subgroups of M are semisubnormal in G, then G is soluble.*

Proof. We use induction on the order of G. Let K be a maximal subgroup of M. By hypothesis, K is semisubnormal in G and by Lemma 1.1 (1), K is semisubnormal in M. By Lemma 1.2 (4), M is supersoluble and consequently is 2-nilpotent. Then K is also 2-nilpotent and by Lemma 1.3, K^G is soluble. Since M is a maximal subgroup of G, then either $MK^G = G$, or $K^G \le M$. If $MK^G = G$, then G is soluble. Let $K^G \le M$. Then M/K^G is a maximal subgroup of G/K^G. Let $\overline{S}$ be a maximal subgroup of M/K^G. Then M has a maximal subgroup S such that $K^G \le S$ and $\overline{S} = S/K^G$. By hypothesis, S is semisubnormal in G. By Lemma 1.1, SK^G/K^G is semisubnormal in G/K^G. Since $K^G \le S$, we have $S = SK^G$ and S/K^G is semisubnormal in G/K^G. By induction, G/K^G is soluble. Then G is soluble. The lemma is proved. □

***Example* 3.1.** In the condition of the Lemma 3.1, the index $|G:M|$ may not be a prime. For example, the group $G = A_4 = A \rtimes B$. The subgroup B has the order 3. Besides, B is maximal in G and all maximal subgroups of B are semisubnormal in G, but $|G:B| = 4$ is not a prime.

***Example* 3.2.** The alternating group $G = A_4$ of degree 4 has two non-conjugate maximal subgroups $A = Z_3$ and $B = Z_2 \times Z_2$. It is clear that all maximal subgroups of A and of B are semisubnormal in G. But G is non-supersoluble.

***Theorem* 3.1.** *Let H and K are non-conjugate maximal subgroups of G. If all maximal subgroups of H and of K are semisubnormal in G, then the second derived subgroup $(G')'$ is nilpotent.*

Proof. Note that the nilpotency of the second derived subgroup $(G')'$ is equivalent to $G \in \mathfrak{N} \circ \mathfrak{A}^2$.

Assume that the claim is false and let G be a minimal counterexample. By Lemma 3.1, G is soluble. By Lemma 1.1 (1), every maximal subgroup of H is semisubnormal in H and by Lemma 1.2 (4), H is supersoluble. Similarly, K is supersoluble.

Let N be an arbitrary non-trivial normal subgroup in G. Then either $HN = G$, or $HN = H$. If $HN = G$, then

$$G/N = HN/N \cong H/H \cap N \in \mathfrak{N} \circ \mathfrak{A} \subseteq \mathfrak{N} \circ \mathfrak{A}^2.$$

If $HN = H$, then $N \le H$. Similarly either $KN = G$ and $G/N \in \mathfrak{N} \circ \mathfrak{A}^2$, or $N \le K$. Let $N \le H \cap K$. Then G/N has non-conjugate maximal subgroups H/N and K/N. If $\overline{S}$ is a maximal subgroup of H/N, then H has a maximal subgroup S such that $\overline{S} = S/N$. By hypothesis, S is semisubnormal in G and by Lemma 1.1 (2), $\overline{S} = S/N$ is semisubnormal in G/N. Similarly, if $\overline{T}$ is a maximal subgroup of K/N, then it is semisubnormal in G/N. Therefore for G/N with non-conjugate maximal subgroups H/N and K/N the conditions of the theorem are satisfied. By induction, $G/N \in \mathfrak{N} \circ \mathfrak{A}^2$.

By Lemmas 1.5 and 1.6, G is primitive. Then for G we have Lemma 1.7. Hence $\Phi(G)=1$ и G contains a unique minimal normal subgroup N such that $N = C_G(N)$.

Suppose that at least one of the subgroups H or K is normal in G. For example, let H be normal in G. Then $|G:H|=q$ and by [16, Theorem 1], $G = N \rtimes T$, where T has abelian subgroup of index q. Since $T \in \mathfrak{A}^2$, it follows that $G \in \mathfrak{N} \circ \mathfrak{A}^2$, a contradiction.

Therefore in the future we assume that the subgroups H and K are non-normal. By [16, Theorem 2], $G = N \rtimes T$, where

$$T / C_T(N) \cong \overline{T} = \langle y \rangle \rtimes (\langle t \rangle \langle z \rangle)$$

and $\langle z \rangle = Z(\overline{T})$. Since $N = C_G(N)$, we have $C_T(N) = 1$ and $T = (\langle y \rangle \times \langle z \rangle) \langle t \rangle$, because $\langle z \rangle = Z(T)$. Thus $T \in \mathfrak{A}^2$ and $G \in \mathfrak{N} \circ \mathfrak{A}^2$, a contradiction. The theorem is proved. □

***Corollary* 3.1.1.** *If all* 2*-maximal subgroups of* G *are semisubnormal in* G, *then the derived subgroup* G' *is nilpotent.*

Proof. Note that the nilpotency of the derived subgroup G' is equivalent to $G \in \mathfrak{N} \circ \mathfrak{A}$.

Assume that the claim is false and let G be a minimal counterexample. It is easy to show that G/N satisfies the hypothesis of the corollary, where N is an arbitrary non-trivial normal subgroup of G. By induction, $G/N \in \mathfrak{N} \circ \mathfrak{A}$. Hence by Lemmas 1.5 and 1.6, G is primitive.

Let M be an arbitrary maximal subgroup of G. Then by Lemmas 1.1 (1) and 1.2 (4), M is supersoluble. Hence either G is supersoluble, or G is a minimal non-supersoluble group.

If G is supersoluble, then $G \in \mathfrak{N} \circ \mathfrak{A}$ by [11, Theorem 4.52], a contradiction.

Let G be a minimal non-supersoluble group. By Lemmas 1.7 and 1.8, G is soluble, P is a unique minimal normal subgroup of G, $|P| > p$ and P is a Sylow p-subgroup of G such that $G = P \rtimes M$, where M is a maximal subgroup of G. Besides, M is a Hall p'-subgroup of G. Let P_1 be a subgroup of prime order p of P.

If M is abelian, then $G \in \mathfrak{N} \circ \mathfrak{A}$, a contradiction. Therefore we assume that M is non-abelian. Hence M has maximal subgroups M_1 and M_2 such that $M = \langle M_1, M_2 \rangle$. If at least one of the subgroups M_1 or M_2 is subnormal in G, then $O_{p'}(G) \neq 1$, a contradiction. Thus M_1 and M_2 are seminormal in G. Hence there are the subgroups V_1 and V_2 such that

$$M_1V_1 = M_2V_2 = G, M_1P_1 = P_1M_1, M_2P_1 = P_1M_2,$$

because $P \leq V_1 \cap V_2$. Then $M_1 \leq N_G(P_1)$ and $M_2 \leq N_G(P_1)$. Therefore $P_1 \lhd G = P\langle M_1, M_2 \rangle$, a contradiction. The corollary is proved. □

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