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NEW CRITERIONS OF EXISTENCE AND CONJUGACY OF HALL SUBGROUPS OF FINITE GROUPS

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Abstract. In the paper, new criterions for existence and conjugacy of Hall subgroups of finite groups are given. In particular, the Schur-Zassenhaus theorem, Hall theorem and Čunihin theorem are generalized.

1. Introduction

Throughout this paper, all groups are finite, G denotes a finite group and π denotes a non-empty subset of the set of all primes. A subgroup H is said to be permutable with a subgroup B if $HB = BH$. The notation and terminology are standard, as in [10] and [3].

The famous Schur-Zassenhaus Theorem asserts that: *If G has a normal Hall π-subgroup A, then G is an $E_{\pi'}$-group (that is, G has a Hall π'-group). Moreover, if either A or G/A is soluble, then A is a $C_{\pi'}$-subgroup (that is, any two Hall π'-subgroups of G are conjugate).*

In 1928, Hall [6] proved that: *A finite soluble group has a Hall π-subgroup and any two Hall π-subgroups are conjugate in G.*

In 1949, Čunihin developed further the Schur-Zassenhaus and Hall theorems and proved the following classical result.

Theorem (S. A. Čunihin [1]). *If G is π-separable, then G is an E_π-group and an $E_{\pi'}$-group. Moreover, if G is π-soluble, then G is a C_π-group and a $C_{\pi'}$-group.*

Note that a group G is said to be π-separable if G has a normal series

$$(*) \qquad 1 = G_0 \leq G_1 \leq \ldots \leq G_{t-1} \leq G_t = G,$$

where each index $|G_i : G_{i-1}|$ is either a π-number or a π'-number. A group G is said to be π-soluble if each index $|G_i : G_{i-1}|$ of Series $(*)$ is either a π-prime power (that is, a power of some prime in π) or a π'-number.

The example of the group $PSL(2, 7)$ shows that the condition of normality for the members of Series $(*)$ could not be omitted.

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It is well known that the above Schur-Zassenhaus theorem, Hall theorem and Čunihin theorem are truly fundamental results of group theory. In connection with these important results, the following two problems have naturally arisen:

Problem I. Whether the conclusion of the Schur-Zassenhaus Theorem holds if the Hall subgroup A of G is not normal. In other words, can we weaken the condition of normality for the Hall subgroup A of G so that the conclusion of the Schur-Zassenhaus Theorem is still true?

Problem II. Whether we can replace the condition of normality for the members of Series $(*)$ by some weaker condition, for example, by permutability of the members of Series $(*)$ with some systems of subgroups of G.

Some results pertaining to Problem I have been obtained in [4, 5]. In Section 3 of this paper, we give the following further generalization of the Schur-Zassenhaus Theorem.

Theorem A. *Let A be a Hall π-subgroup of G. Let $G = AT$ for some subgroup T of G, and let q be a prime. If A permutes with every Sylow p-subgroup of T, for all primes $p \neq q$, and either A or T is soluble, then T contains a complement of A in G and any two complements of A in G are conjugate.*

Notice that the well known Feit-Thompson theorem about solvability of groups of odd order is not used in the proof of Theorem A. By using the Feit-Thompson theorem, we obtain the following stronger version of Theorem A.

Theorem A*. *Let A be a Hall π-subgroup of G. Let $G = AT$ for some subgroup T of G, and let q be a prime. If A permutes with every Sylow p-subgroup of T for all primes $p \neq q$, then T contains a complement of A in G and any two complements of A in G are conjugate.*

Recall that a subgroup H of G is said to be a supplement of a subgroup A in G if $AH = G$. Let

$$(**) \qquad 1 = H_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

be some subgroup series of G. We say that a subgroup series

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1 \leq T_0 = G$$

is a *supplement of Series* $(**)$ *in* G if T_i is a supplement of H_i in G for all $i = 0, 1, \ldots, t$.

Another purpose of this paper is to give a positive answer to Problem II. We will prove the following results.

Theorem B. *Suppose that G has a subgroup series*

$$1 = H_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a supplement

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1 \leq T_0 = G$$

of this series in G such that H_i permutes with every Sylow subgroup of T_i for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ is either a π-number or a π'-number, then G is an E_π-group and an $E_{\pi'}$-group. Moreover, if each π-index $|H_{i+1} : H_i|$ is a prime power, then G has a soluble Hall π-subgroup.

Corollary 1.1. *Suppose that G has a subgroup series*

$$1 = H_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a supplement

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1 \leq T_0 = G$$

of this series in G such that H_i permutes with all Sylow subgroups of T_i for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ $(i = 0, 1, \ldots, t-1)$ is a prime power, then G is an E_π-group, for any set π of primes.

Corollary 1.2. *Suppose that G has a subgroup series*

$$1 = H_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a supplement

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1 \leq T_0 = G$$

of this series in G such that H_i permutes with all Sylow subgroups of T_i for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ is a prime power, then G is soluble.

Theorem C. *Suppose that G has a subgroup series*

$$1 = H_0 < H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a subgroup T such that $G = H_1T$ and H_i permutes with all subgroups of T for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ is either a π-number or a π'-number, then G is a C_π-group and a $C_{\pi'}$-group.

The following example shows that, under the conditions of Theorems A, B or C, the group G is not necessarily π-separable.

Example 1.1. Let $G = A_5 \times C_7$, where C_7 is a group of order 7 and A_5 is the alternating group of degree 5. Let C_5 be a Sylow 5-subgroup of A_5. Consider the subgroup series

$$(***) \qquad 1 = H_0 < H_1 < H_2 < H_3 = G,$$

where $H_1 = A_4$ and $H_2 = A_5$. Then the series $1 = T_3 < T_2 < T_1 < T_0 = G$, where $T_2 = C_7$ and $T_1 = C_5 \times C_7$, is a supplement of Series $(***)$ in G. It is clear also that H_i permutes with all subgroups of T_i, for all i. Let $\pi = \{5, 7\}$. Then every index of Series $(***)$ is either a π-number or a π'-number. However, G is not π-separable.

2. Preliminaries

In this section, we cite some known results which are used in our proofs.

Lemma 2.1 (S. A. Čunihin [2, Theorem 1.4.2])**.** *Let N be a normal subgroup of G. If N and G/N are C_π-groups, then G is a C_π-group.*

Lemma 2.2 (O. Kegel [8, Theorem 3])**.** *Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^xA$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

Let A be a subgroup of G. A subgroup T is said to be a minimal supplement of A in G if $AT = G$ but $AT_0 \neq G$ for all proper subgroups T_0 of G.

The following lemma is obvious.

Lemma 2.3. *If N is normal in G and T is a minimal supplement of N in G, then $N \cap T \leq \Phi(T)$.*

Lemma 2.4 (P. Hall [7]). *Suppose that G has a Hall p'-subgroup for each prime p dividing $|G|$. Then G is soluble.*

Let A and B be subgroups of G and $\emptyset \neq X \subseteq G$. Following [4], we say that A is X-permutable (or A X-permutes) with B if $AB^x = B^x A$ for some $x \in X$.

The following lemma is also evident.

Lemma 2.5. *Let A, B, X be subgroups of G and $K \trianglelefteq G$. If A is X-permutable with B, then AK/K is XK/K-permutable with BK/K in G/K.*

Lemma 2.6 (O. Kegel [9, Theorem 3]). *If a subgroup A of G permutes with all Sylow subgroups of G, then A is subnormal in G.*

Lemma 2.7 (H. Wielandt [11]). *If a π-subgroup A of G is subnormal in G, then $A \leq O_\pi(G)$.*

3. Proofs of Theorems A and A*

Theorem A is a special case (when $X = 1$) of the following theorem.

Theorem 3.1. *Let X be a normal π-separable subgroup of G and A a Hall π-subgroup of G. Let $G = AT$ for some subgroup T of G, and let q be a prime. If A is X-permutable with every Sylow p-subgroup of T for all primes $p \neq q$ and either A is soluble or every π'-subgroup of T is soluble, then T contains a complement of A in G and any two complements of A in G are conjugate.*

Proof. Suppose that this theorem is false and let G be a counterexample of minimal order. Then, clearly, T is not a subgroup of G with prime power order and $|\pi'| \geq 2$. We now proceed with the proof via the following steps.

(1) $X = 1$.

Suppose that $X \neq 1$ and let D be a minimal normal subgroup of G contained in X. Then D is either a π-group or a π'-group. In the former case we have $D \leq A$. Otherwise, $D \leq T$. We first claim that the hypothesis is still true for G/D. Clearly, $G/D = (AD/D)(TD/D)$, where AD/D is a Hall π-subgroup of G/D and X/D is a normal π-separable subgroup of G/D. Moreover, if A is soluble, then $AD/D \simeq A/(A \cap D)$ is soluble. Suppose that every π'-subgroup of T is soluble. Let V/D be a π'-subgroup of TD/D. Then $V = V \cap TD = D(V \cap T)$. Since $V/D = D(V \cap T)/D \simeq (V \cap T)/(V \cap T \cap D)$, $(V \cap T)/(V \cap T \cap D)$ is a π'-group. If D is a π'-group, then $D \leq T$ and so V is a π'-subgroup of T. Hence V is soluble and thereby V/D is soluble. Now assume that D is a π-group. Then $V \cap T = [V \cap T \cap D]E$ for a Hall π'-subgroup E of $V \cap T$ by the Schur-Zassenhaus Theorem. Since E is soluble by hypothesis, V/D is soluble. Thus every π'-subgroup of TD/D is soluble. Now let Q/D be a Sylow p-subgroup of TD/D, where $p \neq q$. Then for some Sylow p-subgroup P of T, we have $Q/D = DP/D$. By hypothesis, A X-permutes with P. Hence, AD/D is XD/D-permutable with $Q/D = DP/D$ in G/D by Lemma 2.5. Therefore, our claim holds.

Since $|G/D| < |G|$, the minimal choice implies that TD/D contains a complement V/D of AD/D in G/D and every two complements of AD/D in G/D are conjugate. Obviously, $V/D = (V \cap TD)/D = D(V \cap T)/D \simeq (V \cap T)/(V \cap T \cap D)$. Since AD/D is a Hall π-subgroup of G/D, V/D is a Hall π'-subgroup of G/D. If D is a π'-group, then V is a Hall π'-subgroup of G. Hence V is a complement of A in G. If D is a π-group, then by the Schur-Zassenhaus Theorem, $V \cap T = [V \cap T \cap D]E$, for a Hall π'-subgroup E of $V \cap T$. It follows that $V = D(V \cap T) = DE$ and so $G = AE$. Thus, E is a complement of A in G since E is a Hall π'-subgroup of G.

Now let T_1 and T_2 be Hall π'-subgroups of G, where $T_2 \leq T$. Then $T_1D/D = {T_2}^xD/D$, for some $x \in G$. If D is a π'-group, then $T_1 = {T_2}^x$, which contradicts the choice of G. Hence D is a π-group. By hypothesis, either D or ${T_2}^x$ is soluble. Therefore, by the Schur-Zassenhaus Theorem, T_1 and ${T_2}^x$ are conjugate in T_1D. This implies that every Hall π'-subgroup of G is conjugate with T_2 and hence every two complements of A in G are conjugate, which contradicts the choice of G.

(2) $O_\pi(G) = 1$ and $O_{\pi'}(G) = 1$ (see the proof of (1)).

(3) *A permutes with every Sylow p-subgroup P of G, for all $p \neq q$ such that a Sylow p-subgroup of T is a Sylow p-subgroup of G.*

Since a Sylow p-subgroup T_p of T is a Sylow p-subgroup of G, $P = {T_p}^x$, for some $x \in G$. Because $G = AT$, $x = ta$, where $a \in A$ and $t \in T$. Since A permutes with the Sylow subgroup ${T_p}^t$ of T, we obtain that A permutes with ${T_p}^x = P$.

(4) *G is not simple.*

Let P be any Sylow p-subgroup of G, where $q \neq p \in \pi'$. Then by (2), $AP^x = P^xA$, for all $x \in G$. Besides, $AP \neq G$ since $|\pi'| \geq 2$. Hence G is not simple by Lemma 2.2.

(5) *T has a Hall π'-subgroup.*

Suppose that it is false. Then $D = A \cap T \neq 1$. Obviously, $T = DT$. Since A is a Hall π-subgroup of G, D is a Hall π-subgroup of T. Let P be a Sylow p-subgroup of T, where $p \neq q$. Since $AP = PA$ by (1), $(A \cap T)P = AP \cap T = PA \cap T = P(A \cap T)$. Hence the hypothesis holds for (D, T). If $T \neq G$, then T is a $C_{\pi'}$-group by the choice of G. In particular, T has a Hall π'-subgroup E, which, evidently, is a Hall π'-subgroup G.

Now assume that $T = G$. First suppose that A is a q-group. Let D be a proper normal subgroup of G. We show that D is a C_π-group. Let $p \neq q$ be a prime dividing $|D|$, P a Sylow p-subgroup of D and G_p a Sylow p-subgroup of G containing P. Then by hypothesis, $AG_p = G_pA$. Hence, $AG_p \cap D$ is a Hall $\{q, p\}$-subgroup of D. Besides, $A \cap D$ is a Sylow q-group of D and $P = G_p \cap D$. Since $(A \cap D)P \leq AG_p \cap D$ and $|AG_p \cap D| = |A \cap D||P|$, we have that $(A \cap D)P = AG_p \cap D = P(A \cap D)$. Therefore the hypothesis holds for $D = (A \cap D)D$. This implies that D is a $C_{\pi'}$-group by the choice of G. Let $D_{\pi'}$ be a Hall π'-subgroup of D. By the Frattini argument, $G = DN$, where $N = N_G(D_{\pi'})$. It follows from $|G : N| = |D : N \cap D|$ that $|G : N| = q^a$. Let A_0 be a Sylow q-subgroup of N. Since $T = G$, by (1), A permutes with any Sylow p-subgroup P of G, where $p \neq q$. Hence A^x also permutes with all Sylow p-subgroups of G, where $p \neq q$. We may, therefore, assume that $A_0 \leq A$. Then $A \cap N = A_0$ and, clearly, the hypothesis holds on $N = A_0N$. In view of (2), $N \neq G$. Hence N is a $C_{\pi'}$-group by the choice of G. Let E be a Hall π'-subgroup of N. Then, evidently, E is also a Hall π'-subgroup of G since $|G : N| = q^a$.

Now let T_1 and T_2 be Hall π'-subgroups of G. Then $D_1 = T_1 \cap D$ and $D_2 = T_2 \cap D$ are Hall π'-subgroups of D. Hence D_1 and D_2 are conjugate in D. It follows that $N_G(D_1) = N_G(D_2)^x$ for some $x \in G$. Since $T_1 \leq N_G(D_1)$ and $T_2 \leq N_G(D_2)$, T_1 is a conjugate of some Hall π'-subgroup of $N_G(D_2)$. Hence T_1 and T_2 are conjugate in G. This contradiction shows that A is not a Sylow q-subgroup of G. Let P be any Sylow p-subgroup of G, where $p \neq q$ is a prime dividing $|A|$. Since $T = G$, by (2), $AP^x = P^xA = A$, for all $x \in G$. Hence $P^G \leq A$, which contradicts (2).

(6) *G/D is a $C_{\pi'}$-group, for every non-trivial normal subgroup D of G.*

In view of (5), we may, without loss of generality, assume that T is a Hall π'-subgroup of G. Hence, as in the proof of (1), we obtain that G/D satisfies the hypothesis. The minimal choice of G implies that G/D is a $C_{\pi'}$-group.

(7) *Every proper normal subgroup D of G is a $C_{\pi'}$-group.*

By (5), we may assume that $G = AT$, where A is a Hall π-subgroup and T is a Hall π'-subgroup. Then $D = (D\cap A)(D\cap T)$. Let $p \neq q$ be a prime dividing $|T\cap D|$, P a Sylow p-subgroup of $T \cap D$ and G_p a Sylow p-subgroup of G containing P. Then by hypothesis, $AG_p = G_pA$. Hence $AG_p \cap D$ is a Hall $\pi \cup \{p\}$-subgroup of D. Besides, $A\cap D$ is a Hall π-subgroup of D and $P = G_p\cap D$. Since $(A\cap D)P \leq AG_p\cap D$ and $|AG_p \cap D| = |A \cap D||P|$, $(A \cap D)P = AG_p \cap D = P(A \cap D)$. Therefore the hypothesis holds for $D = (A \cap D)(D \cap T)$. The minimal choice implies that D is a $C_{\pi'}$-group.

Final contradiction. By (4), G has a proper normal subgroup $D \neq 1$. By (6) and (7) both D and G/D are $C_{\pi'}$-groups. Hence G is a $C_{\pi'}$-group by Lemma 2.1. The final contradiction completes the proof.

Proof of Theorem A.* In view of the Feit-Thompson Theorem about solvability of groups of odd order, we know that either every π-group or every π'-group is soluble. Hence Theorem A* is a corollary of Theorem A.

4. Proof of Theorem B

Theorem B is a special case (when $X = 1$) of the following theorem.

Theorem 4.1. *Let X be a normal π-separable subgroup of G. Suppose that G has a subgroup series*

$$1 = H_0 \leq H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a supplement

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1 \leq T_0 = G$$

of this series in G such that H_i X-permutes with every Sylow subgroup of T_i for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ is either a π-number or a π'-number, then G is an E_π-group and an $E_{\pi'}$-group. Moreover, if each π-index $|H_{i+1} : H_i|$ is a prime power, then G has a soluble Hall π-subgroup.

Proof. Suppose that this theorem is false and let G be a counterexample of minimal order. Without loss of generality, we may assume that $H_1 \neq 1$. We proceed with the proof by proving the following claims:

(1) *The assertions of the theorem hold for every non-trivial quotient G/N of G.*

We consider the series

$$1 = H_0N/N \leq H_1N/N \leq \ldots \leq H_{t-1}N/N \leq H_tN/N = G/N \tag{1}$$

and its supplement

$$1 = T_tN/N \leq T_{t-1}N/N \leq \ldots \leq T_1N/N \leq T_0N/N = G/N$$

in G/N. By Lemma 2.5, H_iN/N is XN/N-permutable with any Sylow subgroup of T_iN/N for all $i = 1, 2, \ldots, t$. On the other hand, since $|H_{i+1}N/N : H_iN/N| = |H_{i+1}N : H_iN| = |H_{i+1} : H_i| : |N \cap H_{i+1} : N \cap H_i|$, every index of the series (1) is either a π-number or a π'-number (a π-prime power or a π'-number). Moreover, obviously, $XN/N \simeq X/(X \cap N)$ is π-separable. This shows that the hypothesis holds on G/N. Hence in the case $N \neq 1$, the assertions of the theorem hold for G/N by the choice of G.

(2) $O_{\pi'}(G) = 1 = O_\pi(G)$.

Suppose that $D = O_{\pi'}(G) \neq 1$. Then by (1), G/D has a Hall π'-subgroup A/D and a Hall π-subgroup B/D. Then, obviously, A is a Hall π'-subgroup of G. By the Schur-Zassenhaus Theorem, D has a complement V in B, which, clearly, is a Hall π-subgroup of G. Hence G is an E_π-group and an $E_{\pi'}$-group. Besides, if every π-index of series (1) is a prime power, then B/D has a soluble Hall π-subgroup B/D. It follows that V is also soluble. This contradiction shows that $O_{\pi'}(G) = 1$. Analogously, we may prove that $O_\pi(G) = 1$.

(3) $X = 1$.

Indeed, if N is a minimal normal subgroup of G contained in X, then N is either a π-group or a π'-group, which contradicts (2).

(4) $T_1 \neq G$.

Suppose that $T_1 = G$. Then by hypothesis and (3), H_1 permutes with all Sylow subgroups of G. It follows from Lemma 2.6 that H_1 is subnormal in G. Since H_1 is either a π-group or a π'-group, $H_1 \leq O_\pi(G)$ or $H_1 \leq O_{\pi'}(G)$ by Lemma 2.7. It follows from (2) that $H_1 = 1$, which contradicts $H_1 \neq 1$. Hence (4) holds.

(5) *The assertions of the theorem hold for* T_1.

We consider the series

$$1 = H_0 \cap T_1 \leq H_1 \cap T_1 \leq \ldots \leq H_{t-1} \leq H_t \cap T_1 = T_1. \tag{2}$$

Then the series

$$1 = T_t \leq T_{t-1} \leq \ldots \leq T_1$$

is a supplement of the series (2) in T_1 since $(H_i \cap T_1)T_i = H_iT_i \cap T_1 = G \cap T_1 = T_1$. Since $H_{i+1} = H_iT_1 \cap H_{i+1} = H_i(H_{i+1} \cap T_1)$, $|H_{i+1} : H_i| = |H_{i+1} \cap T_1 : H_i \cap T_1|$, for all $i = 1, 2, \ldots, t-1$ and $|H_1 \cap T_1 : H_0 \cap T_1| = |H_1 \cap T_1| \leq |H_1 : 1|$, we see that every index of the series (2) is either a π-number or a π'-number. Moreover, if every π-index of the series (1) is a prime power, then every π-index of the series (2) is a prime power. Now let E be a Sylow subgroup of T_i. By (3) and the hypothesis, $H_iE = EH_i$. Hence $H_iE \cap T_1 = E(H_i \cap T_1) = (H_i \cap T_1)E$. This shows that the hypothesis holds for T_1. The minimal choice of G implies that (5) holds.

Final contradiction. Let $(T_1)_\pi$ and $(T_1)_{\pi'}$ be a Hall π-subgroup and a Hall π'-subgroup of T_1, respectively. By (3) and the hypothesis, H_1 permutes with all Sylow subgroups of $(T_1)_\pi$. Hence H_1 permutes with $(T_1)_\pi$. Similarly, H_1 permutes with $(T_1)_{\pi'}$. By hypothesis, H_1 is either a π-group or a π'-group. Assume that H_1 is a π-group. Since $G = H_1T_1$, we see that $G_\pi = H_1(T_1)_\pi$ is a Hall π-subgroup of G and $(T_1)_{\pi'}$ is a Hall π'-subgroup of G. If H_1 is a π'-group, then $G_{\pi'} = H_1T_{\pi'}$ is a Hall π'-subgroup of G and T_π is a Hall π-subgroup of G. Finally, we prove that if every π-index of the series (1) is a prime power, then G has a soluble Hall π-subgroup. In fact, by (5), we see that $(T_1)_\pi$ is soluble. If H_1 is a π'-group, then $(T_1)_\pi$ is a soluble Hall π-subgroup of G since $G = H_1T_1$. If H_1 is a p-group, then $H_1(T_1)_\pi$ is a Hall π-subgroup of G. Since $(T_1)_\pi$ is soluble and H_1 permutes with every Sylow subgroup of $(T_1)_\pi$, we see that $H_1(T_1)_\pi$ is soluble by Lemma 2.4. The contradiction completes the proof.

5. Proof of Theorem C

Theorem C is a special case (when $X = 1$) of the following theorem.

Theorem 5.1. *Let X be a normal π-separable subgroup of G. Suppose that G has a subgroup series*

$$1 = H_0 < H_1 \leq \ldots \leq H_{t-1} \leq H_t = G,$$

where $H_1 \neq 1$, and a subgroup T such that $TH_1 = G$ and H_i X-permutes with all nilpotent subgroups of T for all $i = 1, 2, \ldots, t$. If the index $|H_{i+1} : H_i|$ is either a π-number or a π'-number, then G is a C_π-group and a $C_{\pi'}$-group as well.

Proof. Suppose that this theorem is false and let G be a counterexample of minimal order. By Theorem 3.2, G has a Hall π-subgroup S and a Hall π'-subgroup J. Hence we may assume that either some Hall π-subgroup S_1 of G is not conjugate with S or some Hall π'-subgroup J_1 of G is not conjugate with J. We may, without loss of generality, assume that $2 \notin \pi$. Then S is soluble by the Feit-Thompson Theorem of groups of odd order. We proceed with the proof via the following steps.

(1) *The assertion of the theorem holds for every non-trivial quotient G/N of G.*

Consider the series

$$1 = H_0N/N \leq H_1N/N \leq \ldots \leq H_{t-1}N/N \leq H_tN/N = G/N \tag{3}$$

of G/N. Then $(H_1N/N)(TN/N) = G/N$. Let V/N be any nilpotent subgroup of TN/N. Then $V = N(V \cap T)$ and so $V/N \simeq V \cap T/V \cap T \cap N$. Let V_0 be a minimal supplement of $V \cap T \cap N$ in $V \cap T$. Then by Lemma 2.3, V_0 is nilpotent. Hence H_i X-permutes with V_0 by hypothesis. Besides, $V/N = N(V \cap T) = NV_0(V \cap T \cap N)/N = NV_0/N$. Hence by Lemma 2.5, H_iN/N is XN/N-permutable with any nilpotent subgroup of TN/N, for all $i = 1, 2, \ldots, t$. This shows that the hypothesis holds for G/N. Hence, in the case $N \neq 1$, the assertion of the theorem holds for G/N by the choice of G.

(2) $O_{\pi'}(G) = 1 = O_\pi(G)$.

Suppose that $D = O_{\pi'}(G) \neq 1$. Then by (1), there are elements $x, y \in G$ such that ${S_1}^xD = SD$ and ${J_1}^yD = JD$. Since $SD/D \simeq S$ is soluble, by the Schur-Zassenhaus Theorem, ${S_1}^x$ and S are conjugate in SD. On the other hand, since $D \subseteq J$, ${J_1}^y = J$. This contradiction shows that $O_{\pi'}(G) = 1$. Analogously, we can prove that $O_\pi(G) = 1$.

(3) $X = 1$.

Indeed, if N is a minimal normal subgroup of G contained in X, then P is either a π-group or a π'-group, which contradicts (2).

(4) $T \neq G$ (see the proof of (4) in the proof of Theorem 3.2).

(5) *The assertions of the theorem hold for T* (see the proof of (5) in the proof of Theorem 3.2).

(6) *If D is a normal subgroup of G and $H_1 \leq D$, then $D = G$.*

Suppose that $D \neq G$. Let $D_i = H_i \cap D$, for all $i = 1, 2, \ldots, t$. Consider the series

$$1 = D_0 \leq D_1 \leq \ldots \leq D_{t-1} \leq D_t = D. \tag{4}$$

First note that $D = D \cap H_1T = H_1(D \cap T)$. Let E be a nilpotent subgroup of $D \cap T$. Then $H_iE = EH_i$ and so $H_iE \cap D = E(H_i \cap D) = ED_i = D_iE$. Thus D_i permutes with every nilpotent subgroup of $D \cap T$ for all $i = 1, 2, \ldots, t-1$. On the other hand, $|D_i : D_{i-1}| = |(D \cap H_i)H_{i-1} : H_{i-1}|||H_i : H_{i-1}|$. Hence each index $|D_i : D_{i-1}|$ is either a π-number or a π'-number. Therefore D is a C_π-group and a $C_{\pi'}$-group by the choice of G. Since $1 \neq H_1 \leq D$, G/D is a C_π-group and a $C_{\pi'}$-group by (1) and the choice of G. It follows from Lemma 2.1 that G is a C_π-group and a $C_{\pi'}$-group, which contradicts the choice of G. Hence, (6) holds.

(7) *If H_1 is a π-group (π'-group) and D is a normal subgroup of G containing a Hall π-subgroup of T (containing a Hall π'-subgroup of T, respectively), then the hypothesis holds for D.*

Suppose, for example, that H_1 is a π-group. We claim that $D = (H_1 \cap D)(T \cap D)$. In fact, let E be a Hall π-subgroup of T contained in D and $T_{\pi'}$ a Hall π'-subgroup of T. Since H_1 is a π-group and $G = H_1T$, $T_{\pi'}$ is also a Hall π'-subgroup of G. Clearly H_1E is a Hall π-subgroup of G. Hence $D = (D \cap H_1E)(D \cap T_{\pi'}) = E(D \cap H_1)(D \cap T_{\pi'}) = (D \cap H_1)(E(D \cap T_{\pi'})) = (D \cap H_1)(T \cap D)$. Thus, our claim holds. Now, by similar inference as in (6), we see that the hypothesis holds for D.

(8) *If H_1 is a π-group (π'-group) and E is a Hall π-subgroup of T (a Hall π'-subgroup of T), then $E^G \neq G$.*

Assume, for example, that H_1 is a π-group. Since $G = H_1T$, we have that $x = ht$, where $h \in H_1$ and $t \in T$, for any $x \in G$. Because H_1 permutes with all Sylow subgroups of T, $H_1E^t = E^tH_1$. Hence $H_1E^x = H_1E^{th} = E^{th}H_1$. Now by Lemma 2.2, either ${H_1}^G \neq G$ or $E^G \neq G$. But in view of (7), the former case is impossible. Hence $E^G \neq G$.

Final contradiction. In view of (1), (7), (8) and Lemma 2.1, G is a C_π-group and a $C_{\pi'}$-group. The contradiction completes the proof.

Remark. We prove Theorem C on the base of the Feit-Thompson Theorem of groups of odd order. The following fact may be proved without using this deep result.

Theorem. *Suppose that G has a subgroup series*

$$1 = H_0 < H_1 \leq \ldots \leq H_{t-1} \leq H_t = G$$

and a subgroup T such that $G = H_1T$ and H_i permutes with all subgroups of T for all $i = 1, 2, \ldots, t$. If each index $|H_{i+1} : H_i|$ is either a π-number or a π'-number, then G is an E_π-group and an $E_{\pi'}$-group. Moreover, if each π-index $|H_{i+1} : H_i|$ is a prime power, then G is a C_π-group and a $C_{\pi'}$-group.

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