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On two sublattices of the subgroup lattice of a finite group

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Abstract. Let $\mathfrak{F}$ be a non-empty class of groups, let G be a finite group and let $\mathcal{L}(G)$ be the lattice of all subgroups of G. A chief H/K factor of G is $\mathfrak{F}$-*central* in G if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. Let $\mathcal{L}_{c\mathfrak{F}}(G)$ be the set of all subgroups A of G such that every chief factor H/K of G between A_G and A^G is $\mathfrak{F}$-central in G; $\mathcal{L}_{\mathfrak{F}}(G)$ denotes the set of all subgroups A of G with $A^G/A_G \in \mathfrak{F}$. We prove that the set $\mathcal{L}_{c\mathfrak{F}}(G)$ and, in the case when $\mathfrak{F}$ is a Fitting formation, the set $\mathcal{L}_{\mathfrak{F}}(G)$ are sublattices of the lattice $\mathcal{L}(G)$. We also study conditions under which the lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ and the lattice of all subnormal subgroup of G are modular.

1 Introduction

Throughout this paper, all groups are finite, and G always denotes a finite group. Moreover, $\mathcal{L}(G)$ denotes the lattice of all subgroups of G; $\mathcal{L}_n(G)$ is the lattice of all normal subgroups of G; $\mathcal{L}_{\mathrm{sn}}(G)$ denotes the lattice of all subnormal subgroups of G. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G; C_n denotes a cyclic group of order n. We use $\mathfrak{N}$, $\mathfrak{N}^*$, $\mathfrak{U}$ and $\mathfrak{S}$ to denote the classes of all nilpotent, of all quasinilpotent, of all supersoluble and of all soluble groups, respectively.

In what follows, $\mathfrak{F}$ is a class of groups containing all identity groups; $G^{\mathfrak{F}}$ denotes the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$; $G_{\mathfrak{F}}$ is the product of all normal subgroups N of G with $N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if, for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ and a *Fitting formation* if $\mathfrak{F}$ is a formation such that, for every group G, every normal subgroup of $G_{\mathfrak{F}}$ belongs to $\mathfrak{F}$. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \le \Phi(G)$ and *(normally) hereditary* if $H \in \mathfrak{F}$ whenever $H \le G \in \mathfrak{F}$ (respectively whenever $H \trianglelefteq G \in \mathfrak{F}$).

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Note, in passing, that $\mathfrak{N}$, $\mathfrak{N}^*$ and $\mathfrak{S}$ are Fitting formations, and $\mathfrak{N}$, $\mathfrak{U}$ and $\mathfrak{S}$ are hereditary saturated formations [2].

If $K \leq H$ are normal subgroups of G and $C \leq C_G(H/K)$, then one can form the semidirect product $(H/K) \rtimes (G/C)$ putting

$$(hK)^{gC} = g^{-1}hgK \quad \text{for all } hK \in H/K \text{ and } gC \in G/C.$$

We say, following [10], that a chief H/K factor of G is $\mathfrak{F}$-*central* in G if

$$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}.$$

In this paper, we deal with the following two sets of subgroups of G: $\mathcal{L}_{c\mathfrak{F}}(G)$ is the set of all subgroups A of G such that every chief factor H/K of G between A_G and A^G is $\mathfrak{F}$-central in G; $\mathcal{L}_{\mathfrak{F}}(G)$ denotes the set of all subgroups A of G with $A^G/A_G \in \mathfrak{F}$.

Before continuing, consider some well-known examples.

Example 1.1. (i) A subgroup A of G is said to be *quasinormal* or *permutable* (respectively S-*quasinormal* or S-*permutable* [1, 5]) in G if A permutes with all subgroups (respectively with all Sylow subgroups) H of G, that is, $AH = HA$. In view of [8] (see also [1, Corollary 1.5.6]), every quasinormal subgroup of G belongs to $\mathcal{L}_{c\mathfrak{N}}(G)$. On the other hand, in view of the results of Kegel [7] and Deskins [3] (see also [1, Theorem 1.2.17]), every S-permutable subgroup of G belongs to $\mathcal{L}_{\mathfrak{N}}(G)$.

(ii) A subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [9, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is,

(a) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$,

(b) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

In view of [9, Theorem 5.2.5], every modular subgroup of G belongs to $\mathcal{L}_{c\mathfrak{U}}(G)$.

(iii) Let $G = (C_7 \rtimes \mathrm{Aut}(C_7)) \times A_5 \times P$, where A_5 is an alternating group of degree 5 and P is a non-abelian group of order p^3 and exponent p for some prime $p > 2$. Let A be a subgroup of $\mathrm{Aut}(C_7)$ of order 2. Then A is not modular in G, but $A \in \mathcal{L}_{c\mathfrak{U}}(G)$ since every chief factor of G below $A^G = C_7 \rtimes A$ is cyclic. Now let L be a subgroup of P of order p with $L \nleq Z(P)$. Then L is not quasinormal in P, so L is not quasinormal in G, but $L \in \mathcal{L}_{c\mathfrak{N}}(G)$ since, for every chief factor H/K of G below $L^G = P$, we have $C_G(H/K) = G$.

(iv) Let $G = (A_5 \wr C_7) \times A_4 = (K \rtimes C_7) \times A_4$, where K is the base group of the regular wreath product $A_5 \wr C_7$ and A_5 and A_4 are the alternating groups of degree 5 and 4, respectively. Let L be the first copy of A_5 in K. Then L is subnormal in G, but $L \notin \mathcal{L}_{\mathfrak{N}}(G)$. Finally, if Z is a subgroup of order 2 in A_4 and $H = (KC_7)Z$, then the subgroup H is not S-quasinormal in G, but $H \in \mathcal{L}_{\mathfrak{N}}(G)$.

It is well known that the set of all quasinormal subgroups and the set of all modular subgroups do not form sublattices of the lattice $\mathcal{L}(G)$. Nevertheless, our first result shows that the set $\mathcal{L}_{c\mathfrak{N}}(G)$ and the set $\mathcal{L}_{c\mathfrak{U}}(G)$ are sublattices of $\mathcal{L}(G)$.

Theorem 1.2. *The set $\mathcal{L}_{c\mathfrak{F}}(G)$ and, in the case when $\mathfrak{F}$ is a Fitting formation, the set $\mathcal{L}_{\mathfrak{F}}(G)$ are sublattices of the lattice $\mathcal{L}(G)$.*

Applying the lattice $\mathcal{L}_{\mathfrak{N}^*}(G)$, we can give permutable conditions under which the lattice $\mathcal{L}_{\mathrm{sn}}(G)$ is modular.

Theorem 1.3. *The lattice $\mathcal{L}_{\mathrm{sn}}(G)$ is modular if and only if, for every two subnormal subgroups $L \leq T$ of G, where $L \in \mathcal{L}_{\mathfrak{N}^*}(T)$, L permutes with every subnormal subgroup M of T, that is, $LM = ML$.*

Finally, we describe the conditions under which the lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ is modular or distributive.

Theorem 1.4. *The following statements hold:*

(i) *The lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ is modular if and only if two subgroups $A, B \in \mathcal{L}_{c\mathfrak{N}}(G)$ permute.*

(ii) *The lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ is distributive if and only if $\mathcal{L}_{c\mathfrak{N}}(G) = \mathcal{L}_n(G)$ is distributive.*

2 Proof of Theorem 1.2

Let $D = M \rtimes A$ and $R = N \rtimes B$. Then the pairs (M, A) and (N, B) are said to be *equivalent* provided there are isomorphisms $f\colon M \to N$ and $g\colon A \to B$ such that $f(a^{-1}ma) = g(a^{-1})f(m)g(a)$ for all $m \in M$ and $a \in A$.

In fact, the following lemma is known (see for example [10, Lemma 3.27]), and it can be proved by direct verification.

Lemma 2.1. *Let $D = M \rtimes A$ and $R = N \rtimes B$. If the pairs (M, A) and (N, B) are equivalent, then $D \simeq R$.*

Lemma 2.2. *Let N, M and $K < H \leq G$ be normal subgroups of G, where H/K is a chief factor of G.*

(1) *If $N \leq K$, then*

$$(H/K) \rtimes (G/C_G(H/K)) \simeq ((H/N)/(K/N)) \rtimes ((G/N)/C_{G/N}((H/N)/(K/N))).$$

(2) *If T/L is a chief factor of G and H/K and T/L are G-isomorphic, then*

$$C_G(H/K) = C_G(T/L),$$
$$(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L)).$$

(3) $(MN/N) \rtimes (G/C_G(MN/N)) \simeq (M/M \cap N) \rtimes (G/C_G(M/M \cap N))$.

Proof. (1) In view of the G-isomorphisms $H/K \simeq (H/N)/(K/N)$ and

$$\begin{aligned} G/C_G(H/K) &\simeq (G/N)/(C_G(H/K)/N) \\ &\simeq (G/N)/(C_{G/N}((H/N)/(K/N))), \end{aligned}$$

the pairs

$$(H/K, G/C_G(H/K)),$$
$$((H/N)/(K/N), (G/N)/C_{G/N}((H/N)/(K/N)))$$

are equivalent. Hence statement (1) is a corollary of Lemma 2.1.

(2) A direct check shows that $C = C_G(H/K) = C_G(T/L)$ and that the pairs $(H/K, G/C)$ and $(T/L, G/C)$ are equivalent. Hence statement (2) is also a corollary of Lemma 2.1.

(3) This follows from the G-isomorphism $MN/N \simeq M/M \cap N$ and part (2). □

Lemma 2.3. *Let $K \leq H$, $K \leq V$, $W \leq V$ and $N \leq H$ be normal subgroups of G. Suppose that every chief factor of G between K and H is $\mathfrak{F}$-central in G.*

(1) *If every chief factor of G between N and H is $\mathfrak{F}$-central in G, then every chief factor G between $K \cap N$ and H is $\mathfrak{F}$-central in G.*

(2) *If every chief factor of G between K and V is $\mathfrak{F}$-central in G, then every chief factor G between K and HV is $\mathfrak{F}$-central in G.*

(3) *If every chief factor of G between K and KN is $\mathfrak{F}$-central in G, then every chief factor G between $K \cap N$ and N is $\mathfrak{F}$-central in G.*

(4) *If every chief factor of G between W and V is $\mathfrak{F}$-central in G, then every chief factor G between $K \cap W$ and $H \cap V$ is $\mathfrak{F}$-central in G.*

(5) *If every chief factor of G between W and V is $\mathfrak{F}$-central in G, then every chief factor G between KW and HV is $\mathfrak{F}$-central in G.*

Proof. (1) In view of Lemma 2.2 (1), we can assume without loss of generality that $K \cap N = 1$. Let T/L be any chief factor of G below H. First suppose that $T \cap K \leq L$. From the G-isomorphism $KT/KL \simeq T/L(T \cap K) = T/L$ and

Lemma 2.1 (2), we get that T/L is $\mathfrak{F}$-central in G. Now assume that $T \cap K \nleq L$. Then $T = L(T \cap K)$, and so, from the G-isomorphism

$$T/L = L(T \cap K)/L \simeq (T \cap K)/(L \cap K),$$

we get that $(T \cap K)/(L \cap K)$ is a chief factor of G, and this factor is G-isomorphic with $(T \cap K)N/(L \cap K)N$ since $(T \cap K) \cap N = 1$. Thus both G-isomorphic factors T/L and $(T \cap K)N/(L \cap K)N$ are $\mathfrak{F}$-central in G by Lemma 2.2 (2) since every chief factor of G between N and H is $\mathfrak{F}$-central in G by hypothesis. Therefore, we have (1).

(2) We can assume without loss of generality that $K = 1$. Let T/L be any chief factor of G such that $H \leq L < T \leq HV$. Then $T = L(T \cap V)$, and so, from the G-isomorphism $T/L = L(T \cap V)/L \simeq (T \cap V)/(L \cap V)$ and Lemma 2.2 (2), we get that T/L is $\mathfrak{F}$-central in G since every chief factor of G between $K = 1$ and V is $\mathfrak{F}$-central in G by hypothesis. Therefore, there is a chief series of G with $\mathfrak{F}$-central in HV, so we have (2) by the Jordan–Hölder theorem for the chief series.

(3) Let $K \cap N = N_0 < N_1 < \cdots < N_{t-1} < N_t = N$ be a chief series of G between $K \cap N$ and N. Then, from the G-isomorphism

$$\begin{aligned} N_i K/N_{i-1}K \simeq N_i/N_i \cap N_{i-1}K &= N_i/N_{i-1}(N_i \cap K) \\ &= N_i/N_{i-1}, \end{aligned}$$

it follows that $K = KN_0 < KN_1 < \cdots < KN_{t-1} < KN_t = KN$ is a chief series of G between K and KN. Therefore, in view of the Jordan–Hölder theorem for the chief series and Lemma 2.2 (2), statement (3) is true.

(4) This follows from the G-isomorphisms

$$(H \cap V)/(K \cap V) \simeq (H \cap V)K/K,$$
$$(V \cap H)/(W \cap H) \simeq (V \cap H)W/W$$

and part (1).

(5) As $HV/KW = (HKW/KW)(VKW/KW)$, statement (5) follows from Lemma 2.2, the G-isomorphisms

$$\begin{aligned} HKW/KW \simeq H/H \cap KW &= H/K(H \cap W) \\ &\simeq (H/K)/(K(H \cap W)/K), \\ VKW/KW &\simeq (V/W)/(W(V \cap K)/W) \end{aligned}$$

and part (2). $\square$

Proof of Theorem 1.2. Let A and B be subgroups of G such that $A, B \in \mathcal{L}_{c\mathfrak{F}}(G)$ (respectively $A, B \in \mathcal{L}_{\mathfrak{F}}(G)$). Then every chief factor of G between A_G and A^G is $\mathfrak{F}$-central in G (then, respectively, $A^G/A_G \in \mathfrak{F}$). Therefore, in view of the G-isomorphisms

$$\begin{aligned} A^G(A_G B_G)/A_G B_G &\simeq A^G/(A^G \cap A_G B_G) = A^G/A_G(A^G \cap B_G) \\ &\simeq (A^G/A_G)/(A_G(A^G \cap B_G)/A_G), \end{aligned}$$

we get that every chief factor of G between $A_G B_G$ and $A^G(A_G B_G)$ is $\mathfrak{F}$-central in G (respectively, we get that $A^G(A_G B_G)/A_G B_G \in \mathfrak{F}$ since $\mathfrak{F}$ is closed under taking homomorphic images). Similarly, we can get that every chief factor of G between $A_G B_G$ and $B^G(A_G B_G)$ is $\mathfrak{F}$-central in G (respectively, we get that $B^G(A_G B_G)/A_G B_G \in \mathfrak{F}$). Moreover,

$$A^G B^G/A_G B_G = (A^G(A_G B_G)/A_G B_G)(B^G(A_G B_G)/A_G B_G),$$

and so every chief factor of G between $A_G B_G$ and $A^G B^G$ is $\mathfrak{F}$-central in G by Lemma 2.3 (2) (respectively, we have $A^G B^G/A_G B_G \in \mathfrak{F}$ since $\mathfrak{F}$ is a Fitting formation).

Next note that $\langle A, B\rangle^G = A^G B^G$ and $A_G B_G \leq \langle A, B\rangle_G$. So every chief factor of G between $\langle A, B\rangle_G$ and $\langle A, B\rangle^G = A^G B^G$ is $\mathfrak{F}$-central in G (respectively, we get that $\langle A, B\rangle^G/\langle A, B\rangle_G \in \mathfrak{F}$ since $\mathfrak{F}$ is closed under taking homomorphic images). Hence $\langle A, B\rangle \in \mathcal{L}_{c\mathfrak{F}}(G)$ (respectively $\langle A, B\rangle \in \mathcal{L}_{\mathfrak{F}}(G)$).

Now note that $(A \cap B)_G = A_G \cap B_G$. On the other hand, from the G-isomorphism

$$\begin{aligned} (A^G \cap B^G)/(A_G \cap B^G) &= (A^G \cap B^G)/(A_G \cap B^G \cap A^G) \\ &\simeq A_G(B^G \cap A^G)/A_G \leq A^G/A_G, \end{aligned}$$

we get that every chief factor of G between $A_G \cap B^G$ and $A^G \cap B^G$ is $\mathfrak{F}$-central in G by Lemma 2.3 (5) (respectively, we get that $(A^G \cap B^G)/(A_G \cap B^G) \in \mathfrak{F}$ since $\mathfrak{F}$ is closed under taking normal subgroups). Similarly, we get that every chief factor of G between $B_G \cap A^G$ and $B^G \cap A^G$ is $\mathfrak{F}$-central in G (respectively, we have $(B^G \cap A^G)/(B_G \cap A^G) \in \mathfrak{F}$). But then we get that every chief factor of G between $(A_G \cap B^G) \cap (B_G \cap A^G) = A_G \cap B_G$ and $A^G \cap B^G$ is $\mathfrak{F}$-central in G by Lemma 2.3 (3) (respectively, we get that $(A^G \cap B^G)/(A_G \cap B_G) \in \mathfrak{F}$ since $\mathfrak{F}$ is a formation). It is clear also that $(A \cap B)^G \leq A^G \cap B^G$. Therefore, every chief factor of G between $(A \cap B)_G = A_G \cap B_G$ and $(A \cap B)^G$ is $\mathfrak{F}$-central in G (respectively, we get that $(A \cap B)^G/(A \cap B)_G \in \mathfrak{F}$). Therefore, we have $A \cap B \in \mathcal{L}_{c\mathfrak{F}}(G)$ (respectively $A \cap B \in \mathcal{L}_{\mathfrak{F}}(G)$). Hence the set $\mathcal{L}_{c\mathfrak{F}}(G)$ (respectively the set $\mathcal{L}_{\mathfrak{F}}(G)$) is a sublattice of the lattice $\mathcal{L}(G)$. □

3 Proofs of Theorems 1.3 and 1.4

We use $Z_{\mathfrak{F}}(G)$ to denote the product of all normal subgroups A of G such that either $A = 1$ or every chief factor of G below A is $\mathfrak{F}$-central in G. In view of Lemma 2.3 (2), every chief factor G below $Z_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-central in G.

Lemma 3.1. *Let $A \in \mathcal{L}_{c\mathfrak{F}}(G)$, and let N and E be subgroups of G, where N is normal in G.*

(1) *If $L \leq T$ are normal subgroups of G such that all chief factors of G between L and T are $\mathfrak{F}$-central in G, then $\mathcal{L}(T/L)$ is isomorphic to the interval $[T, L]$ in $\mathcal{L}_{c\mathfrak{F}}(G)$.*

(2) *$AN/N \in \mathcal{L}_{c\mathfrak{F}}(G/N)$.*

(3) *If $H/N \in \mathcal{L}_{c\mathfrak{F}}(G/N)$, then $H \in \mathcal{L}_{c\mathfrak{F}}(G)$.*

(4) *$\mathcal{L}_{c\mathfrak{F}}(G/N)$ is isomorphic to the interval $[G/N]$ in $\mathcal{L}_{c\mathfrak{F}}(G)$.*

(5) *If $\mathfrak{F}$ is a hereditary saturated formation, then $A \cap E \in \mathcal{L}_{c\mathfrak{F}}(E)$.*

(6) *If $\mathfrak{F}$ is a normally hereditary saturated formation and E is subnormal in G, then $A \cap E \in \mathcal{L}_{c\mathfrak{F}}(E)$.*

Proof. (1) This statement follows from the fact that, for every subgroup H of G with $L \leq H \leq T$, we have $L \leq H_G$ and $H^G \leq T$.

(2) From the G-isomorphisms

$$(A^G N/N)/(A_G N/N) \simeq (A^G/A_G)/(A_G(A^G \cap N)/A_G)$$

and Lemma 2.2, we get that every chief factor of G/N between $A_G N/N$ and $A^G N/N$ is $\mathfrak{F}$-central in G since every chief factor of G between A_G and A^G is $\mathfrak{F}$-central in G by hypothesis. On the other hand, we have

$$(AN/N)^{G/N} = (AN)^G/N = A^G N/N \quad \text{and} \quad A_G N/N \leq (AN/N)_{G/N}.$$

Hence every chief factor of G between $(AN/N)_{G/N}$ and $(AN/N)^{G/N}$ is $\mathfrak{F}$-central in G/N, so $AN/N \in \mathcal{L}_{c\mathfrak{F}}(G/N)$.

(3) This follows from the G-isomorphism

$$H^G/H_G \simeq (H^G/N)/(H_G/N) = (H/N)^{G/N}/(H/N)_{G/N}.$$

(4) This follows from parts (2) and (3).

(5) First note that, by [5, Chapter 1, Theorem 2.7 (a)],

$$(A^G/A_G) \cap (EA_G/A_G) = A_G(A^G \cap E)/A_G \leq Z_{\mathfrak{F}}(EA_G/A_G)$$

since, by hypothesis, we have $A^G/A_G \leq Z_{\mathfrak{F}}(G/A_G)$. On the other hand, we have $f(Z_{\mathfrak{F}}(EA_G/A_G)) = Z_{\mathfrak{F}}(E/E \cap A_G)$, where $f\colon EA_G/A_G \to E/E \cap A_G$ is the canonical isomorphism from EA_G/A_G onto $E/E \cap A_G$. Hence

$$f(A_G(A^G \cap E)/A_G) = (A^G \cap E)/(A_G \cap E) \leq Z_{\mathfrak{F}}(E/A_G \cap E),$$

where $A_G \cap E \leq (A \cap E)_E \leq A \cap E \leq (A \cap E)^E \leq A^G \cap E$, and so

$$(A \cap E)^E/(A \cap E)_E \leq Z_{\mathfrak{F}}(E/(A \cap E)_E).$$

Hence $A \cap E \in \mathcal{L}_{c\mathfrak{F}}(E)$.

(6) See the proof of (5). □

Lemma 3.2. *Suppose that $\mathfrak{F}$ is a Fitting formation. Let $A \in \mathcal{L}_{\mathfrak{F}}(G)$, and let N and E be subgroups of G, where N is normal in G.*

(1) *If $L \leq T$ are normal subgroups of G such that $T/L \in \mathfrak{F}$, then $\mathcal{L}(T/L)$ is isomorphic to the interval $[T, L]$ in $\mathcal{L}_{\mathfrak{F}}(G)$.*

(2) $AN/N \in \mathcal{L}_{\mathfrak{F}}(G/N)$.

(3) *If $H/N \in \mathcal{L}_{\mathfrak{F}}(G/N)$, then $H \in \mathcal{L}_{\mathfrak{F}}(G)$.*

(4) *$\mathcal{L}_{\mathfrak{F}}(G/N)$ is isomorphic to the interval $[G/N]$ in $\mathcal{L}_{\mathfrak{F}}(G)$.*

(5) *If $\mathfrak{F}$ is hereditary, then $A \cap E \in \mathcal{L}_{\mathfrak{F}}(E)$.*

(6) *If E is subnormal in G, then $A \cap E \in \mathcal{L}_{\mathfrak{F}}(E)$.*

Proof. Statements (1), (2), (3) and (4) can be proved similarly to statements (1), (2), (3) and (4) of Lemma 3.1, respectively.

(5) First note that

$$(A^G/A_G) \cap (EA_G/A_G) = A_G(A^G \cap E)/A_G \leq (EA_G/A_G)_{\mathfrak{F}}$$

since $\mathfrak{F}$ is hereditary and we have $A^G/A_G \in (G/A_G)_{\mathfrak{F}}$ by hypothesis. On the other hand, we have

$$f((EA_G/A_G)_{\mathfrak{F}}) = (E/E \cap A_G)_{\mathfrak{F}},$$

where $f\colon EA_G/A_G \to E/E \cap A_G$ is the canonical isomorphism from EA_G/A_G onto $E/E \cap A_G$. Hence

$$f(A_G(A^G \cap E)/A_G) = (A^G \cap E)/(A_G \cap E) \leq (E/A_G \cap E)_{\mathfrak{F}},$$

where $A_G \cap E \leq (A \cap E)_E \leq A \cap E \leq (A \cap E)^E \leq A^G \cap E$, and so

$$(A \cap E)^E/(A \cap E)_E \leq (E/(A \cap E)_E)_{\mathfrak{F}}.$$

Hence $A \cap E \in \mathcal{L}_{\mathfrak{F}}(E)$.

(6) See the proof of (5). □

In fact, the following lemma is known.

Lemma 3.3. *Let A be a subnormal subgroup of G.*

(1) *If A is a p-group, then $A \leq O_p(G)$.*

(2) *If $|G : A|$ is a power of p, then G/A_G is a p-group.*

Proof. There is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that $A_{i-1} \trianglelefteq A_i$ for all $i = 1, \ldots, n$. We can assume without loss of generality that $M = A_{n-1} < G$.

(1) By induction, we have that $A \leq O_p(M)$. On the other hand, the subgroup $O_p(M)$ is normal in G since it is characteristic in M, so $O_p(M) \leq O_p(G)$. Hence we have (1).

(2) It is enough to show that, for every p'-element x of G, we have $x \in A$. But, since evidently $|G : M|$ is a power of p, $x \in M$, so we have $x \in A$ by induction. □

Lemma 3.4 (Wielandt [11]). *If $A = A'$ is a perfect subnormal subgroup of G, then $AB = BA$ for all subnormal subgroups B of G.*

Proof of Theorem 1.4. (i) First suppose that the lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ is modular. We show that, in this case, every pair of subgroups $A, B \in \mathcal{L}_{c\mathfrak{N}}(G)$ is permutable. Suppose that this is false, and let G be a counterexample with $|G| + |A| + |B|$ minimal. Then $AB \neq BA$, but $A_1 B_2 = B_1 A_1$ for all $A_1 \leq A$ and $B_1 \leq B$ such that $A_1, B_1 \in \mathcal{L}_{c\mathfrak{N}}(G)$ and either $A_1 < A$ or $B_1 < B$.

(1) *$AN/N, BN/N \in \mathcal{L}_{c\mathfrak{N}}(G/N)$ for every normal subgroup N of G.* (This follows from Lemma 3.1 (2).)

(2) *$\mathcal{L}_{c\mathfrak{N}}(G/N)$ is isomorphic to the interval $[G/N]$ in $\mathcal{L}_{c\mathfrak{N}}(G)$. Hence the lattice $\mathcal{L}_{c\mathfrak{N}}(G/N)$ is modular.* (This follows from Lemma 3.1 (4).)

(3) *RAB is a subgroup of G for every minimal normal subgroup R of G. Hence $A_G = 1 = B_G$.*

By claim (1), we have $AR/R, BR/R \in \mathcal{L}_{c\mathfrak{N}}(G/R)$, so the choice of G and claim (2) imply that

$$(AR/R)(BR/R) = (BR/R)(AR/R) = RAB/R$$

is a subgroup of G. Therefore, since $AB \neq BA$, it follows that $A_G = 1 = B_G$.

(4) *Let $V = A^G B^G$. Then $V \leq Z_\infty(G)$ and the lattice $\mathcal{L}(V)$ is a sublattice in $\mathcal{L}_{c\mathfrak{N}}(G)$.*

Indeed, claim (3) and Lemma 2.3 (2) imply that $V \leq Z_\infty(G)$. Hence, for every subgroup $H \leq V$, we have $H \in \mathcal{L}_{c\mathfrak{N}}(G)$. Hence claim (4) holds.

The final contradiction for the necessity. Claim (4) implies that V is nilpotent and the lattice $\mathcal{L}(V)$ of all subgroups of V is modular. Let $\pi(V) = \{p_1, \dots, p_t\}$, and let respectively P_i, A_i and B_i be the Sylow p_i-subgroups of V, A and B for all $i = 1, \dots, t$. Then $A_i \le P_i$ and $B_j \le P_j$ for all i and j. Hence $A_i B_j = B_j A_i$ for all $i \ne j$ since $[P_i, P_j] = 1$. It is clear also that $\mathcal{L}(P_i)$ is a sublattice in $\mathcal{L}_{c\mathfrak{N}}(G)$, so $\mathcal{L}(P_i)$ is modular too. But then $A_i B_i = B_i A_i$ for all i by [9, Lemma 2.3.2], so

$$\begin{aligned} AB &= (A_1 \times \cdots \times A_t)(B_1 \times \cdots \times B_t) \\ &= (B_1 \times \cdots \times B_t)(A_1 \times \cdots \times A_t) = BA, \end{aligned}$$

contrary to the choice of (G, A, B). The necessity of the condition of the theorem is proved.

Sufficiency. This follows from the fact that if, for the subgroups $A \le H$ and B of G, we have $AB = BA$, then $H \cap \langle A, B\rangle = \langle A, H \cap B\rangle$.

(ii) The sufficiency is evident. Now suppose that the lattice $\mathcal{L}_{c\mathfrak{N}}(G)$ is distributive. Then, for every subgroup $A \in \mathcal{L}_{c\mathfrak{N}}(G)$, the lattice $\mathcal{L}(A^G/A_G)$ is a sublattice in $\mathcal{L}_{c\mathfrak{N}}(G)$ by Lemma 3.1 (1), so $\mathcal{L}(A^G/A_G)$ is distributive. Hence A^G/A_G is cyclic by the Ore theorem [9, Theorem 1.2.3]. Hence A/A_G is normal in G/A_G. Therefore, $\mathcal{L}_{c\mathfrak{N}}(G) = \mathcal{L}_n(G)$ is distributive. □

Proof of Theorem 1.3. First note that, by [6, X, Lemma 13.3 and Corollary 13.11], $\mathfrak{N}^*$ is a Fitting formation.

Necessity. Suppose that this is false, and let G be a counterexample with subnormal subgroups $L, B \le T$, where

$$L \in \mathcal{L}_{\mathfrak{N}^*}(T) \quad \text{and} \quad LB \ne BL,$$

for which $|G| + |L|$ is minimal. Therefore, for every proper subnormal subgroup V of L with $V \in \mathcal{L}_{\mathfrak{N}^*}(T)$, we have $VB = BV$.

(1) $\langle L, B\rangle = G$. *Hence* $T = G$.

Assume that $\langle L, B\rangle < G$. The subgroup $\langle L, B\rangle$ is subnormal in G by [4, Chapter A, Section 14.4]. Hence the lattice $\mathcal{L}_{\mathrm{sn}}(\langle L, B\rangle)$ is modular since it is a sublattice of the lattice $\mathcal{L}_{\mathrm{sn}}(G)$. Moreover, Lemma 3.2 (6) implies $L \in \mathcal{L}_{\mathfrak{N}^*}(\langle L, B\rangle)$. Hence the choice of G implies that $LB = BL$, a contradiction. Hence we have $T = \langle L, B\rangle = G$.

(2) $L_G = 1 = B_G$.

Indeed, suppose that, for some minimal normal subgroup N of G, we have either $N \le L$ or $N \le B$. It is clear that the lattice $\mathcal{L}_{\mathrm{sn}}(G/N)$ is isomorphic to the interval $[G/N]$ in the modular lattice $\mathcal{L}_{\mathrm{sn}}(G)$. Hence $\mathcal{L}_{\mathrm{sn}}(G/N)$ is modular. Moreover, LN/N and BN/N are subnormal subgroups of G/N by [4, Chapter A, Section 14.1]. On the other hand, Lemma 3.2 (2) and claim (1) imply that

$$LN/N \in \mathcal{L}_{\mathfrak{N}^*}(G/N) = \mathcal{L}_{\mathfrak{N}^*}(T/N).$$

Therefore, the choice of G implies that

$$LB/N = (LN/N)(BN/N) = (BN/N)(LN/N) = BL/N,$$

so $LB = BL$. This contradiction shows that $L_G = 1 = B_G$.

(3) *L is a p-group for some prime p.*

First note that $L^G \simeq L^G/L_G$ is quasinilpotent since $T = G$, $L_G = 1$ and $L \in \mathcal{L}_{\mathfrak{N}^*}(T) = \mathcal{L}_{\mathfrak{N}^*}(G)$ by claims (1) and (2). Therefore, for every proper subnormal subgroup V of L, we have $V \in \mathcal{L}_{\mathfrak{N}^*}(T)$, so $VB = BV$. Since $LB \neq BL$, it follows that L has a normal subgroup W such that L/W is simple and every proper subnormal subgroup of L is contained in W.

By [6, Chapter X, Theorem 13.6], either $L/F(L) = A_1 \times \cdots \times A_t$ for some non-abelian simple groups $A_1, \ldots, A_t$ or $L = F(L)$ is nilpotent. In the former case, we have $t = 1$ and $W = F(L)$, so $L = L'$ is perfect, and hence $LB = BL$ by Lemma 3.4. Therefore, we have the second case, so L is a cyclic p-group for some prime p.

(4) $G = O_p(G)B$. (Since L is a subnormal p-subgroup of G by claim (3), this follows from Lemma 3.3 (1) and claim (1).)

The final contradiction for the necessity. Since B is subnormal in G and $|G : B|$ is a power of p by claim (4), G/B_G is a p-group by Lemma 3.3 (2). But then G is a p-group by claim (2). Hence $\mathcal{L}(G) = \mathcal{L}_{\mathrm{sn}}(G)$ is modular. Therefore, $LB = BL$ by [9, Lemma 2.3.2]. This contradiction completes the proof of the necessity.

Sufficiency. Suppose that this is false, and let G be a counterexample of minimal order. The hypothesis holds for every proper subnormal subgroup V of G, so the lattice $\mathcal{L}_{\mathrm{sn}}(V)$ is modular by the choice of G.

First we show that the lattice $\mathcal{L}_{\mathrm{sn}}(G)$ is upper semimodular. In view of [9, p. 46], it is enough to show that if $A, B \in \mathcal{L}_{\mathrm{sn}}(G)$ such that A and B cover $A \cap B$ in $\mathcal{L}_{\mathrm{sn}}(G)$, then $\langle A, B\rangle$ covers A in $\mathcal{L}_{\mathrm{sn}}(G)$. Assume that this is false.

(1) $G = \langle A, B\rangle$.

Suppose that $\langle A, B\rangle < G$. Then the lattice $\mathcal{L}_{\mathrm{sn}}(\langle A, B\rangle)$ is modular. Hence this lattice is upper semimodular by [9, Theorem 2.1.10], so $\langle A, B\rangle$ covers A, a contradiction. Hence we have (1).

(2) $AB \neq BA$.

Indeed, if $AB = BA$, then $G = AB$ by claim (1). Hence, for every subgroup $T \in \mathcal{L}_{\mathrm{sn}}(G)$ satisfying $A \leq T \leq G$, we have

$$T = A(T \cap B) \quad \text{and either} \quad T \cap B = B \quad \text{or} \quad A \cap B = T \cap B.$$

But then, in the former case, we have $T = A(T \cap B) = A(A \cap B) = A$. In the second case, we have $T = G$. Hence $G = \langle A, B\rangle$ covers A, a contradiction. Hence we have (2).

(3) $A \cap B = 1$. *Hence A and B are minimal subnormal subgroups of G.*

Let $N = A \cap B$. We show that $N = 1$. Suppose that $N \neq 1$. Note that, in view of claim (1), $G = \langle A, B \rangle \le N_G(N)$ since A and B cover N in $\mathcal{L}_{\mathrm{sn}}(G)$. Now let L/N, B/N and T/N be subnormal subgroups of G/N such that

$$L/N, B/N \le T/N \quad \text{and} \quad L/N \in \mathcal{L}_{\mathfrak{N}^*}(T/N).$$

Then L, B and T are subnormal subgroups of G such that $B \le T$ and $L \in \mathcal{L}_{\mathfrak{N}^*}(T)$ by Lemma 3.2 (3). Then $LB = BL$ by hypothesis, so

$$(L/N)(B/N) = (B/N)(L/N).$$

Hence the hypothesis holds for G/N, so the lattice $\mathcal{L}_{\mathrm{sn}}(G/N)$ is modular by the choice of G. Therefore, $G/N = \langle A, B \rangle /N$ covers A/N in $\mathcal{L}_{\mathrm{sn}}(G/N)$, and hence $G = \langle A, B \rangle$ covers A in $\mathcal{L}_{\mathrm{sn}}(G)$. This contradiction shows that $N = 1$, so A and B are minimal subnormal subgroups of G.

The final contradiction for the sufficiency. In view of claim (3), A and B are simple groups. If $A = A'$ is perfect, then $AB = BA$ by Lemma 3.4, contrary to claim (2). Hence A is abelian, so $A \le O_p(G)$ for some prime p, which implies that $A \in \mathcal{L}_{\mathfrak{N}^*}(G)$. But then $AB = BA$ by hypothesis. This contradiction shows that the lattice $\mathcal{L}_{\mathrm{sn}}(G)$ is upper semimodular. On the other hand, this lattice is lower semimodular by [9, Theorem 2.1.8]. From [9, Theorem 2.1.10], we get now that the lattice $\mathcal{L}_{\mathrm{sn}}(G)$ is modular, contrary to the choice of G. Therefore, the sufficiency of the condition of the theorem is true. □

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