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# О $\mathfrak{F}_h$-НОРМАЛЬНЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

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# ON $\mathfrak{F}_h$-NORMAL SUBGROUPS OF FINITE GROUPS

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Пусть $G$ конечная группа и $\mathfrak{F}$ – формация конечной группы. Мы говорим, что подгруппа $H$ группы $G$ является $\mathfrak{F}_h$-нормальной в $G$ если существует такая нормальная подгруппа $T$ группы $G$, что $HT$ – нормальная холловская подгруппа в $G$ и $(H\cap T)H_G/H_G$ содержится в $\mathfrak{F}$-гиперцентре $Z_\infty^{\mathfrak{F}}(G/H_G)$ группы $G/H_G$. В данной работе мы получаем некоторые результаты о $\mathfrak{F}_h$-нормальных подгруппах и используем их для изучения конечных групп.

***Ключевые слова:*** *конечная группа, $\mathfrak{F}_h$-нормальная подгруппа, подгруппа Силова, максимальная подгруппа, минимальная подгруппа.*

Let $G$ be a finite group and $\mathfrak{F}$ a formation of finite groups. We say that a subgroup $H$ of $G$ is $\mathfrak{F}_h$-normal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a normal Hall subgroup of $G$ and $(H\cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of $G/H_G$. In this paper, we obtain some results about the $\mathfrak{F}_h$-normal subgroups and use them to study the structure of finite groups.

***Keywords:*** *finite groups, $\mathfrak{F}_h$-normal subgroup, Sylow subgroup, maximal subgroup, minimal subgroup.*

### *Introduction*

Throughout this paper, all groups are finite and $G$ denotes a group. The notation and terminology are standard, as in [1] and [2].

The relationship between the subgroups and the structure of $G$ has been extensively studied in the literature. Many useful results of finite groups have been obtained under the assumption that some certain subgroups of $G$ of prime power orders are well situated in $G$. Ito [3] has proved that if $G$ is a group of odd order and all minimal subgroups of $G$ lie in the center of $G$, then $G$ is nilpotent. Buckley [4] showed that a group $G$ of odd order is supersoluble if all minimal subgroups of $G$ are normal in $G$. Srinivasan [5] proved that a group $G$ is supersoluble if every maximal subgroup of every Sylow subgroup of $G$ is normal in $G$.

Recently, by considering some special supplemented subgroups, people have obtained a series of new interesting results. For example, Wang [6] introduced $c$-normal subgroup: a subgroup $H$ of $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $G=HK$ and $H\cap K\le H_G$, where $H_G$ is the maximal normal subgroup of $G$ contained in $H$. Later, Yang and Guo [7] gave the concept of $\mathfrak{F}_n$-supplemented subgroup: a subgroup $H$ of $G$ is said to be $\mathfrak{F}_n$-supplemented in $G$ if there exists a normal subgroup $K$ of $G$ such that $G=HK$ and $(H\cap K)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of $G/H_G$. Many facts have shown that $c$-normal and $\mathfrak{F}_n$-supplemented properties of some subgroups can give a good insight into the structure of supersoluble groups and $p$-nilpotent groups (see [6]–[12]).

As a development of this topic, the authors introduce the following new concept.

***Definition* 0.1** [13]. Let $\mathfrak{F}$ be a class of groups. A subgroup $H$ of $G$ is said to be $\mathfrak{F}_h$-normal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a normal Hall subgroup of $G$ and $(H\cap T)H_G/H_G\le Z_\infty^{\mathfrak{F}}(G/H_G)$.

Recall that, for a class $\mathfrak{F}$ of groups, a chief factor $H/K$ of $G$ is called $\mathfrak{F}$-central (see [14] or [1, Definition 2.4.3]) if $[H/K](G/C_G(H/K))\in\mathfrak{F}$.

The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group $G$, that is, the product of all such normal subgroups $H$ of $G$ whose $G$-chief factors are $\mathfrak{F}$-central. A subgroup $H$ of $G$ is said to be $\mathfrak{F}$-hypercenter in $G$ if $H \le Z_\infty^{\mathfrak{F}}(G)$.

A class $\mathfrak{F}$ of groups is called a formation if it is closed under homomorphic image and every group $G$ has a smallest normal subgroup (called $\mathfrak{F}$-residual of $G$ and denoted by $G^{\mathfrak{F}}$) with quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group $G$ with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{N}$, $\mathfrak{U}$ to denote the formations of all nilpotent groups and supersoluble groups, respectively.

Obviously, all normal subgroups, $c$-normal subgroups and $\mathfrak{F}_n$-supplemented subgroups are all $\mathfrak{F}_h$-normal in $G$, for any nonempty saturated formation $\mathfrak{F}$. However, the converse is not true in general (see [13, Example 1.2]).

In this paper, we will use $\mathfrak{F}_h$-normal subgroups to give some new characterizations of some classes of groups. Some previously known results are generalized.

### *1 Preliminaries*

A formation $\mathfrak{F}$ is said to be $S$-closed ($S_n$-closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups.

For the sake of convenience, we cite the following lemmas which are useful in this paper.

***Lemma* 1.1** [15, Lemma 2.1]. *Let $\mathfrak{F}$ be a non-empty saturated formation, $A \le G$ and $Z = Z_\infty^{\mathfrak{F}}(G)$. Then*

*(1) If $A$ is normal in $G$, then $AZ/A \le Z_\infty^{\mathfrak{F}}(G/A)$.*

*(2) If $\mathfrak{F}$ is S-closed, then $Z \cap A \le Z_\infty^{\mathfrak{F}}(A)$.*

*(3) If $\mathfrak{F}$ is $S_n$-closed and $A$ is normal in $G$, then $Z \cap A \le Z_\infty^{\mathfrak{F}}(A)$.*

*(4) If $G \in \mathfrak{F}$, then $Z = G$.*

***Lemma* 1.2** [12, Lemma 2.5]. *Let $p$ be a prime number such that $(|G|, p^2-1) = 1$. If $G/L$ is $p$-nilpotent and $p^3 \nmid |L|$, then $G$ is $p$-nilpotent.*

***Lemma* 1.3** [16, II 7.9]. *Let $N$ be a nilpotent normal subgroup of $G$. If $N \ne 1$ and $N \cap \Phi(G) = 1$, then $N$ is a direct product of some minimal normal subgroups of $G$.*

***Lemma* 1.4** [3, VI 14.3]. *Let $G$ be a finite group. If $G$ has an abelian Sylow $p$-subgroup $P$ of $G$, then $Z(G) \cap G' \cap P = 1$.*

***Lemma* 1.5** [17, Theorem 1]. *Let $\mathfrak{F}$ be a saturated formation and $G$ be a minimal non-$\mathfrak{F}$-group such that $(G^{\mathfrak{F}})'$ is a proper subgroup of $G^{\mathfrak{F}}$, then $G^{\mathfrak{F}}$ is a solvable group.*

***Lemma* 1.6** [1, Corollary 3.2.9]. *If $\mathfrak{F}$ is a local formation, then $[G^{\mathfrak{F}}, Z_\infty^{\mathfrak{F}}(G)] = 1$, for any group $G$.*

***Lemma* 1.7** [13, Theorem 3.2]. *Let $\mathfrak{F}$ be a S-closed saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathfrak{F}$. If all cyclic subgroups of $E$ of prime order and order 4 are $\mathfrak{U}_h$-normal in $G$, then $G \in \mathfrak{F}$.*

***Lemma* 1.8** [13, Theorem 4.1]. *A group $G$ is solvable if and only if every minimal subgroup of $G$ is $\mathfrak{U}_h$-normal in $G$.*

***Lemma* 1.9** [13, Theorem 3.1]. *A group $G$ is supersoluble if and only if there exists a normal subgroup $E$ of $G$ such that $G/E$ is supersoluble and every maximal subgroup of every non-cyclic Sylow subgroup of $E$ is $\mathfrak{U}_h$-normal in $G$.*

***Lemma* 1.10.** *Let $R$ be a soluble minimal normal subgroup of $G$. If there exists a maximal subgroup $R_1$ of $R$ such that $R_1$ is $\mathfrak{U}_h$-normal in $G$, then $R$ is a group of prime order.*

*Proof.* Since $R$ is a minimal normal subgroup of $G$, $R$ is an elementary abelian group and $(R_1)_G = 1$. By hypothesis, there exists a normal subgroup $K$ of $G$ such that $R_1K$ is a normal Hall subgroup of $G$ and $R_1 \cap K \le Z_\infty^{\mathfrak{U}}(G)$. Since $R \cap KG$, $R \cap K = 1$ or $R \cap K = R$. If $R \cap K = 1$, then $R = R \cap R_1K = R_1(R \cap K) = R_1$, a contradiction. If $R \cap K = R$, then $R \le K$, and so $R_1 \le K$. It follows that $R_1 = R_1 \cap K \le Z_\infty^{\mathfrak{U}}(G)$. If $R$ is not a group of prime order, then $1 \ne R_1 \le Z_\infty^{\mathfrak{U}}(G) \cap R$. Hence $Z_\infty^{\mathfrak{U}}(G) \cap R \ne 1$ and $R \le Z_\infty^{\mathfrak{U}}(G)$. It follows that $R$ is a group of prime order. This contradiction completes the proof.

***Lemma* 1.11** [13, Lemma 2.6]. *Let $G$ be a group and $H \le K \le G$. Then*

*(1) $H$ is $\mathfrak{F}_h$-normal in $G$ if and only if $G$ has a normal subgroup $T$ such that $HT$ is a normal Hall subgroup of $G$, $H_G \le T$ and $H/H_G \cap T/H_G \le Z_\infty^{\mathfrak{F}}(G/H_G)$.*

*(2) Suppose that $H$ is normal in $G$. If $K$ is $\mathfrak{F}_h$-normal in $G$, then $K/H$ is $\mathfrak{F}_h$-normal in $G/H$.*

*(3) Suppose that $H$ is normal in $G$. Then for every $\mathfrak{F}_h$-normal subgroup $E$ of $G$ satisfying $(|H|, |E|)=1$, $HE/H$ is $\mathfrak{F}_h$-normal in $G/H$.*

*(4) If $H$ is $\mathfrak{F}_h$-normal in $G$ and $\mathfrak{F}$ is S-closed, then $H$ is $\mathfrak{F}_h$-normal in $K$.*

*(5) If $H$ is $\mathfrak{F}_h$-normal in $G$ and $\mathfrak{F}$ is $S_n$-closed, then $H$ is $\mathfrak{F}_h$-normal in $K$.*

*(6) If $G \in \mathfrak{F}$, then every subgroup of $G$ is $\mathfrak{F}_h$-normal in $G$.*

## 2 Main Results

**Theorem 2.1.** *Let $p$ be a prime divisor of $|G|$ with $(|G|, p-1)=1$. Then $G$ is p-nilpotent if and only if there exists a normal subgroup $N$ of $G$ such that $G/N$ is $p$-nilpotent and every maximal subgroup of every Sylow subgroup of $N$ is $\mathfrak{U}_h$-normal in G.*

*Proof.* The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $G$ be a counterexample of minimal order. we proceed the proof via the following steps.

(1) $G$ is soluble, $G$ has a minimal normal subgroup $L \leq N$ and $L$ is an elementary abelian $r$-group, where $r$ is the largest prime in $\pi(N)$.

If $p>2$, then $G$ is soluble since $(|G|, p-1)=1$. Now we assume that $p=2$. Then $G/N$ is 2-nilpotent and so $G/N$ is soluble. Since every maximal subgroup of every Sylow subgroup of $N$ is $\mathfrak{U}_h$-normal in $G$, it is $\mathfrak{U}_h$-normal in $N$ by Lemma 1.11 (4). Applying Lemma 1.9 for the case $G=N$, we get that $N$ is supersoluble and so $G$ is soluble. Hence, for the largest prime number $r$ in $\pi(N)$, the Sylow $r$-subgroup $R$ of $N$ is normal in $N$. Since $R$ char $N \trianglelefteq G$, $R$ is normal in $G$. Thus, $G$ has a minimal normal subgroup $L \leq N$ and $L$ is an elementary abelian $r$-group.

(2) $G/L$ is $p$-nilpotent and $L=R \in Syl_p(N)$.

Obviously, $(G/L)/(N/L) \simeq G/N$ is $p$-nilpotent. Let $R_1/L$ be a maximal subgroup of a Sylow $r$-subgroup of $N/L$. Then $R_1$ is a maximal subgroup of the Sylow $r$-subgroup $R$ of $N$. By hypothesis and Lemma 1.11 (2), $R_1/L$ is $\mathfrak{U}_h$-normal in $G/L$. Let $Q_1/L$ be a maximal subgroup of a Sylow $q$-subgroup of $N/L$, where $q \neq r$. It is clear that $Q_1 = Q_1^* L$, where $Q_1^*$ is a maximal subgroup of a Sylow $q$-subgroup of $N$. By hypothesis and Lemma 1.11 (3), $Q_1/L = Q_1^* L/L$ is $\mathfrak{U}_h$-normal in $G/L$. Hence by the minimal choice of $G$, $G/L$ is $p$-nilpotent. If $p \nmid |L|$, then $G$ is $p$-nilpotent, a contradiction. So $L$ is a $p$-group. Since the class of all $p$-nilpotent groups is a saturated formation, $L$ is the unique minimal normal subgroup of $G$ contained in $N$ and $L \nleq \Phi(G)$. By Lemma 1.3, $F(N)=L$. Since $N$ is soluble, $L \leq C_N(F(N)) \leq F(N)$ and so $C_N(L)=L=F(N)$. Because $RG$ and $R \leq F(N)$, we have that $L=R \in Syl_p(N)$.

(3) Final contradiction.

Let $L_1$ be a maximal subgroup of $L$. By (2) and the hypothesis, $L_1$ is $\mathfrak{U}_h$-normal in $G$. Then by Lemma 1.10, we have that $|L|=p$. Since $G/L$ is $p$-nilpotent, $G/L$ has a normal $p$-complement $H/L$. By Schur Zassenhaus theorem, $H=G_{p'}L$, where $G_{p'}$ is a Hall $p'$-subgroup of $G$. Since $p$ is the prime divisor of $|G|$ with $(|G|, p-1)=1$ and $N_H(L)/C_H(L) \lesssim Aut(L)$ is a cyclic subgroup of order $p-1$. By the well known Burnside theorem, we have that $H$ is $p$-nilpotent. Hence, $G_{p'}$ char $H \trianglelefteq G$ and so $G_{p'} \trianglelefteq G$. Clearly, $G_{p'}$ is a normal $p$-complement of $G$, which implies that $G$ is $p$-nilpotent. The final contradiction completes the proof.

**Theorem 2.2.** *Let $p$ be a prime divisor of $|G|$ with $(|G|, p-1)=1$. Then $G$ is $p$-nilpotent if and only if $G$ has a soluble normal subgroup $H$ such that $G/H$ is $p$-nilpotent and every maximal subgroup of every Sylow subgroup of $F(H)$ is $\mathfrak{U}_h$-normal in G.*

*Proof.* The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $G$ be a counterexample with $|G||H|$ is minimal. Let $P$ be an arbitrary Sylow $r$-subgroup of $F(H)$. Since $P$ char $F(H)$ char $H \trianglelefteq G$, $P \trianglelefteq G$. We proceed the proof via the following steps.

(1) $\Phi(G) \cap P = 1$.

If not, then $1 \neq \Phi(G) \cap P \trianglelefteq G$. Let $R = \Phi(G) \cap P$. Clearly, $(G/R)/(H/R) \simeq G/H$ is $p$-nilpotent. By [3, Theorem III.3.5], we have that $F(H/R)=F(H)/R$. Assume that $P/R$ is a Sylow $r$-subgroup of $F(H/R)$ and $P_1/R$ a maximal subgroup of $P/R$. Then $P_1$ is a maximal subgroup of $P$. By hypothesis, $P_1$ is $\mathfrak{U}_h$-normal in $G$. Then by Lemma 1.11 (2), $P_1/R$ is $\mathfrak{U}_h$-normal in $G/R$. Now, let $Q/R$ be a maximal subgroup of some Sylow $q$-subgroup of $F(H/R)=F(H)/R$, where $q \neq r$. Then $Q=Q_1R$, where $Q_1$ is a maximal subgroup of the Sylow $q$-subgroup of $F(H)$. By hypothesis, $Q_1$ is $\mathfrak{U}_h$-normal in $G$. Hence $Q/R=Q_1R/R$ is $\mathfrak{U}_h$-normal in $G/R$ by Lemma 1.11 (3). This shows that $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of $(G,H)$ implies that $G/R$ is $p$-nilpotent. Since $G/\Phi(G) \simeq (G/R)/(\Phi(G)/R)$ is $p$-nilpotent and the class of all $p$-nilpotent groups is a saturated formation, $G$ is $p$-nilpotent, a contradiction. Hence (1) holds.

(2) $P = \langle x_1 \rangle \times \cdots \times \langle x_m \rangle$, where every $\langle x_i \rangle$

($i = 1,\cdots,m$) is a normal subgroup of $G$ of order $r$.

By (1) and Lemma 1.3, $P = R_1 \times\cdots\times R_m$, where $R_i$ ($i = 1,\cdots,m$) is a minimal normal subgroup of $G$. We now prove that all $R_i$ are of order $r$.

Assume that $|R_i| > r$, for some $i$. Without loss of generality, we let $|R_1| > r$. Let $R_1^*$ be a maximal subgroup of $R_1$. Then $R_1^* \times R_2 \times\cdots\times R_m = P_1$ is a maximal subgroup of $P$. Set $T = R_2 \times\cdots\times R_m$, then, clearly $(P_1)_G = T$. By hypothesis, $P_1$ is $\mathfrak{U}_h$-normal in $G$. Hence by Lemma 1.11 (1), there exists a normal subgroup $N$ of $G$, such that $(P_1)_G \le N$, $P_1N$ is a normal Hall subgroup of $G$ and $(P_1 \cap N)/(P_1)_G \le Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. It follows that $P_1N = R_1^*TN = R_1^*N$. If $R_1^* \cap N \ne 1$, then $1 < R_1 \cap N \trianglelefteq G$. By the minimal normality of $R_1$, $R_1 \cap N = R_1$ and so $R_1 \le N$. Hence $P_1N = R_1^*N = N$. Consequently $P_1 \le N$. It follows that $P_1/(P_1)_G \le Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. If $(P_1)_G = P_1$, then $R_1^* = 1$, which contradicts $R_1^* \cap N \ne 1$. Hence $(P_1)_G < P_1$ and so $1 \ne P_1/(P_1)_G \le Z_\infty^{\mathfrak{U}}(G/(P_1)_G) \cap P/(P_1)_G$. Since $P/(P_1)_G \simeq R_1$ and $R_1$ is a minimal normal subgroup of $G$, $P/(P_1)_G$ is a chief factor of $G$. This implies that $Z_\infty^{\mathfrak{U}}(G/(P_1)_G) \cap P/(P_1)_G = P/(P_1)_G$ and so $P/(P_1)_G \le Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. It follows that $|P/(P_1)_G| = r$. Hence $|R_1| = r$, a contradiction. Now assume that $R_1^* \cap N = 1$. Then $(R_1^*)_G = 1 \le N \trianglelefteq G$, $R_1^*N = P_1N$ is a normal Hall subgroup and $(R_1^* \cap N)/(R_1^*)_G = 1 \le Z_\infty^{\mathfrak{U}}(G/(R_1^*)_G)$. This shows that $R_1^*$ is $\mathfrak{U}_h$-normal in $G$. Hence $R_1$ is a cyclic group of order $r$ by Lemma 1.10, a contradiction again. Thus (2) holds.

(3) $G/F(H)$ is $p$-nilpotent.

From (2), $F(H) = \langle y_1\rangle \times\cdots\times \langle y_n\rangle$, where every $\langle y_i\rangle$ ($i = 1,\cdots,n$) is a normal subgroup of $G$ of prime order. Since $G/C_G(\langle y_i\rangle)$ is isomorphic to a subgroup of $Aut(\langle y_i\rangle)$, $G/C_G(\langle y_i\rangle)$ is cyclic and so it is $p$-nilpotent for each $i$. It follows that $G/\cap_{i=1}^n C_G(\langle y_i\rangle)$ is $p$-nilpotent. Obviously, $C_G(F(H)) = \cap_{i=1}^n C_G(\langle y_i\rangle)$. Hence $G/C_G(F(H))$ is $p$-nilpotent. Consequently $G/(H \cap C_G(F(H))) = G/C_H(F(H))$ is $p$-nilpotent. Because $F(H)$ is abelian, we have that $F(H) \le C_H(F(H))$. On the other hand, $C_H(F(H)) \le F(H)$ for $H$ is soluble. Thus $F(H) = C_H(F(H))$ and so $G/F(H)$ is $p$-nilpotent.

(4) Final contradiction.

In view of Theorem 2.1, we have that $G$ is $p$-nilpotent. The final contradiction completes the proof.

***Theorem 2.3.*** *Let $\mathfrak{F}$ be a $S$-closed saturated formation which satisfies that every minimal non-$\mathfrak{F}$-group is soluble. Then $G$ is an $\mathfrak{F}$-group if and only if $G$ has a normal subgroup $N$ such that $G/N \in \mathfrak{F}$ and every cyclic subgroup of order 4 of $N$ is $\mathfrak{F}_h$-normal in $G$ and every minimal subgroup of $N$ is contained in $Z_\infty^{\mathfrak{F}}(G)$.*

*Proof.* The necessity is obvious. We only need to prove the sufficiency. Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Then, obviously $N \ne 1$.

Let $L$ be a proper subgroup of $G$. Then $L/L \cap N \simeq LN/N \le G/N$ implies that $L/L \cap N \in \mathfrak{F}$. Since $L \cap N \le N$, by hypothesis, every cyclic subgroup of $L \cap N$ of order 4 is $\mathfrak{F}_h$-normal in $G$ and hence is $\mathfrak{F}_h$-normal in $L$ by Lemma 1.11. On the other hand, since every minimal subgroup of $L$ is a minimal subgroup of $G$, every minimal subgroup of $L$ is contained in $Z_\infty^{\mathfrak{F}}(G) \cap L \subseteq Z_\infty^{\mathfrak{F}}(L)$ by Lemma 1.1. This shows that $(L, N \cap L)$ satisfies the hypothesis. By the minimal choice of $G$, $L \in \mathfrak{F}$ and so $G$ is a minimal non-$\mathfrak{F}$-group. By [1, Theorem 3.4.2] and the hypothesis, we know that $G$ is soluble and $G$ has the following properties: (1) $G^{\mathfrak{F}}$ is a $p$-group, for some prime $p$; (2) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of $G$; (3) If $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is an elementary abelian $p$-group; (4) If $p > 2$, then the exponent of $G^{\mathfrak{F}}$ is $p$; If $p = 2$, then the exponent of $G^{\mathfrak{F}}$ is 2 or 4.

Since $G/N \in \mathfrak{F}$, $G^{\mathfrak{F}} \le N$. Suppose that the exponent of $G^{\mathfrak{F}}$ is a prime. Then by hypothesis, $G^{\mathfrak{F}} \subseteq Z_\infty^{\mathfrak{F}}(G)$ and so $G \in \mathfrak{F}$, a contradiction.

Now assume that $G^{\mathfrak{F}}$ is not abelian and $p = 2$. We claim that there is no an element of order 4 in $G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$. Assume that there exists an element $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$ with $|\langle x\rangle| = 4$. Then by hypothesis, $\langle x\rangle$ is $\mathfrak{F}_h$-normal in $G$. Hence by Lemma 1.11 (1), there exists a normal subgroup $T$ of $G$ such that $\langle x\rangle T$ is a normal Hall subgroup of $G$ and $\langle x\rangle/\langle x\rangle_G \cap T/\langle x\rangle_G \le Z_\infty^{\mathfrak{F}}(G/\langle x\rangle_G)$. Thus $G^{\mathfrak{F}} \le \langle x\rangle T$ by (1). Let $P_1 = G^{\mathfrak{F}} \cap T$. Then $P_1 \trianglelefteq G$. If $P_1 \le \Phi(G^{\mathfrak{F}})$, then $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap \langle x\rangle T = \langle x\rangle (G^{\mathfrak{F}} \cap T) = \langle x\rangle P_1 = \langle x\rangle$, a contradiction. So $P_1 \not\le \Phi(G^{\mathfrak{F}})$. By (2) $P_1\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$. It follows that

$P_1 = G^{\mathfrak{F}}$ and so $G^{\mathfrak{F}} \le T$. Thus $\langle x\rangle \le T$ and $\langle x\rangle = \langle x\rangle \cap T$. We first assume that $\langle x\rangle/\langle x\rangle_G \cap T/\langle x\rangle_G = 1$. Then $\langle x\rangle = \langle x\rangle_G \trianglelefteq G$. Hence $\langle x\rangle\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}}) \trianglelefteq G/\Phi(G^{\mathfrak{F}})$. Then by (2), $\langle x\rangle\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}}$ and so $\langle x\rangle = G^{\mathfrak{F}}$, a contradiction. Hence $\langle x\rangle/\langle x\rangle_G \cap T/\langle x\rangle_G \ne 1$, that is $\langle x\rangle_G < \langle x\rangle$, and so $|\langle x\rangle_G| \le 2$. If $|\langle x\rangle_G| = 1$, then $\langle x\rangle \le Z_\infty^{\mathfrak{F}}(G)$. By hypothesis, $G^{\mathfrak{F}} \le Z_\infty^{\mathfrak{F}}(G)$ and consequently $G \in \mathfrak{F}$, a contradiction. If $|\langle x\rangle_G| = 2$, then $\langle x\rangle/\langle x\rangle_G \le Z_\infty^{\mathfrak{F}}(G/\langle x\rangle_G)$ and $\langle x\rangle_G \le Z_\infty^{\mathfrak{F}}(G)$. It follows that $Z_\infty^{\mathfrak{F}}(G/\langle x\rangle_G) = Z_\infty^{\mathfrak{F}}(G)/\langle x\rangle_G$. Hence $\langle x\rangle \le Z_\infty^{\mathfrak{F}}(G)$. This implies that $G^{\mathfrak{F}} \le Z_\infty^{\mathfrak{F}}(G)$. Consequently $G \in \mathfrak{F}$. This final contradiction completes the proof.

***Corollary* 2.3.1.** *Let $\mathfrak{F}$ be a $S$-closed saturated formation which satisfies that every minimal non-$\mathfrak{F}$-group is soluble. Then $G$ is an $\mathfrak{F}$-group if and only if every cyclic subgroup of order 4 of $G$ is $\mathfrak{F}_h$-normal in $G$ and every minimal subgroup of $G$ is contained in $Z_\infty^{\mathfrak{F}}(G)$.*

***Corollary* 2.3.2** (Miao, Guo [18]). *Let $\mathfrak{F}$ be a $S$-closed saturated formation which satisfies that a minimal non-$\mathfrak{F}$-group is soluble and its $\mathfrak{F}$-residual is a Sylow subgroup. If every cyclic subgroup of order 4 of $G$ is $c$-normal in $G$ and every minimal subgroup of $G$ is contained in the $\mathfrak{F}$-hypercenter of $G$, then $G$ is an $\mathfrak{F}$-group.*

***Corollary* 2.3.3** (Miao, Guo [18]). *Let $\mathfrak{F}$ be a $S$-closed saturated formation which satisfies that a minimal non-$\mathfrak{F}$-group is soluble and its $\mathfrak{F}$-residual is a Sylow subgroup. Let $N$ be a normal subgroup of $G$ and $G/N \in \mathfrak{F}$. If every cyclic subgroup of order 4 of $N$ is $c$-normal in $G$ and every minimal subgroup of $N$ is contained in the $\mathfrak{F}$-hypercenter of $G$, then $G$ is an $\mathfrak{F}$-group.*

***Theorem* 2.4.** *Let $\mathfrak{F}$ be a $S$-closed saturated formation containing $\mathfrak{U}$ and $G$ a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup $N$ of $G$ such that $G/N \in \mathfrak{F}$ and all elements of $N$ of odd prime order are $\mathfrak{U}_h$-normal in $G$ and $N$ has an abelian Sylow 2-subgroup and every subgroup of $N$ of order 2 is contained in $Z_\infty^{\mathfrak{U}}(G)$.*

*Proof.* Assume that the theorem is false and let $G$ be a counterexample of minimal order.

First we show that $M \in \mathfrak{F}$ for every maximal subgroup $M$ of $G$. If $N \not\le M$, then $G = MN$ and $M/M \cap N \simeq MN/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is S-closed, $M \cap Z_\infty^{\mathfrak{F}}(G) \le Z_\infty^{\mathfrak{F}}(M)$ by Lemma 1.1. Then by Lemma 1.11, we see that $(M, M \cap N)$ satisfies the hypothesis. Hence $M \in \mathfrak{F}$ by the choice of $G$. Therefore $G$ is a minimal non-$\mathfrak{F}$-group. Let $R = G^{\mathfrak{F}}$. Then $R \le N$. Assume that $R' < R$, where $R'$ is the derived subgroup of $R$. Then $R$ is soluble by Lemma 1.5. Hence by [1, Theorem 3.4.2] and since $R$ has an abelian Sylow 2-subgroup, $R$ is a $p$-group of exponent $p$. If $p \ne 2$, then $G \in F$ by Lemma 1.7, a contradiction. Suppose that $p = 2$, then $R$ is an elementary abelian 2-group. Thus, by hypothesis, $R \le Z_\infty^{\mathfrak{F}}(G)$ and so $G \in \mathfrak{F}$, a contradiction. Now assume that $R = R'$. Let $T$ be a Sylow 2-group of $R$. Then $T$ is abelian and so $T \cap Z(R) = 1$ by Lemma 1.4. Assume that $T \ne 1$. Then there exists an element $r \in T$ with $|r| = 2$. Hence $r \in Z_\infty^{\mathfrak{F}}(G)$ and so $r \in Z_\infty^{\mathfrak{F}}(G) \cap R$. Since $Z_\infty^{\mathfrak{F}}(G) \cap R$ is contained in $Z(R)$ by Lemma 1.6, $r \in Z(R) \cap T \ne 1$. That is $Z(R) \cap T \ne 1$. This contradiction shows that $R$ is of odd order. Therefore by Feit-Thompson theorem, $R$ is soluble, which contradicts $R = R'$.

These contradictions show that the counterexample of minimal order does not exist. Therefore the Theorem holds.

***Theorem* 2.5.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$.. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if one of the following conditions holds:*

*(a) $G$ is 2-nilpotent and every element $x$ of odd prime order of $H$ is $\mathfrak{U}_h$-normal in $G$.*

*(b) $H$ has an abelian Sylow 2-subgroup and every subgroup of prime order of $H$ is $\mathfrak{U}_h$-normal in G.*

*Proof.* (a) If $G$ is 2-nilpotent, then $H$ is 2-nilpotent. Let $K$ be the 2-complement of $H$. Then $K \trianglelefteq G$. Since $(G/K)/(H/K) \simeq G/H \in \mathfrak{F}$ and $H/K$ is a 2-group, $H/K$ has no element of odd order . Hence $G/K \in \mathfrak{F}$ by induction on $|G|$. Since $K$ is a 2-complement of $H$, $K$ has no cyclic subgroup of order 4. Thus $G \in \mathfrak{F}$ by Lemma 1.7.

(b) Let $E = G^{\mathfrak{F}}$. Then, obviously, $E \le H$ and $E$ has abelian Sylow 2-subgroups. By hypotheses, every subgroup $\langle x\rangle$ of prime order of $E$ is $\mathfrak{U}_h$-normal in $G$. Hence, by Lemma 1.11, $\langle x\rangle$ is also $\mathfrak{U}_h$-normal in $E$. It follows from Lemma 1.8 that $E$ is soluble. Let $M$ be a maximal subgroup of $G$ such that $E \not\le M$. Then $ME/E \simeq M/M \cap E \in \mathfrak{F}$. It is easy to see that $(M, M \cap E)$ satisfies the hypothesis. Therefore $M \in \mathfrak{F}$ by induction. Then, applying [1, Theorem 3.4.2], we see that $E$ is a $p$-group of exponent $p$. Thus $G \in \mathfrak{F}$ by Lemma 1.7.

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