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On Identities and m-Neutral Sequences of n-Ary Groups

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Abstract In this paper, we study such polyadic analog of an identity of a group as m-neutral sequence. In particular, we prove that all Post's equivalence classes of the free covering group of any n-ary group [where $n = k(m-1)+1$ and $k \geq 1$] defined by m-neutral sequences form the $(k+1)$-ary group, which is isomorphic to the n-ary subgroup of all identities of the n-ary group in the case when $m = 2$.

Keywords n-Ary group · Identity · m-Neutral sequence

Mathematics Subject Classification 20N15

1 Introduction

One of the main results of the paper [1] states that the set of all identities of any n-ary group (in the case when thus set is not empty) is a characteristic n-ary subgroup contained in the center of the n-ary group. In the binary case (when $n = 2$) it is trivial. But in the case when $n > 2$ this statement is more informative, since in this case there can be more than one identity in the n-ary group. Moreover, there are non-one-element n-ary groups in which every element is identity. Different informations on the structure

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of the n-ary subgroup of all identities of an n-ary group can be found in [2]. Note only that in the case when $n = 3$ the following conditions hold:

(1) If a finite ternary group has more than one identity, then the orders of the center, of the ternary subgroup of all identities of this ternary group, and of this ternary group itself are even;
(2) If a finite ternary group has more than two identities, then the orders of the center, of the ternary subgroup of all identities of this ternary group, and of this ternary group itself are divisible by 4.

Conditions (1) and (2) are special cases of the following result in [3]:

(3) If $k \geq 1$, p is a prime and a finite $(p+1)$-ary group $<A, [\ ]>$ has more than p^{k-1} identities, then the orders of the center, of the $(p+1)$-ary subgroup of all identities of this $(p+1)$-ary group, and of this $(p+1)$-ary group itself are divisible by p^k.

In this paper, we continue to study the polyadic analogs of an identity of a group. In particular, we show that the main result in [1] is a corollary of the main result of this paper.

2 Preliminaries

All information of this section can be found in [2,4,5].

Recall that a universal algebra $< A, [\ ] >$ with one n-ary operation $[\ ] : A^n \to A$ $(n \geq 2)$ is called an n-ary group (Dörnte [6]) if $[\ ]$ is associative, i.e.,

$$[[a_1 \dots a_n]a_{n+1} \dots a_{2n-1}] = [a_1 \dots a_i[a_{i+1} \dots a_{i+n}]a_{i+n+1} \dots a_{2n-1}]$$

for all $i = 1, 2, \dots, n-1$ and all $a_1, \dots, a_{2n-1} \in A$, and the equation $[a_1 \dots a_{i-1}x_i a_{i+1} \dots a_n] = b$ can be uniquely solved in A for every $i = 1, 2, \dots, n$ and all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in A$.

It is evident that every group is an n-ary group, where $n = 2$.

Remark 2.1 In [7], Post noted that the unique solvability of the equation in Dörnte's definition can be replaced by the solvability of this equation. Other different definitions of an n-ary group can be found in [2]. Note only that by definition of Skiba and Tjutin [8], a universal algebra $< A, [\ ] >$ with one n-ary operation $[\ ] : A^n \to A$ $(n \geq 2)$ is called an n-ary group if $[\ ]$ is associative and either two equations $[x \underbrace{a \dots a}_{n-1}] = b$ and $[\underbrace{a \dots a}_{n-1} x] = b$ are solved for any $a, b \in A$ or (in the case when $n \geq 3$) only one equation $[\underbrace{a \dots a}_{i-1} x \underbrace{a \dots a}_{n-i}] = b$ is solved for any $a, b \in A$.

Remark 2.2 If $k \geq 1$, then

$$[[\dots[[a_1 \dots a_n]a_{n+1} \dots a_{2n-1}] \dots]a_{(k-1)(n-1)+2} \dots a_{k(n-1)+1}] = [a_1 \dots a_{k(n-1)+1}].$$

Since an n-ary operation [ ] is associative, all inner operation in the left part of this equality can be placed by different ways.

Recall that an element e of an n-ary group $< A, [\ ] >$ is called an identity of $< A, [\ ] >$(Dörnte [6]) if

$$[x\underbrace{e\ldots e}_{n-1}] = [ex\underbrace{e\ldots e}_{n-2}] = \cdots = [\underbrace{e\ldots e}_{n-2}xe] = [\underbrace{e\ldots e}_{n-1}x] = x$$

for every $x \in A$ and $i = 1, 2, \ldots, n$. This definition is an n-ary generalization of the definition of the identity of a group A as the element $e \in A$ such that $ex = xe = x$ for any $x \in A$.

One more n-ary generalization of an identity of a group is a neutral sequence. Recall that a sequence $e_1 \ldots e_{s(n-1)}$ of the elements of an n-ary group $<A, [\ ]>$, where $s \geq 1$, is said to be neutral (Post [7]) if $[e_1 \ldots e_{s(n-1)}x] = [xe_1 \ldots e_{s(n-1)}] = x$ for every $x \in A$.

Recall also that a sequence β of the elements of an n-ary group $<A, [\ ]>$ is said to be inverse for a sequence α of the elements of this group (Post [7]) if the sequences $\alpha\beta$ and $\beta\alpha$ are neutral.

Remark 2.3 The following statements can be proved:

(1) If e is an element of an n-ary group $<A, [\ ]>$ such that $[x\underbrace{e\ldots e}_{n-1}] = [ex\underbrace{e\ldots e}_{n-2}] = x$ for every $x \in A$, then e is an identity of A;
(2) If $e_1 \ldots e_{s(n-1)}$ is a sequence of the elements of an n-ary group $<A, [\ ]>$ such that for any $x \in A$ either $[e_1 \ldots e_{s(n-1)}x] = x$ or $[xe_1 \ldots e_{s(n-1)}] = x$, then $e_1 \ldots e_{s(n-1)}$ is neutral;
(3) If α and β are the sequences of an n-ary group $<A, [\ ]>$ such that one of the sequences $\alpha\beta$ or $\beta\alpha$ is neutral, then the sequence β is inverse for α.

Let $<A, [\ ]>$ be an n-ary group, F_A a free semigroup over an alphabet A and θ_A a Post's equivalence relation [7] defined on F_A by the rule $(\alpha, \beta) \in \theta_A$ iff there are the sequences $\gamma, \delta \in F_A$ such that $[\gamma\alpha\delta] = [\gamma\beta\delta]$. It is easy to show that A is a congruence on the semigroup F_A and the semigroup $A^* = F_A/\theta_A$ is a group which is said to be a free covering group [or abstract containing group (Post)] for the n-ary group $<A, [\ ]>$. The class $\theta_A(\varepsilon)$ is an identity of A, where ε is any neutral sequence of A, and $\theta_A(\beta)$ is an inverse sequence for $\theta_A(\alpha)$, where β is any inverse sequence for a sequence α.

In the future, we use the symbol θ to denote θ_A.

For every $i = 1, \ldots, n-1$, we put

$$A^{(i)} = \{\theta(\alpha) \in A^* | l(\alpha) = s(n-1) + i, s \geq 0\},$$

where $\theta(\alpha)$ is a class of the congruence θ containing a sequence α, $l(\alpha)$ is the length of the sequence α.

It is evident that $A^{(i)} = \{\theta(\alpha) \in A^* | l(\alpha) = i\}$. In particular, $A' = \{\theta(a) | a \in A\}$.

If we fix the elements $a_1, \dots, a_{i-1} \in A$, then

$$A^{(i)} = \{\theta(aa_1 \dots a_{i-1}) | a \in A\} = \{\theta(a_1 \dots a_{i-1}a) | a \in A\}.$$

The symbol A_0 is used to denote the set $A^{(n-1)}$.

Remark 2.4 It is easy to prove that in the case when $n = k(m-1) + 1$, where $n \geq 3$ and $m \geq 2$, the set $A^{(m-1)}$ is a $(k+1)$-ary group with $(k+1)$-ary operation

$$[\theta(\alpha_1)\theta(\alpha_2) \dots \theta(\alpha_{k+1})]_{k+1} = \theta(\alpha_1\alpha_2 \dots \alpha_{k+1}).$$

If $m = 2$, then $k = n - 1$ and we get the n-ary operation

$$[\theta(a_1)\theta(a_2) \dots \theta(a_n)]_n = \theta(a_1a_2 \dots a_n) = \theta([a_1a_2 \dots a_n]), a_1, a_2, \dots, a_n \in A.$$

In this case, the map $a \to \theta(a)$ is an isomorphism of an n-ary group $<A, [\ ]>$ on an n-ary group $<A', [\ ]_n>$.

If $m = n$, then $k = 1$ and we get the binary operation

$$\begin{aligned}&[\theta(a_1a_2 \dots a_{n-1})\theta(b_1b_2 \dots b_{n-1})]_2 \\ &\quad = \theta(a_1a_2 \dots a_{n-1}b_1b_2 \dots b_{n-1}) = \theta([a_1a_2 \dots a_{n-1}b_1]b_2 \dots b_{n-1}),\end{aligned}$$

where $a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1} \in A$. The group $< A^{(n-1)} = A_0, [\ ]_2 >$ is called an associated group for an n-ary group $<A, [\ ]>$ (Post) and denoted by A_0.

Recall that the center of an n-ary group $<A, [\ ]>$ [5] is the set

$$Z(A) = \{z \in A | (zx, xz) \in \theta \text{ for all } x \in A\}.$$

The center of an n-ary group $<A, [\ ]>$ can be defined as follows:

$$Z(A) = \{z \in A | [zx_1x_2 \dots x_{n-1}] = [x_1zx_2 \dots x_{n-1}] \text{ for all } x_1, \dots, x_{n-1} \in A\}.$$

Recall also that an n-ary subgroup $<B, [\ ]>$of an n-ary group $<A, [\ ]>$ is said to be invariant in $<A, [\ ]>$ (Post [7]) if $[\alpha B\gamma] = B$ for any sequence α of the elements of A, where γ is any inverse sequence for α (see also [2,4,5]).

The center of any n-ary group is an example of an invariant n-ary subgroup.

Finally, note that the invariance is preserved under all isomorphisms of the n-ary groups.

Remark 2.5 If $<B, [\ ]>$ is an n-ary subgroup of an n-ary group $< A, [\ ] >$ such that $[\alpha B\gamma] \subseteq B$ for any sequence α of the elements of A, where γ is any inverse sequence for α, then B is invariant in A.

3 m-Neutral Sequences and Their Properties

The following definition combines the concepts of a neutral sequence and an identity of an n-ary group.

Definition 3.1 (see [4]). Let $<A, [\ ]>$ be an n-ary group, where $n = k(m-1)+1$ and $k \geq 1$. A sequence $\alpha = e_1 \dots e_{t(n-1)+m-1}$ of the elements of A, where $t \geq 0$, is said to be m-neutral, if $[\underbrace{\alpha \dots \alpha}_{j-1} x \underbrace{\alpha \dots \alpha}_{k-j+1}] = x$ for every $x \in A$ and $j = 1, \dots, k+1$.

It is evident that the n-neutral sequences of the elements of an n-ary group are the neutral sequences of this group. Moreover, every identity of an n-ary group is a 2-neutral sequence and every 2-neutral sequence $e_1 \dots e_{t(n-1)+1}$ is either an identity (in the case when $t = 0$) or identified with identity $[e_1 \dots e_{t(n-1)+1}]$ (in the case when $t \geq 1$).

The following proposition can be proved by the simple calculations.

Proposition 3.2 *Let $< A, [\ ] >$ be an n-ary group, where $n = k(m-1)+1$. A sequence α of the elements of A is m-neutral if and only if the following conditions hold*:

(1) *For any $x \in A$, the sequences αx and $x\alpha$ are equivalent, that is, $\theta(\alpha)\theta(x) = \theta(x)\theta(\alpha)$;*
(2) *The sequence $[\underbrace{\alpha \dots \alpha}_{k}]$ is neutral.*

Proposition 3.3 *Let α be an m-neutral sequence of an n-ary group $< A, [\ ] >$, where $n = k(m-1)+1$, γ any inverse sequence for α and β any sequence from A. The following hold*:

(1) *The sequences $\alpha\beta$ and $\beta\alpha$ are equivalent*;
(2) *The sequences $\gamma\beta$ and $\beta\gamma$ are equivalent*;
(3) $\theta(\underbrace{\alpha \dots \alpha}_{j-1} \beta \underbrace{\alpha \dots \alpha}_{k-j+1}) = \theta(\beta)$ *and* $\theta(\underbrace{\gamma \dots \gamma}_{j-1} \beta \underbrace{\gamma \dots \gamma}_{k-j+1}) = \theta(\beta)$ *for any* $j = 1, \dots, k+1$.

Proof (1) This assertion directly follows from Proposition 3.2.
(2) Let δ be any inverse sequence for the sequence β. By Claim (1), $\theta(\alpha)\theta(\delta) = \theta(\delta)\theta(\alpha)$ and so

$$\begin{aligned}
(\theta(\alpha)\theta(\delta))^{-1} &= (\theta(\delta)\theta(\alpha))^{-1},\\
(\theta(\delta))^{-1}(\theta(\alpha)^{-1}) &= (\theta(\alpha))^{-1}(\theta(\delta))^{-1},\\
\theta(\beta)\theta(\gamma) &= \theta(\gamma)\theta(\beta),\\
\theta(\beta\gamma) &= \theta(\gamma\beta).
\end{aligned}$$

(3) By Claim (1), we have

$$\theta(\underbrace{\alpha \dots \alpha}_{j-1} \beta \underbrace{\alpha \dots \alpha}_{k-j+1}) = \theta(\underbrace{\alpha \dots \alpha}_{k})\theta(\beta).$$

Since $\underbrace{\alpha\ldots\alpha}_{k}$ is neutral by Proposition 3.2(2), $\theta(\underbrace{\alpha\ldots\alpha}_{j-1}\beta\underbrace{\alpha\ldots\alpha}_{k-j+1}) = \theta(\beta)$. Hence

$$\theta(\underbrace{\gamma\ldots\gamma}_{j-1})\theta(\underbrace{\alpha\ldots\alpha}_{j-1}\beta\underbrace{\alpha\ldots\alpha}_{k-j+1})\theta(\underbrace{\gamma\ldots\gamma}_{k-j+1} = \theta(\underbrace{\gamma\ldots\gamma}_{j-1})\theta(\beta)\theta(\underbrace{\gamma\ldots\gamma}_{k-j+1},$$
$$\theta(\underbrace{\gamma\ldots\gamma}_{j-1}\underbrace{\alpha\ldots\alpha}_{j-1}\beta\underbrace{\alpha\ldots\alpha}_{k-j+1}\underbrace{\gamma\ldots\gamma}_{k-j+1}) = \theta(\underbrace{\gamma\ldots\gamma}_{j-1}\beta\underbrace{\gamma\ldots\gamma}_{k-j+1}).$$

Since γ is an inverse sequence for α, it follows that

$$\theta(\underbrace{\gamma\ldots\gamma}_{j-1}\beta\underbrace{\gamma\ldots\gamma}_{k-j+1}) = \theta(\beta).$$

The proposition is proved. □

From Proposition 3.3(3) we get

Corollary 3.4 *If α is an m-neutral sequence of an n-ary group $< A, [\ ] >$, where $n = k(m-1)+1$, γ is any inverse sequence for α, then $[\underbrace{\gamma\ldots\gamma}_{j-1}x\underbrace{\gamma\ldots\gamma}_{k-j+1}] = x$ for any $x \in A$ and $j = 1, \ldots, k+1$.*

Proposition 3.5 *Let $r \geq 1, k \geq 1, s = kr, m = r(l-1)+1, n = k(m-1)+1 = s(l-1)+1$, β be an l-neutral sequence of an n-ary group $<A, [\ ]>$. Then $\alpha = \underbrace{\beta\ldots\beta}_{r}$ is an m-neutral sequence of $< A, [\ ] >$.*

Proof Since β is an l-neutral sequence of $< A, [\ ] >$, βx and $x\beta$ are equivalent in $< A, [\ ] >$ for any $x \in A$ in view of Proposition 3.2(1). Then the sequences $\underbrace{\beta\ldots\beta}_{r}x$ and $x\underbrace{\beta\ldots\beta}_{r}$ are also equivalent, that is, αx and $x\alpha$ are equivalent. Moreover, by Proposition 3.2(2),

$$\underbrace{\beta\ldots\beta}_{s} = \underbrace{\beta\ldots\beta}_{kr} = \underbrace{\underbrace{\beta\ldots\beta}_{r}\ldots\underbrace{\beta\ldots\beta}_{r}}_{k} = \underbrace{\alpha\ldots\alpha}_{k}$$

is neutral in $< A, [\ ] >$. Thus α is m-neutral in $< A, [\ ] >$ by Proposition 3.2. The proposition is proved. □

Proposition 3.6 *For any m-neutral sequences $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ of an n-ary group $< A, [\ ] >$, where $n = k(m-1)+1$, the sequence $\alpha = \alpha_1\alpha_2\ldots\alpha_{k+1}$ is also m-neutral.*

Proof Let $l(\alpha_j) = t_j(n-1)+m-1$ be the length of the sequence α_j, $j = 1, \ldots, k+1$. Then the length of the sequence α is $l(\alpha) = t(n-1)+m-1$, where $t = (t_1 + \ldots + t_{k+1} + 1)(n-1) + m - 1$. Moreover, by Proposition 3.3(1) and the definition of an m-neutral sequence, we get

$$\begin{aligned}
&[\underbrace{\alpha \ldots \alpha}_{j-1} x \underbrace{\alpha \ldots \alpha}_{k-j+1}] \\
&\quad = [\underbrace{\alpha_1\alpha_2 \ldots \alpha_{k+1} \ldots \alpha_1\alpha_2 \ldots \alpha_{k+1}}_{j-1} x \underbrace{\alpha_1\alpha_2 \ldots \alpha_{k+1} \ldots \alpha_1\alpha_2 \ldots \alpha_{k+1}}_{k-j+1}] \\
&\quad = [\underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{j-1} \ldots [\underbrace{\alpha_2 \ldots \alpha_2}_{j-1} [\underbrace{\alpha_1 \ldots \alpha_1}_{j-1} x \underbrace{\alpha_1 \ldots \alpha_1}_{k-j+1}] \underbrace{\alpha_2 \ldots \alpha_2}_{k-j+1}] \ldots \underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{k-j+1}] \\
&\quad = [\underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{j-1} \ldots [\underbrace{\alpha_2 \ldots \alpha_2}_{j-1} x \underbrace{\alpha_2 \ldots \alpha_2}_{k-j+1}] \ldots \underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{k-j+1}] \\
&\quad \ldots = [\underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{j-1} x \underbrace{\alpha_{k+1} \ldots \alpha_{k+1}}_{k-j+1}] = x.
\end{aligned}$$

Hence α is neutral in $< A, [\ ] >$. The proposition is proved. □

Remark 3.7 If in Proposition 3.6 we put $m = 2$ and $m = n$, then we get respectively the following known facts:

(1) The set $E(A)$ of all identities of an n-ary group $< A, [\ ] >$ is closed under n-ary operation [ ], that is, $< E(A), [\ ] >$ is an n-ary subsemigroup of $< A, [\ ] >$;
(2) If α and β are the neutral sequences of an n-ary group $< A, [\ ] >$, then the sequence $\alpha\beta$ is also neutral in $< A, [\ ] >$.

Denote by $N(A, m)$ the set of all m-neutral sequences of an n-ary group $< A, [\ ] >$. It is evident that $E(A) \subseteq N(A, 2)$ and $N(A, n)$ is a set of all neutral sequences of $< A, [\ ] >$. The equality $E(A) = N(A, 2)$ is true only in the case when there are no identities in $< A, [\ ] >$. Moreover, in the case when $E(A)$ is not empty, for every identity e of A, $e_1 \ldots e_{t(n-1)+1} \in N(A, 2)$, where $e_1 \ldots e_{t(n-1)+1}$ is any equivalent sequence of length $t(n-1)+1$ for e, and $e_1 \ldots e_{t(n-1)+1} \notin E(A)$ for $t \geq 1$.

In a $(k+1)$-ary group $< A^{(m-1)}, [\ ]_{k+1} >$, select

$$N(A^{(m-1)}) = \{\theta(\alpha) | \alpha \in N(A, m)\}.$$

It is evident that $N(A') = E(A')$. Since every n-ary group contains the neutral sequences, $N(A_0) = N(A^{(n-1)})$ is not empty. Furthermore, since any two neutral sequences are equivalent in the sence of Post, $N(A_0)$ consists of the single class $\theta(\alpha)$, where α is any neutral sequence, and $\theta(\alpha)$ is an identity of the free covering group A^* (that is, $N(A_0) = E(A_0)$).

If $m \neq n$, then $N(A^{(m-1)})$ may be empty. In particular, $N(A')$ may be empty.

Since for every element $\theta(\alpha)$ of the non-empty set $N(A^{(m-1)})$ we can choose a sequence α of length $m-1$,

$$N(A^{(m-1)}) = \{\theta(\alpha) | \alpha \in N(A, m), l(\alpha) = m - 1\}.$$

In particular, $N(A') = \{\theta(a) | a \in E(A)\}$.

From Proposition 3.3(3) we get also the following

Corollary 3.8 *Let* $< A, [\ ] >$ *be an* n*-ary group,* $n = k(m-1)+1, U \in N(A^{(m-1)}), V \in A^{(m-1)}$. *Then* $[\underbrace{U \dots U}_{j-1} V \underbrace{U \dots U}_{k-j+1}]_{k+1} = V$ *and* $[\underbrace{U^{-1} \dots U^{-1}}_{j-1} V \underbrace{U^{-1} \dots U^{-1}}_{k-j+1}]_{k+1} = V$ *for all* $j = 1, \dots, k+1$, *where* U^{-1} *is an inverse element for* U *in the free covering group* A^*.

From the first equality of Corollary 3.8, we get the result about the connection between the m-neutral sequences of an n-ary group $< A, [\ ] >$ and the identities of a $(k+1)$-ary group $< A^{(m-1)}, [\ ]_{k+1} >$.

Proposition 3.9 *If* α *is an* m*-neutral sequence of an* n*-ary group* $< A, [\ ] >$, *where* $n = k(m-1)+1$, *then the class* $\theta(\alpha)$ *is an identity of a* $(k+1)$*-ary group* $< A^{(m-1)}, [\ ]_{k+1} >$ *and* $N(A^{(m-1)}) \subseteq E(A^{(m-1)})$.

In general case, the converse inclusion $E(A^{(m-1)}) \subseteq N(A^{(m-1)})$ is false (see Example 5.3 below).

Proposition 3.10 *Let* $< A, [\ ] >$ *be an* n*-ary group,* $n = k(m-1)+1, U \in N(A^{(m-1)}), V \in A^{(m-1)}$ *and* ϕ *be an automorphism of an* $(k+1)$*-ary group* $< A^{(m-1)}, [\ ]_{k+1} >$. *Then* $[\underbrace{U^\phi \dots U^\phi}_{j-1} V \underbrace{U^\phi \dots U^\phi}_{k-j+1}]_{k+1} = V$ *for all* $j = 1, \dots, k+1$ *and* $N^\phi(A^{(m-1)}) \subseteq E(A^{(m-1)})$.

Proof Since $V^{\phi^{-1}} \in A^{(m-1)}$, it follows in view of Corollary 3.8 that

$$[\underbrace{U \dots U}_{j-1} V^{\phi^{-1}} \underbrace{U \dots U}_{k-j+1}]_{k+1} = V^{\phi^{-1}}.$$

Hence

$$([\underbrace{U \dots U}_{j-1} V^{\phi^{-1}} \underbrace{U \dots U}_{k-j+1}]_{k+1})^\phi = (V^{\phi^{-1}})^\phi,$$

$$[\underbrace{U^\phi \dots U^\phi}_{j-1} (V^{\phi^{-1}})^\phi \underbrace{U^\phi \dots U^\phi}_{k-j+1}]_{k+1} = V,$$

$$[\underbrace{U^\phi \dots U^\phi}_{j-1} V \underbrace{U^\phi \dots U^\phi}_{k-j+1}]_{k+1} = V.$$

The last equality means that U^ϕ is an identity of a $(k+1)$-ary group $< A^{(m-1)}, [\ ]_{k+1} >$. Hence $N^\phi(A^{(m-1)}) \subseteq E(A^{(m-1)})$ (it follows also from Proposition 3.9). The proposition is proved. □

Proposition 3.11 *Let α be an m-neutral sequence of an n-ary group $<A, [\ ]>$, where $n = k(m-1)+1$ and $Z(A)$ is not empty. If ϕ is an automorphism of a $(k+1)$-ary group $< A^{(m-1)}, [\ ]_{k+1} >$ and $U = \theta(\alpha)$, $U^\phi = \theta(\delta)$, then δ is an m-neutral sequence of $<A, [\ ]>$.*

Proof Let x be any element of A and $c_1, \ldots, c_{m-2}$ be fixed elements of $Z(A)$. Let $V = \theta(xc_1 \ldots c_{m-2})$. Since $U = \theta(\alpha) \in N(A^{(m-1)})$ and $V \in A^{(m-1)}$, by Proposition 3.10,

$$[\underbrace{U^\phi \ldots U^\phi}_{j-1} V \underbrace{U^\phi \ldots U^\phi}_{k-j+1}]_{k+1} = V.$$

Then, in view of $c_1, \ldots, c_{m-2} \in Z(A)$, we get

$$\begin{aligned}
&[\underbrace{\theta(\delta) \ldots \theta(\delta)}_{j-1} \theta(xc_1 \ldots c_{m-2}) \underbrace{\theta(\delta) \ldots \theta(\delta)}_{k-j+1}]_{k+1} = \theta(xc_1 \ldots c_{m-2}),\\
&\theta(\underbrace{\delta \ldots \delta}_{j-1} xc_1 \ldots c_{m-2} \underbrace{\delta \ldots \delta}_{k-j+1}) = \theta(xc_1 \ldots c_{m-2}),\\
&\theta(\underbrace{\delta \ldots \delta}_{j-1} x \underbrace{\delta \ldots \delta}_{k-j+1} c_1 \ldots c_{m-2}) = \theta(xc_1 \ldots c_{m-2}),\\
&\theta(\underbrace{\delta \ldots \delta}_{j-1} x \underbrace{\delta \ldots \delta}_{k-j+1})\theta(c_1 \ldots c_{m-2}) = \theta(x)\theta(c_1 \ldots c_{m-2}),\\
&\theta(\underbrace{\delta \ldots \delta}_{j-1} x \underbrace{\delta \ldots \delta}_{k-j+1}) = \theta(x),\\
&\theta([\underbrace{\delta \ldots \delta}_{j-1} x \underbrace{\delta \ldots \delta}_{k-j+1}]) = \theta(x),\\
&[\underbrace{\delta \ldots \delta}_{j-1} x \underbrace{\delta \ldots \delta}_{k-j+1}] = x.
\end{aligned}$$

Hence δ is an m-neutral sequence of $<A, [\ ]>$. The proposition is proved. □

4 Main Results

In view of Proposition 3.9, $N(A^{(m-1)}) \subseteq E(A^{(m-1)})$ for any n-ary group $<A, [\ ]>$, where $n = k(m-1)+1$. The following theorem shows that in the case when $Z(A)$ is not empty the converse inclusion is true.

Theorem 4.1 *Let $<A, [\ ]>$ be an n-ary group, where $n = k(m-1)+1$ and $N(A^{(m-1)})$ is not empty. Then:*

(1) *$< N(A^{(m-1)}), [\ ]_{k+1} >$ is an invariant $(k+1)$-ary subgroup of $< A^{(m-1)}, [\ ]_{k+1} >$ and $N(A^{(m-1)}) \subseteq Z(A^{(m-1)})$;*
(2) *If the center $Z(A)$ is not empty, then $< N(A^{(m-1)}), [\ ]_{k+1}>$ is a characteristic subgroup of $< A^{(m-1)}, [\ ]_{k+1} >$ and $< N(A^{(m-1)}), [\ ]_{k+1} > = < E(A^{(m-1)}), [\ ]_{k+1} >$.*

Proof (1) Let $\theta(\alpha_1), \theta(\alpha_2), \ldots, \theta(\alpha_{k+1})$ be any elements of $N(A^{(m-1)})$. Put $\alpha = \alpha_1\alpha_2\ldots\alpha_{k+1}$ and $U = [\theta(\alpha_1)\theta(\alpha_2)\ldots\theta(\alpha_{k+1})]_{k+1}$. Since

$$[\theta(\alpha_1)\theta(\alpha_2)\ldots\theta(\alpha_{k+1})]_{k+1} = \theta(\alpha_1\alpha_2\ldots\alpha_{k+1}) = \theta(\alpha),$$

it follows that $U = \theta(\alpha)$. Moreover, since $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$ are m-neutral, α is also m-neutral by Proposition 3.6. Therefore $U = \theta(\alpha) \in N(A^{(m-1)})$. Hence $N(A^{(m-1)})$ is closed under $(k+1)$-ary operation $[\ ]_{k+1}$.

Now we show that the equation

$$[\theta(\alpha_1)\ldots\theta(\alpha_{i-1})x\theta(\alpha_{i+1})\ldots\theta(\alpha_{k+1})]_{k+1} = \theta(\beta)$$

is solved in $N(A^{(m-1)})$ for every $i = 1, \ldots, k+1$, where $\theta(\alpha_1), \ldots, \theta(\alpha_{i-1}), \theta(\alpha_{i+1}), \ldots, \theta(\alpha_{k+1}), \theta(\beta) \in N(A^{(m-1)})$. Let $W = \theta(\gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1})$, where $\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{k+1}$ are the inverse sequences for the sequences $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{k+1}$, respectively. It is evident that W is a solution of our equation. Let $\tau = \gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1}$. In view of Proposition 3.3(2), m-neutrality of β and Corollary 3.4, we get

$$\begin{aligned}
&[\underbrace{\tau\ldots\tau}_{j-1}\,x\,\underbrace{\tau\ldots\tau}_{k-j+1}]\\
&= [\underbrace{\gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1}\ldots\gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1}}_{j-1}\,x\\
&\quad\times\underbrace{\gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1}\ldots\gamma_{i-1}\ldots\gamma_1\beta\gamma_{k+1}\ldots\gamma_{i+1}}_{k-j+1}]\\
&= [\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{j-1}\ldots[\underbrace{\gamma_2\ldots\gamma_2}_{j-1}[\underbrace{\gamma_1\ldots\gamma_1}_{j-1}[\underbrace{\beta\ldots\beta}_{j-1}\,x\,\underbrace{\beta\ldots\beta}_{k-j+1}]\underbrace{\gamma_1\ldots\gamma_1}_{k-j+1}]\underbrace{\gamma_2\ldots\gamma_2}_{k-j+1}]\ldots\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{k-j+1}]\\
&= [\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{j-1}\ldots[\underbrace{\gamma_2\ldots\gamma_2}_{j-1}[\underbrace{\gamma_1\ldots\gamma_1}_{j-1}\,x\,\underbrace{\gamma_1\ldots\gamma_1}_{k-j+1}]\underbrace{\gamma_2\ldots\gamma_2}_{k-j+1}]\ldots\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{k-j+1}]\\
&= [\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{j-1}\ldots[\underbrace{\gamma_2\ldots\gamma_2}_{j-1}\,x\,\underbrace{\gamma_2\ldots\gamma_2}_{k-j+1}]\ldots\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{k-j+1}]\\
&\ldots\ldots\ldots\\
&= [\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{j-1}\,x\,\underbrace{\gamma_{k+1}\ldots\gamma_{k+1}}_{k-j+1}] = x.
\end{aligned}$$

Hence τ is m-neutral, $W = \theta(\tau) \in N(A^{(m-1)})$ and the universal algebra $< N(A^{(m-1)}), [\ \ ]_{k+1} >$ is a $(k+1)$-ary subgroup of a $(k+1)$-ary group $< A^{(m-1)}, [\ ]_{k+1} >$.

Let $U = \theta(\alpha_1)\ldots\theta(\alpha_i)$ be any sequence of $< A^{(m-1)}, [\ \ ]_{k+1} >$, $V = \theta(\beta_1)\ldots\theta(\beta_j)$ an inverse sequence for U in $< A^{(m-1)}, [\ ]_{k+1} >$ and $\theta(\alpha)$ any element of $< N(A^{(m-1)}), [\ ]_{k+1} >$. Then, in view of definition of a $(k+1)$-ary operation $[\ ]_{k+1}$, Proposition 3.3(1) and m-neutrality of the sequence UV, we have

$$\begin{aligned}
[U\theta(\alpha)V]_{k+1} &= [\theta(\alpha_1)\ldots\theta(\alpha_i)\theta(\alpha)\theta(\beta_1)\ldots\theta(\beta_j)]_{k+1}\\
&= \theta(\alpha_1)\ldots\theta(\alpha_i)\theta(\alpha)\theta(\beta_1)\ldots\theta(\beta_j)\\
&= \theta(\alpha)\theta(\alpha_1)\ldots\theta(\alpha_i)\theta(\beta_1)\ldots\theta(\beta_j)
\end{aligned}$$

$$= [\theta(\alpha)\theta(\alpha_1)\ldots\theta(\alpha_i)\theta(\beta_1)\ldots\theta(\beta_j)]_{k+1}$$
$$= [\theta(\alpha)UV]_{k+1} = \theta(\alpha) \in N(A^{m-1}),$$

hence $[UN(A^{(m-1)})V]_{k+1} \subseteq N(A^{(m-1)})$. Therefore, $< N(A^{(m-1)}), [\ ]_{k+1} >$ is invariant in $< A^{(m-1)}, [\ ]_{k+1} >$by Remark 2.5.

In view of Proposition 3.3(1),

$$[\theta(\alpha)\theta(\beta_1)\ldots\theta(\beta_k)]_{k+1} = \theta(\alpha\beta_1\ldots\beta_k) = \theta(\beta_1\alpha\beta_2\ldots\beta_k)$$
$$= [\theta(\beta_1)\theta(\alpha)\theta(\beta_2)\ldots\theta(\beta_k)]_{k+1}$$

for any $\theta(\alpha) \in N(A^{(m-1)}), \theta(\beta_1), \ldots, \theta(\beta_k) \in A^{(m-1)}$. Therefore, $\theta(\alpha) \in Z(A^{m-1})$ and so $N(A^{(m-1)}) \subseteq Z(A^{(m-1)})$.

(2) If $U \in N(A^{(m-1)})$, then, by Proposition 3.11, $U^{\phi} \in N(A^{(m-1)})$ for any automorphism ϕ of $< A^{(m-1)}, [\ ]_{k+1} >$. Hence $N^{\phi}(A^{(m-1)}) \subseteq N(A^{(m-1)})$. In particular, $N^{\phi^{-1}}(A^{(m-1)}) \subseteq \mathbf{N}(A^{(m-1)})$. Therefore, $(N^{\phi^{-1}}(A^{(m-1)}))^{\phi} \subseteq N^{\phi}(A^{(m-1)})$ and hence $N(A^{(m-1)}) \subseteq N^{\phi}(A^{(m-1)})$. Thus $N^{\phi}(A^{(m-1)}) = N(A^{(m-1)})$ and so $< N(A^{(m-1)}), [\ ]_{k+1} >$ is characteristic in $< A^{(m-1)}, [\ ]_{k+1} >$.

Let x be any element of A and $a_1, \ldots, a_{m-1}$ fixed elements of $Z(A)$. If a class $U = \theta(\alpha)$ is an identity of $< A^{(m-1)}, [\ ]_{k+1} >$, then

$$[\underbrace{U\ldots U}_{j-1}\theta(xa_1\ldots a_{m-1})\underbrace{U\ldots U}_{k-j+1}]_{k+1} = \theta(xa_1\ldots a_{m-1})$$

for any $j = 1, \ldots, k+1$. Since $a_1, \ldots, a_{m-1} \in Z(A)$, it follows that

$$[\underbrace{\theta(\alpha)\ldots\theta(\alpha)}_{j-1}\theta(xa_1\ldots a_{m-1})\underbrace{\theta(\alpha)\ldots\theta(\alpha)}_{k-j+1}]_{k+1} = \theta(xa_1\ldots a_{m-1}),$$
$$\theta(\underbrace{\alpha\ldots\alpha}_{j-1}xa_1\ldots a_{m-1}\underbrace{\alpha\ldots\alpha}_{k-j+1}) = \theta(xa_1\ldots a_{m-1}),$$
$$\theta(\underbrace{\alpha\ldots\alpha}_{j-1}x\underbrace{\alpha\ldots\alpha}_{k-j+1}a_1\ldots a_{m-1}) = \theta(xa_1\ldots a_{m-1}),$$
$$\theta(\underbrace{\alpha\ldots\alpha}_{j-1}x\underbrace{\alpha\ldots\alpha}_{k-j+1})\theta(a_1\ldots a_{m-1}) = \theta(x)\theta(a_1\ldots a_{m-1}),$$
$$\theta(\underbrace{\alpha\ldots\alpha}_{j-1}x\underbrace{\alpha\ldots\alpha}_{k-j+1}) = \theta(x),$$
$$\theta([\underbrace{\alpha\ldots\alpha}_{j-1}x\underbrace{\alpha\ldots\alpha}_{k-j+1}]) = \theta(x),$$
$$[\underbrace{\alpha\ldots\alpha}_{j-1}x\underbrace{\alpha\ldots\alpha}_{k-j+1}] = x.$$

Hence α is m-neutral in $< A, [\ ] >$and so $E(A^{(m-1)}) \subseteq N(A^{(m-1)})$. In view of Proposition 3.9, it follows that $N(A^{(m-1)}) = E(A^{(m-1)})$. The theorem is proved. $\square$

Note that the equality $<N(A^{(m-1)}), [\ ]_{k+1}> = <E(A^{(m-1)}), [\ ]_{k+1}>$ is also true for some polyadic groups $<A, [\ ]>$ with empty center (see Examples 5.4, 5.5 below).

Let $<A, [\ ]>$ be an n-ary group ($n = k(m-1)+1$) and $c_1, \ldots, c_{m-2}$ elements of A. Define in A a $(k+1)$-ary operation $[\ ]_{k+1, c_1 \ldots c_{m-2}}$ such that

$$[a_1 a_2 \ldots a_{k+1}]_{k+1, c_1 \ldots c_{m-2}} = [a_1 c_1 \ldots c_{m-2} a_2 c_1 \ldots c_{m-2} \ldots a_k c_1 \ldots c_{m-2} a_{k+1}].$$

It is easy to show (see, for example, [4]) that $<A, [\ ]_{k+1, c_1 \ldots c_{m-2}}>$ is a $(k+1)$-ary group. Moreover, the following lemma is true.

Lemma 4.2 [9, Lemma 3]. *Let $<A, [\ ]>$ be an n-ary group ($n = k(m-1)+1$) and $c_1, \ldots, c_{m-2} \in A$. Then the map $\tau : A^{(m-1)} \to A$ such that $\tau(\theta(ac_1 \ldots c_{m-2})) = a$ is the isomorphism of a $(k+1)$-ary group $<A^{(m-1)}, [\ ]_{k+1}>$ on a $(k+1)$-ary group $<A, [\ ]_{k+1, c_1 \ldots c_{m-2}}>$.*

For the fixed elements $c_1, \ldots, c_{m-2}$ of an n-ary group $<A, [\ ]>$ put

$$N(A, c_1 \ldots c_{m-2}) = \{a \in A | ac_1 \ldots c_{m-2} \in N(A, m)\}.$$

It is evident that

$$N(A, c_1 \ldots c_{m-2}) = \{a \in A | \theta(ac_1 \ldots c_{m-2}) \in N(A^{(m-1)})\}$$

and

$$N(A^{m-1}) = \{\theta(ac_1 \ldots c_{m-2}) | ac_1 \ldots c_{m-2} \in N(A, m)\}.$$

Therefore, the set $N(A, c_1 \ldots c_{m-2}) = N^{\tau}(A^{(m-1)})$, where τ is the isomorphism from Lemma 4.2.

Denote by $E(A, c_1 \ldots c_{m-2})$ the set of all identities of $<A, [\ ]_{k+1, c_1 \ldots c_{m-2}}>$. It is evident that $E(A, c_1 \ldots c_{m-2}) = E^{\tau}(A^{(m-1)})$, where τ is the isomorphism from Lemma 4.2.

In view of Lemma 4.2, we get the following isomorphic copy of Theorem 4.1.

Theorem 4.3 *Let $<A, [\ ]>$ be an n-ary group, where $n = k(m-1)+1$. If $N(A, c_1 \ldots c_{m-2})$ is not empty, then:*

(1) *$<N(A, c_1 \ldots c_{m-2}), [\ ]_{k+1, c_1 \ldots c_{m-2}}>$ is an invariant $(k+1)$-ary subgroup of $<A, [\ ]_{k+1, c_1 \ldots c_{m-2}}>$ and $N(A, c_1 \ldots c_{m-2}) \subseteq Z(A)$;*
(2) *If $Z(A)$ is not empty, then $<N(A, c_1 \ldots c_{m-2}), [\ ]_{k+1, c_1 \ldots c_{m-2}}>$ is a characteristic subgroup of $<A, [\ ]_{k+1, c_1 \ldots c_{m-2}}>$ and $<N(A, c_1 \ldots c_{m-2}), [\ ]_{k+1, c_1 \ldots c_{m-2}}> = <E(A, c_1 \ldots c_{m-2}), [\ ]_{k+1, c_1 \ldots c_{m-2}}>$.*

5 Corollaries and Examples

If in Theorem 4.1 we put $m = n$, then we get the statement about trivial subgroup of a group $A_0 = A^{(n-1)}$.

If in Theorem 4.1 we put $m = 2$, then we get

Corollary 5.1 *Let* $< A, [\ ] >$*be an* n*-ary group such that* $N(A')$ *is not empty. The following statements hold*:

(1) $< N(A'), [\ ]_n >$ *is an invariant* n*-ary subgroup of an* n*-ary group* $< A', [\ ]_n >$*and* $N(A') \subseteq Z(A')$;
(2) *If* $Z(A)$ *is not empty, then* $< N(A'), [\ ]_n >$ *is characteristic in* $< A', [\ ]_n >$*and* $< N(A'), [\ ]_n > = < E(A'), [\ ]_n >$.

Since the center of an n-ary group with identities is not empty and the map $\psi : \theta(a) \rightarrow a$ is isomorphism of an n-ary group $< A', [\ ]_n >$on an n-ary group $< A, [\ ] >$such that $((N(A'))^{\psi} = E(A)$, from Corollary 5.1 we get the following

Corollary 5.2 (Gal'mak [1,2]). *If* $E(A)$ *is not empty, then* $< E(A), [\ ] >$ *is a characteristic* n*-ary subgroup of an* n*-ary group* $< A, [\ ] >$*and* $E(A) \subseteq Z(A)$.

Corollary 5.2 follows also from Theorem 4.3 in the case when the sequence $c_1 \ldots c_{m-2}$ is empty (that is, $m = 2$).

If in Theorem 4.3 we put $m = n$, then we get the statement about trivial subgroup of a group $< A, [\ ]_{2,c_1 \ldots c_{n-2}} >$ with binary operation $[a_1 a_2]_{2,c_1 \ldots c_{n-2}} = [a_1 c_1 \ldots c_{n-2} a_2]$.

In view of Theorem 4.1(2), if the center of an n-ary group $< A, [\ ] >(n = k(m-1)+1)$ is not empty, then $< N(A^{(m-1)}), [\ ]_{k+1} > = < E(A^{(m-1)}), [\ ]_{k+1} >$. Therefore, if we want to get an n-ary group $< A, [\ ] >$for which $E(A^{(m-1)}) \nsubseteq N(A^{(m-1)})$, then we must consider $< A, [\ ] >$such that $Z(A)$ is empty.

Example 5.3 Let $B_3 = \{(12), (13), (23)\}$ be a set of all odd permutation of the set $\{1, 2, 3\}$, $< B_3, [\ ] >$a 7-ary group with 7-ary operation derived from operation of the symmetric group S_3. It is easy to see that $Z(B_3)$ is empty. Since $7 = 3(3-1)+1$, we can consider the 3-neutral sequences in $< B_3, [\ ] >$. Moreover, in view of Remark 2.4, there is a 4-ary group $< B_3^{(2)}, [\ ]_4 >$. Since every sequence of length 2 of $< B_3, [\ ] >$ is equivalent to one of the sequences $\lambda = (12)(12)$, $\mu = (12)(13)$ or $\nu = (12)(23)$, $B_3^{(2)} = \{\theta(\lambda), \theta(\mu), \theta(\nu)\}$. It is not difficult to show that every element of $< B_3^{(2)}, [\ ]_4 >$ is an identity, that is, $E(B_3^{(2)}) = B_3^{(2)}$, and $N(B_3^{(2)}) = \{\theta(\lambda)\}$. Thus $N(B_3^{(2)}) \neq E(B_3^{(2)})$.

The following two examples show that there are polyadic groups $< A, [\ ] >(n = k(m-1)+1)$ with $Z(A) = \varnothing$ such that $<N(A^{(m-1)}), [\ ]_{k+1} > = < E(A^{(m-1)}), [\ ]_{k+1} >$.

Example 5.4 Let $< B_3, [\ ] >$be a 7-ary group such as in Example 5.3. Since $7 = 2(4-1)+1$, we can consider 4-neutral sequences in $< B_3, [\ ] >$. Furthermore, in view of Remark 2.4, there is a ternary group $< B_3^{(3)}, [\ ]_3 >$. Since every sequence of length 3 of $< B_3, [\ ] >$ is equivalent to one of the sequences $\lambda = (12)(12)(12)$, $\mu = (12)(12)(13)$ or $\nu = (12)(12)(23)$, $B_3^{(3)} = \{\theta(\lambda), \theta(\mu), \theta(\nu)\}$. It is not difficult to show that there are no identities in $< B_3^{(3)}, [\ ]_3 >$and the set $N(B_3^{(3)})$ is also empty. Thus $N(B_3^{(3)}) = E(B_3^{(3)}) = \varnothing$.

Example 5.5 Let B_3 be a set such as in Example 5.3, $< B_3, [\ ] >$ a 5-ary group with 5-ary operation derived from operation of the symmetric group S_3. It is easy to see that $Z(B_3)$ is empty. Since $5 = 2(3-1)+1$, we can consider 3-neutral sequences in $< B_3, [\ ] >$. Moreover, in view of Remark 2.4, there is a ternary group $< B_3^{(2)}, [\ ]_3 >$. As in Example 5.3, we get $B_3^{(2)} = \{\theta(\lambda), \theta(\mu), \theta(\nu)\}$. Furthermore, $N(B_3^{(2)}) = E(B_3^{(2)}) = \{\theta(\lambda)\}$.

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