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Relativistic two-particle Sturm-Liouville problems for *p*-states: exact and numerical solutions in the momentum representation

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Релятивистские связанные состояния двухчастичных систем с орбитальным моментом $l=1$ (*p*-состояния) исследуются в специальном случае оператора взаимодействия, который допускает эквивалентную интегральному уравнению формулировку в виде задач Штурма-Лиувилля непосредственно в импульсном представлении. Численно найден вид условий квантования для *p*-состояний и решений задач Штурма-Лиувилля, а также вид волновых функций. В предельном случае нулевой массы связанного состояния численные решения сравниваются с точными аналитическими, которые также найдены.
Ключевые слова: релятивистская двухчастичная система, связанное состояние, интегральное уравнение, парциальное разложение, задача Штурма-Лиувилля.

Relativistic two-particle system bound states with orbital momentum $l=1$ (*p*-states) are being investigated in a special case of the interaction operator, which allows the formulation in the form of Sturm-Liouville problems, equivalent to the integral equation, directly in the momentum representation. The quantization conditions for *p*-states, solutions of the Sturm-Liouville problems, and behaviors of the wave functions are found numerically. In the limiting case of the zero bound state mass the numerical solutions are compared with exact analytical ones, which are also found.
Keywords: relativistic two-particle system, bound state, integral equation, partial decomposition, Sturm-Liouville problem.

Introduction

Description of relativistic two-particle bound states can be realized with the help of the dynamic equations of local quantum field theory, commonly used examples of which include quasipotential equations by Logunov-Tavkhelidze [1] and Kadyshevsky [2]. It is important to note that quasipotential equations in the momentum representation can be reduced to one-dimensional integral equations for the majority of quasipotentials used in applications, where quasipotentials have the property of spherical symmetry in the relativistic configuration representation [3], [4]. The kernels of three-dimensional dynamic equations (quasipotentials), which are local in the Lobachevski momentum space [3], are relativistic generalizations of the nonrelativistic quantum-mechanical potentials recorded in the momentum representation. This allows for the use of conventional non-relativistic quantum mechanical considerations in the construction of quasipotential interactions and the investigation of two-particle systems with such interactions [4]. One method for solving integral quasipotential equations is based on reducing these equations to differential ones in the rapidity space [5].

Partial decomposition of integral quasipotential equations

Consider the integral quasipotential equations for the bound states of a system of two relativistic spinless particles of mass *m* each [5]

$$G_{0,j}^{-1}(E,E_p)\psi_j(\vec{p}) = \int V(E,\vec{p},\vec{k})\psi_j(\vec{k})\, m\frac{d\vec{k}}{E_k}, \qquad (1)$$

where $\psi_j(\vec{p})$ are the relative motion wave functions, $\vec{p}$ and $\vec{k}$ are the initial and final relative momenta of the particles in the center-of-mass system, $E_p=\sqrt{\vec{p}^{\,2}+m^2}$ and $E_k=\sqrt{\vec{k}^{\,2}+m^2}$ are the initial and final energies of the particles respectively, $2E$ is the two-particle system energy, $G_{0,j}^{-1}$ are the inverse Green functions of the Logunov-Tavkhelidze ($j=1$) and Kadyshevsky ($j=2$) equations and of their modified versions ($j=3$, $j=4$):

$$G_{0,1}^{-1}(E,E_p)=E^2-E_p^2;\qquad G_{0,2}^{-1}(E,E_p)=E_p(E-E_p);$$
$$G_{0,3}^{-1}(E,E_p)=m(E^2-E_p^2)/E_p;\quad G_{0,4}^{-1}(E,E_p)=m(E-E_p). \tag{2}$$

If one makes the partial decomposition of a local in the Lobachevski momentum space quasipotential $V(E,\vec{p},\vec{k})=V(E,\vec{p}(-)\vec{k})$ and chooses the wave function in the form of $\psi_j(\vec{p})=\psi_{j,l}(p)Y_l^{\mu}(\vec{n}_p)$, then the three dimensional equation (1), after applying the addition theorem for Legendre polynomials and spherical harmonics reduction, will be reduced to the one-dimensional equation

$$G_{0,j}^{-1}(E,E_p)\psi_{j,l}(p)=\frac{4\pi}{2l+1}m\int_0^{\infty}V_l(E,p,k)\psi_{j,l}(k)\,k^2\,\frac{dk}{E_k}, \tag{3}$$

where

$$V_l(E,p,k)=\frac{2l+1}{2}\int_0^{\pi}V(E,p,k,\cos\theta_{pk})P_l(\cos\theta_{pk})\sin\theta_{pk}\,d\theta_{pk}\,. \tag{4}$$

As an example of relativistic potential $V(E,\vec{p},\vec{k})$, consider a generalization [5] of the non-relativistic quantum mechanical potential $V(\vec{p},\vec{k})=-\lambda\left(4\pi|\vec{p}-\vec{k}|\right)^{-1}$, namely:

$$V(\vec{p},\vec{k})=V(\vec{p}(-)\vec{k})=-\frac{\lambda}{4\pi}\frac{1}{|\vec{\Delta}_{p,k}|}=-\frac{\lambda}{4\pi}\frac{1}{|\vec{p}(-)\vec{k}|}=-\frac{\lambda}{4\pi}\frac{m}{\sqrt{\left(E_pE_k-\vec{p}\,\vec{k}\right)^2-m^4}}\,. \tag{5}$$

Using the explicit form of potential (5) in the partial potential definition (4), introducing the notations $\alpha=E_pE_k-p\,k$, $\beta=E_pE_k+p\,k$, and making the substitution $y=E_pE_k-p\,k\cos\theta_{pk}$, one obtains

$$V_l(p,k)=-\frac{2l+1}{4\pi}\frac{\lambda m}{2}\frac{1}{p\,k}\int_{\alpha}^{\beta}P_l\left(\frac{\beta+\alpha-2y}{\beta-\alpha}\right)\frac{1}{\sqrt{y^2-m^4}}dy \tag{6}$$

Then, substituting the partial potentials (6) into equation (3), one obtains

$$G_{0,j}^{-1}(E,E_p)p\psi_{j,l}(p)=-\lambda m^2\int_0^{\infty}\tilde{V}_l(p,k)k\psi_{j,l}(k)\frac{dk}{E_k}, \tag{7}$$

where

$$\tilde{V}_l(p,k)=-\frac{4\pi}{2l+1}\frac{p\,k}{\lambda m}V_l(p,k)=\frac{1}{2}\int_{\alpha}^{\beta}P_l\left(\frac{\beta+\alpha-2y}{\beta-\alpha}\right)\frac{1}{\sqrt{y^2-m^4}}dy\,. \tag{8}$$

Let us introduce the parameterizations $E=m\cos w$, $p=m\sinh\chi_p$, $k=m\sinh\chi_k$, where $w\in[0;\pi/2]$; and χ_p, χ_k are rapidities, and let us also denote $p\psi_{j,l}(p)=\phi_{j,l}(\chi_p)$ and $\tilde{V}_l(p,k)=I_l(\chi_p,\chi_k)$. Then it is clear that $\alpha=m\cosh(\chi_p-\chi_k)$, $\beta=m\cosh(\chi_p+\chi_k)$ and partial quasipotential equations (7) take the form

$$G_{0,j}^{-1}(m\cos w,m\cosh\chi_p)\phi_{j,l}(\chi_p)=-\lambda m^2\int_0^{\infty}I_l(\chi_p,\chi_k)\phi_{j,l}(\chi_k)\,d\chi_k\,. \tag{9}$$

In order to find the kernels of integral equations explicitly (9), it is necessary to carry out the calculation of values (8).

Partial equations for the p-states

Further in this paper we consider the case $l=1$ (*p*-states), for which formula (8) takes the form

$$I_1(\chi_p,\chi_k)=\frac{1}{2}\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\beta+\alpha-2y}{\sqrt{y^2-m^4}}dy\,. \tag{10}$$

Implementing the integration in (10) one obtains:

$$I_1(\chi_p,\chi_k)=\begin{cases}\coth\chi_k\left(\chi_p\coth\chi_p-1\right), & \chi_k\ge\chi_p;\\ \coth\chi_p\left(\chi_k\coth\chi_k-1\right), & \chi_k\le\chi_p.\end{cases} \quad (11)$$

It is clear from (11) that the quantity $I_1(\chi_p,\chi_k)$ can be interpreted as the Green function of a homogeneous boundary value problem. Therefore, using the derivations of equations (9) with the kernel (11) with respect to the parameter χ_p one can see that the functions $F_{j,1}(\chi_p)=G^{-1}_{0,j}(m\cos w,m\cosh\chi_p)\phi_{j,1}(\chi_p)$ are solutions of boundary value problems containing differential equations, which are similar to the Schrodinger equation (with zero energy):

$$F''_{j,1}(\chi)-\frac{2}{\sinh^2\chi}F_{j,1}(\chi)-\lambda m^2G_{0,j}(m\cos w,m\cosh\chi)F_{j,1}(\chi)=0, \quad (12)$$

where, for all j, the boundary conditions take the form

$$F_{j,1}(0)=0; \qquad F'_{j,1}(\infty)=0. \quad (13)$$

At first consider the Sturm-Liouville problems (12), (13) at $w=\pi/2$, where $E=0$, and differential equations (12) at $j=1$ and $j=2$ take the form

$$F''_{1,2;1}(\chi)-2\sinh^{-2}\chi\,F_{1,2;1}(\chi)+\lambda\cosh^{-2}\chi\,F_{1,2;1}(\chi)=0. \quad (14)$$

Taking into account boundary conditions (13) at point $\chi=0$, solution of equations (14) yields the functions $F_{1,2;1}(\chi)$ in the form of the hypergeometric series [6]:

$$F_{1,2;1}(\chi)=A_{1,s}\tanh^2\chi\;{}_2F_1\!\left(1-s;\frac{3}{2}+s;\frac{5}{2};\tanh^2\chi\right), \quad (15)$$

where the notation $\lambda=2s(2s+1)$ is introduced.

Then from the boundary conditions at infinity one gets

$$s=n+1,\quad \lambda=(2n+2)(2n+3),\quad n=0,1,2,\ldots\,. \quad (16)$$

Thus, (15) and (16) are the exact analytical solutions of problems (12), (13) in the limiting case when the mass of the bound state $2E$ is equal to zero.

At other values of the bound state mass we find solutions to problems (12), (13) numerically. Values of the coupling constant λ can be obtained from the integral equations (9). If one uses the above introduced notation for $F_{j,1}(\chi_p)$, and the notation $f=-\lambda^{-1}$, then equation (9) can be written as

$$m^2\int_0^\infty I_1(\chi_p,\chi_k)G_{0,j}(m\cos w,m\cosh\chi_k)F_{j,1}(\chi_k)\,d\chi_k = f\,F_{j,1}(\chi_p). \quad (17)$$

After choosing a sufficiently large but finite upper limit of integration let us divide the integration domain into elementary segments $[\chi^{i-1};\chi^i=ih]$. Then, replacing the integrals with the composite trapezoidal quadrature formula and introducing the notation

$$\chi_p^i=ih;\qquad F^i_{j,1}=F_{j,1}(\chi_p^i);\quad F^q_{j,1}=F_{j,1}(\chi_k^q);$$

$$T^{i,q}=\coth\chi_p^i\left(\chi_k^q\coth\chi_k^q-1\right);\quad B^{i,q}=\coth\chi_k^q\left(\chi_p^i\coth\chi_p^i-1\right);$$

$$K_j^q(m\cos w)=G_{0,j}(m\cos w,m\cosh\chi_k^q),$$

one obtains a matrix eigenvalue problem $MF_{j,1}=fF_{j,1}$, or in the index form:

$$M_{i,q}F^q_{j,1}=f\,F^i_{j,1} \quad (18)$$

where

$$M_{i,q}=\begin{cases}m^2hT^{i,q}K_j^q(m\cos w), & q\ge i;\\ m^2hB^{i,q}K_j^q(m\cos w), & q<i.\end{cases} \quad (19)$$

Eigenvalues of matrix (19) were found with the help of the computer algebra system Mathematica [7]. The resulting solutions to problem (18), namely the quantities $\lambda = -f^{-1}$, were verified using the Richardson extrapolation [8]. Then, after solving differential equations (12) in the Mathematica package, the eigenfunctions $F_{j,1}(\chi)$ were determined.

As in the case $l = 0$ [9], values of the coupling constant λ obtained numerically, have up to eight correct significant digits. The maximum absolute error of the numerical solution of differential equations with j =1,2 at $w = \pi/2$, for which there exists an analytical solution, is on the order of 10^{-13}. It should be noted that the values of the constant λ for $j = 3$ and $j = 4$, as well as for $j = 1$ and $j = 2$, coincide at $w = \pi/2$. Also, the values of the constant λ for $j = 1$ and $j = 2$, obtained numerically, coincide with the exact values (16).

Figures 1–4 illustrate the dependence of the coupling constant λ, as well as of $F(\chi)$ and $\psi(p)$, on parameter $w \in (0; \pi/2]$, for j = 1:4.

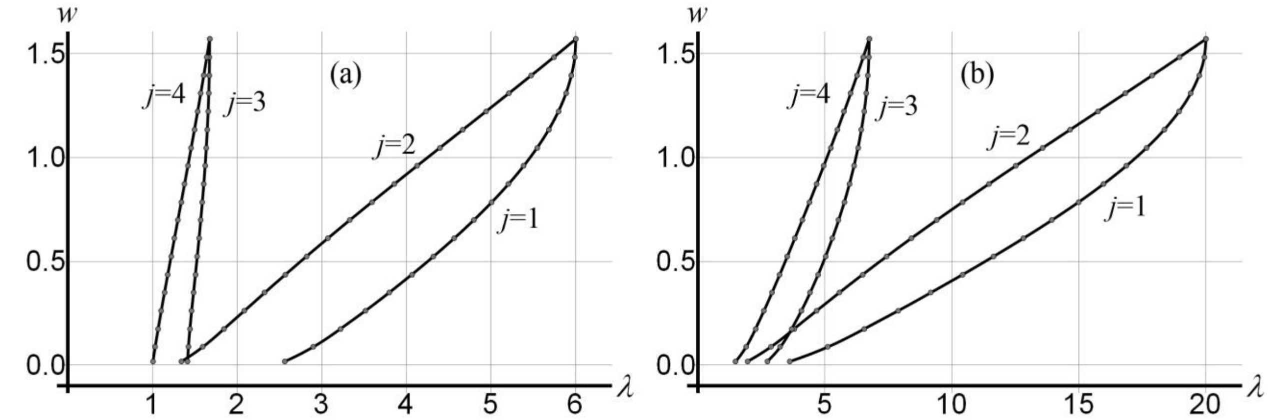

Figure 1 – The dependence of the coupling constant λ on parameter *w* for the ground (a) and the first excited (b) states

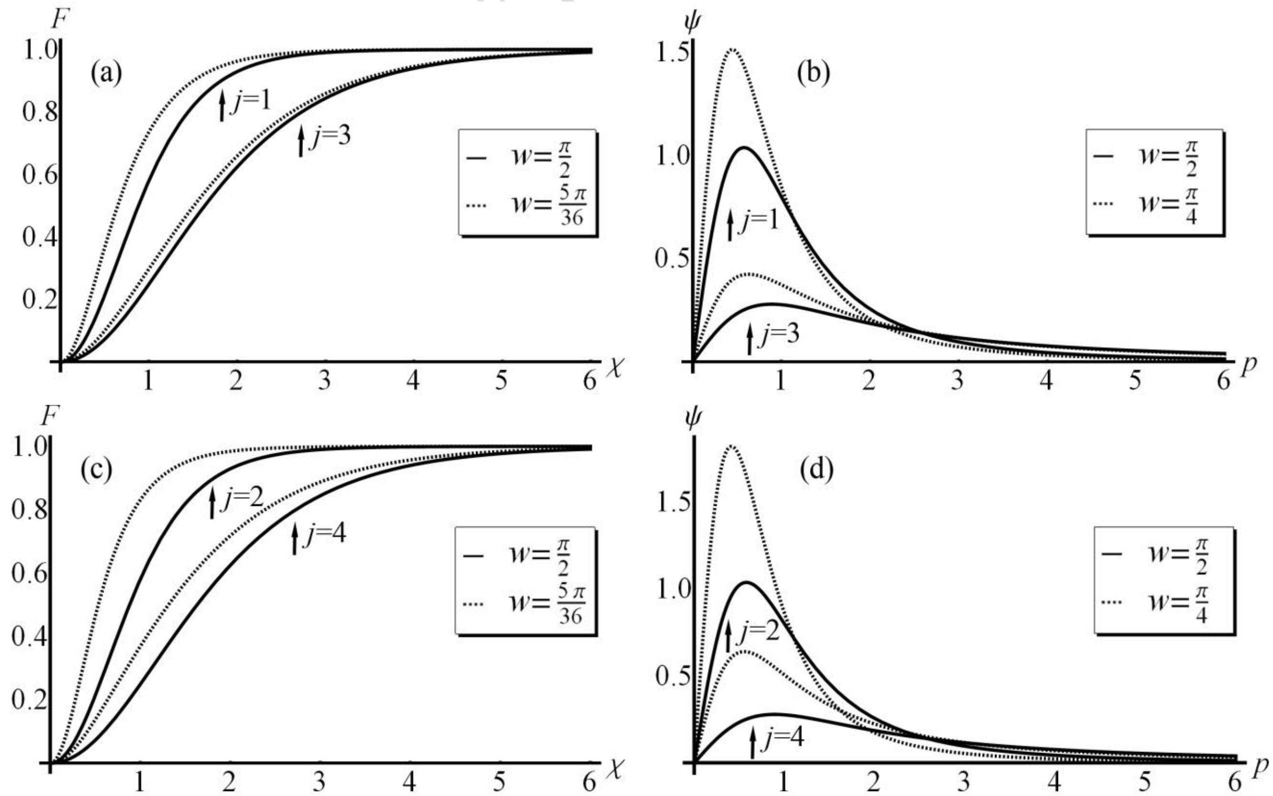

Figure 2 – Solutions of the Sturm-Liouville problems $F(\chi)$ (a; c), and of the integral equations $\psi(p)$ (b; d) for the ground state at j = 1–4

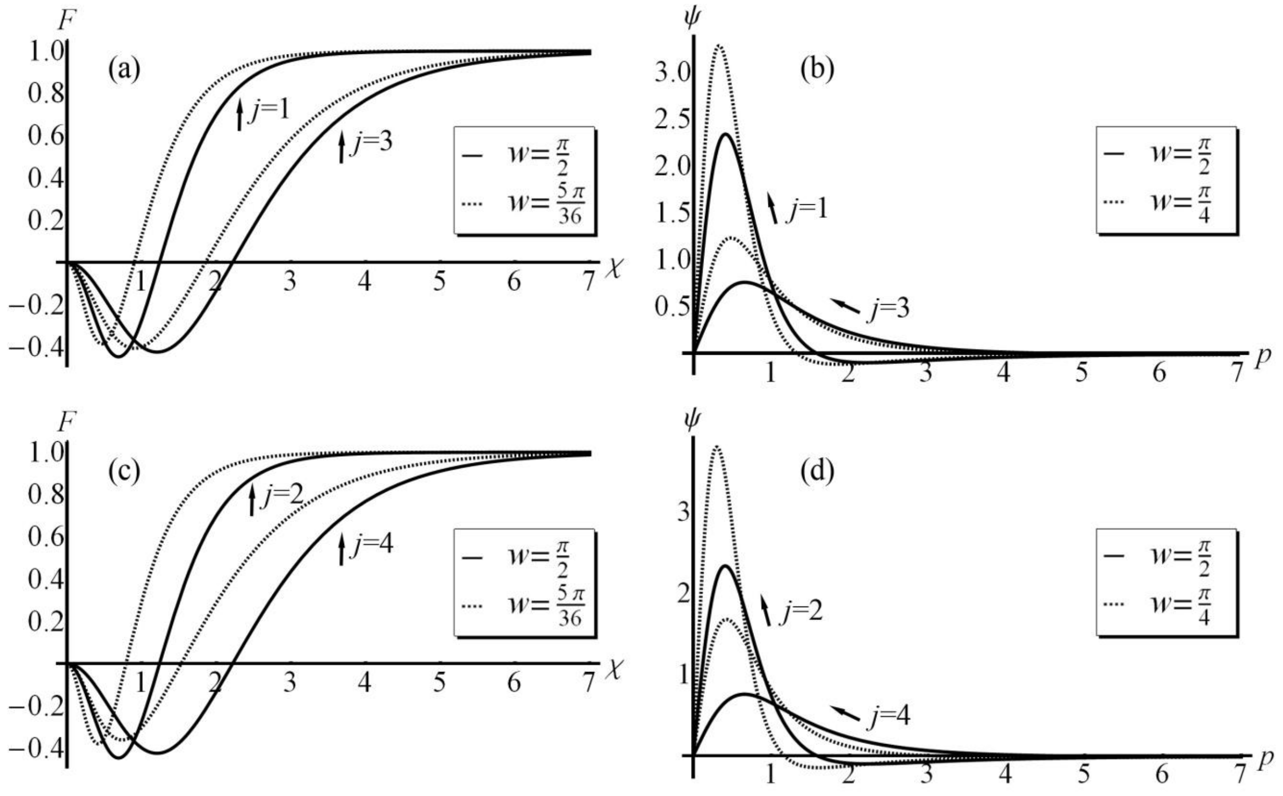

Figure 3 – Solutions of the Sturm-Liouville problems $F(\chi)$ (a; c),
and of the integral equations $\psi(p)$ (b; d) for the first excited state at j = 1–4

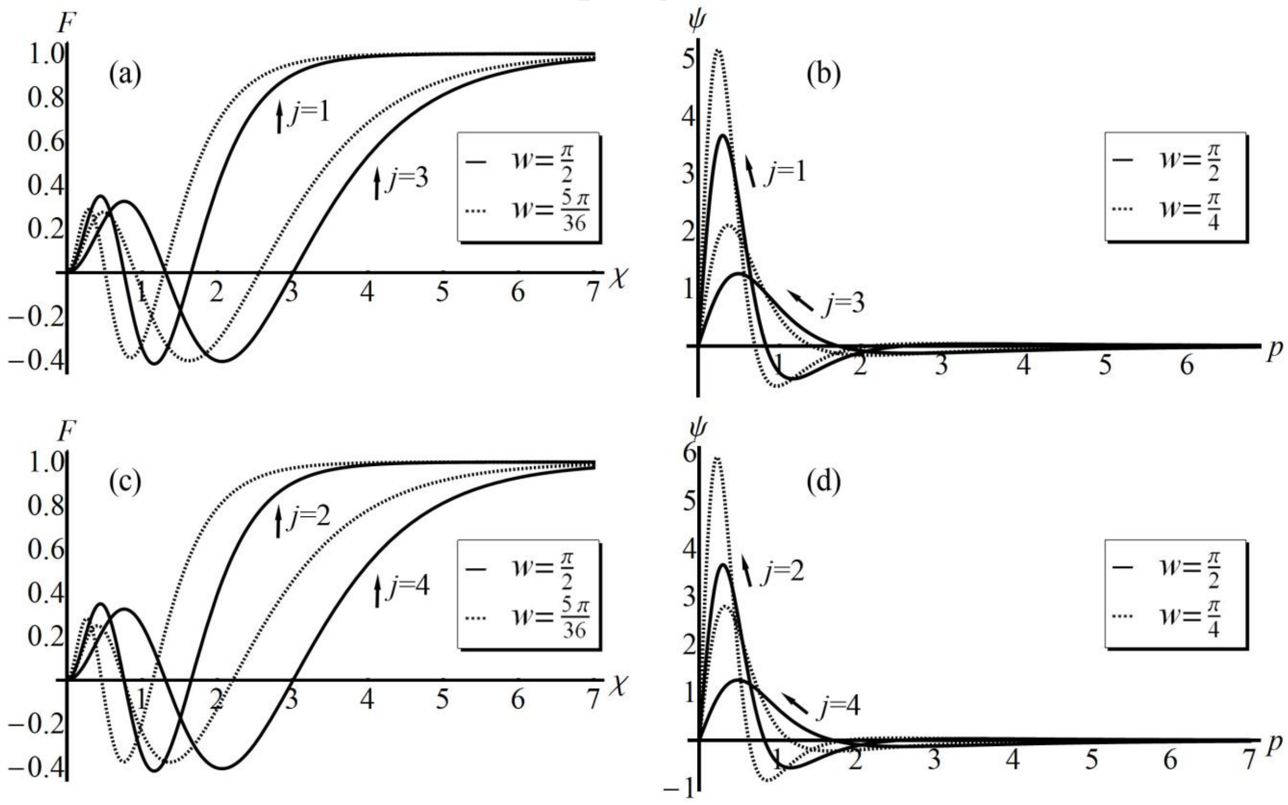

Figure 4 – Solutions of the Sturm-Liouville problems $F(\chi)$ (a; c),
and of the integral equations $\psi(p)$ (b; d) for the second excited state at j = 1–4

Dependence of the coupling constant λ on parameter *w* and corresponding solutions of the Sturm-Liouville problems and integral equations have also been determined in our calculations for larger values of the principal quantum number (up to $n = 5$). These solutions are not represented here due to the limited volume of this article.

It has to be noted that for all four types of quasipotential equations the found functions $F(\chi)$ and wave functions $\psi(p)$ have the number of zeros (except zero at $\chi = 0$), coinciding with the principal quantum number *n*.

Conclusion

In this paper the three dimensional integral quasipotential equations are reduced to one-dimensional equations in the case of the orbital angular momentum $l = 1$ for the potential $V(\vec{p},\vec{k}) = -\lambda\ (4\pi\left|\vec{\Delta}_{p,k}\right|)^{-1}$, which is a relativistic generalization of the nonrelativistic quantum mechanical potential $V(\vec{p},\vec{k}) = -\lambda\left(4\pi\left|\vec{p}-\vec{k}\right|\right)^{-1}$. It is shown that one-dimensional integral equations are equivalent to Sturm-Liouville problems in the momentum or rapidity space. Numerical solutions of the quasipotential equations considered are obtained for the cases of zero and non-zero mass of bound states. In the strong coupling limit of $2E = 0$, exact analytical solutions of the discussed Sturm-Liouville problems are also found. Correspondence between numerical and exact solutions in this limiting case is verified.

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