Журнал Белорусского государственного университета. Математика. Информатика. 2020;2:28–35
Journal of the Belarusian State University. Mathematics and Informatics. 2020;2:28–35

УДК 517.983.3:517.983.34

ОПЕРАТОРЫ ХАУСДОРФА НА ОДНОРОДНЫХ ПРОСТРАНСТВАХ ЛОКАЛЬНО КОМПАКТНЫХ ГРУПП

А. Р. МИРОТИН[1)]

[1)]*Гомельский государственный университет им. Франциска Скорины, ул. Советская, 104, 246019, г. Гомель, Беларусь*

Операторы Хаусдорфа на вещественной прямой и многомерном евклидовом пространстве произошли из классических методов суммирования. Сейчас эта тема активно изучается. Операторы Хаусдорфа на общих группах были определены автором в 2019 г. В настоящей работе исследуются операторы Хаусдорфа на пространствах Лебега и вещественных пространствах Харди над однородными пространствами локально компактных групп. В частности, вводится атомарное пространство Харди над однородными пространствами локально компактных групп и определяются условия ограниченности операторов Хаусдорфа на таких пространствах. Рассмотрено несколько следствий, и сформулированы нерешенные проблемы.

Ключевые слова: оператор Хаусдорфа; локально компактная группа; однородное пространство; атомарное пространство Харди; пространство Лебега; ограниченный оператор.

HAUSDORFF OPERATORS ON HOMOGENEOUS SPACES OF LOCALLY COMPACT GROUPS

A. R. MIROTIN[a]

[a]*Francisk Skorina Gomel State University, 104 Savieckaja Street, Homieĺ 246019, Belarus*

Hausdorff operators on the real line and multidimensional Euclidean spaces originated from some classical summation methods. Now it is an active research area. Hausdorff operators on general groups were defined and studied by the author since 2019. The purpose of this paper is to define and study Hausdorff operators on Lebesgue and real Hardy spaces over homogeneous spaces of locally compact groups. We introduce in particular an atomic Hardy space over homogeneous spaces of locally compact groups and obtain conditions for boundedness of Hausdorff operators on such spaces. Several corollaries are considered and unsolved problems are formulated.

Keywords: Hausdorff operator; locally compact group; homogeneous space; atomic Hardy space; Lebesgue space; bounded operator.

Introduction and preliminaries

Hausdorff operators were introduced by G. Hardy [1, chapter XI] on the segment, and by C. Georgakis [2] and independently by E. Liflyand and F. Moricz [3] on the whole real line. Their multidimensional generalizations were considered later by G. Brown and F. Moricz [4], and E. Liflyand and A. Lerner [5]. Now it is an active research area. It is enough to note that the Google search by request «Hausdorff operator» gives more then 1 200 000 results. See also survey articles [6; 7] for historical remarks and the state of the art up to 2013.

Образец цитирования:
Миротин АР. Операторы Хаусдорфа на однородных пространствах локально компактных групп. *Журнал Белорусского государственного университета. Математика. Информатика.* 2020;2:28–35 (на англ.).
https://doi.org/10.33581/2520-6508-2020-2-28-35

For citation:
Mirotin AR. Hausdorff operators on homogeneous spaces of locally compact groups. *Journal of the Belarusian State University. Mathematics and Informatics.* 2020;2:28–35.
https://doi.org/10.33581/2520-6508-2020-2-28-35

Автор:
Адольф Рувимович Миротин – доктор физико-математических наук, профессор; заведующий кафедрой математического анализа и дифференциальных уравнений факультета математики и технологий программирования.

Author:
Adolf R. Mirotin, doctor of science (physics and mathematics), full professor; head of the department of mathematical analysis and differential equations, faculty of mathematics and technologies of programming.
amirotin@yandex.ru

Hausdorff operators on general groups were defined and studied by the author in [8] and [9]. The purpose of this paper is to define and study Hausdorff operators on Lebesgue and real Hardy spaces over homogeneous spaces of locally compact groups. In what follows G stands for a locally compact group with left Haar measure ν. We denote by $\mathrm{Aut}(G)$ the space of all topological automorphisms of G endowed with its natural topology (see, e. g., [10, chapter 6, section 26]), $\mathcal{L}(Y)$ denotes the space of linear bounded operators on a normed space Y.

In [8] the next definition was proposed.

Definition 1 [8]. Let (T, m) be a measure space, G be a topological group, $A : T \to \mathrm{Aut}(G)$ be a measurable map, and Ψ be a locally integrable function on Ω. We define the *Hausdorff operator* with the kernel Ψ over the group G by the formula

$$\mathcal{H}f(x) = \int_T \Psi(t) f(A(t)(x)) dm(t).$$

By [8, lemma 1] for locally compact G this operator is bounded in $L^p(G)$ $(1 \le p \le \infty)$ provided $\Psi(t)(\mathrm{mod}\, A(t))^{-1/p} \in L^1(T, m)$ and

$$\|\mathcal{H}\|_{\mathcal{L}(L^p(G))} \le \int_T |\Psi(t)| (\mathrm{mod}\, A(t))^{-1/p} dm(t)$$

(for a definition of $\mathrm{mod}\, A(u)$ see, e. g., [11, chapter VII, section 1]).

Moreover, in [8] and [9] conditions for boundedness of Hausdorff operators on the real Hardy space $H^1(G)$ over metrizable locally compact group G with doubling condition were obtained. In this work, we define Hausdorff operators on homogeneous spaces of locally compact groups and prove analogs of aforementioned results for this situation.

Let K be a compact subgroup of G with normalized Haar measure β. Consider the quotient space G/K of left cosets $\dot{x} := xK = \pi_K(x)$ $(x \in G)$ where $\pi_K : G \to G/K$ stands for a natural projection. We shall assume that the measure ν is normalized in such a way that (generalized) Weil's formula

$$\int_G g(x) dx = \int_{G/K} \left(\int_K g(xk) dk \right) d\lambda(\dot{x}) \tag{1}$$

holds for all $g \in L^1(G)$, where λ denotes some left G-invariant measure on G/K (see, e. g., [11, chapter VII, § 2, No. 5, theorem 2] and especially remark c) after this theorem (left G-invariance of λ means that $\lambda(xE) = \lambda(E)$ for every Borel subset E of G/K and for every $x \in G$; this measure is unique up to constant multiplier)). Here and below we write dx instead of $d\nu(x)$ and dk instead of $d\beta(k)$. We shall write also $d\dot{x}$ instead of $d\lambda(\dot{x})$.

The function $g : G \to \mathbb{C}$ is called *right K-invariant* if $g(xk) = g(x)$ for all $x \in G$, $k \in K$. For such a function we put $\dot{g}(\dot{x}) := g(x)$. This definition is correct and for $g \in L^1(G)$ formula (1) implies that

$$\int_G g(x) dx = \int_{G/K} \dot{g}(\dot{x}) d\dot{x} \tag{2}$$

(recall that $\int_K dk = 1$). The map $g \mapsto \dot{g}$ is a bijection between the set of all right K-invariant functions on G (all right K-invariant functions from $L^1(G)$) and the set of all functions on G/K (respectively functions from $L^1(G/K, \lambda)$).

Let an automorphism $A \in \mathrm{Aut}(G)$ maps K onto itself. Since

$$A(\dot{x}) := A(xK) = \{A(x)A(k) : k \in K\} = A(x)K = \pi_K(A(x))$$

we get a homeomorphism $\dot{A} : G/K \to G/K$, $\dot{A}(\dot{x}) := \pi_K(A(x))$. Then for every right K-invariant function g on G we have $\dot{g}(\dot{A}(\dot{x})) = g(A(x))$. From now on we put

$$\mathrm{Aut}_K(G) := \{\dot{A} : A \in \mathrm{Aut}(G),\ A(K) = K\}.$$

Definition 2. Let (Ω, μ) be a measure space, $\left(\dot{A}(u)\right)_{u \in \Omega} \subset \mathrm{Aut}_K(G)$ be a family of homeomorphisms of G/K, and $\Phi \in L^1_{loc}(\Omega, \mu)$. For a function f on G/K we define a *Hausdorff operator* on G/K as follows

$$\left(\mathcal{H}_{\Phi, \dot{A}} f\right)(\dot{x}) := \int_{\Omega} \Phi(u) f\left(\dot{A}(u)(\dot{x})\right) d\mu(u).$$

As was mentioned by G. Hardy in the case $\Omega = [0, 1]$ [1, theorem 217] his Hausdorff operators possess some regularity property. A Hausdorff operator in the sense of definition 2 also enjoys this property as the next proposition shows.

Proposition 1. *Suppose that the conditions of definition 2 are fulfilled and the group G is σ-compact. In order that the transformation $\mathcal{H}_{\Phi, \dot{A}}$ should be regular, i. e. that $f \in C(G/K)$, $f(\dot{x}) \to l$ when $\dot{x} \to \infty$ should imply $\mathcal{H}_{\Phi, \dot{A}} f(\dot{x}) \to l$, it is necessary and sufficient that $\int_{\Omega} \Phi(u) d\mu(u) = 1$.*

P r o o f. If $f(x) = 1$ then $\mathcal{H}_{\Phi, \dot{A}} f(\dot{x}) = \int_{\Omega} \Phi(u) d\mu(u)$. Thus, $\int_{\Omega} \Phi(u) d\mu(u) = 1$ is a necessary condition.

To prove the sufficiency, first note that since $\dot{A}(u)$ has continuous inverse, $f(\dot{x}) \to l$ when $\dot{x} \to \infty$ implies $f\left(\dot{A}(u)\dot{x}\right) \to l$ when $\dot{x} \to \infty$. Indeed, $f(\dot{x}) \to l$ when $\dot{x} \to \infty$ means that for every $\varepsilon > 0$ there is a compact $C_\varepsilon \subset G/K$ such that $\left|f(\dot{x}) - l\right| < \varepsilon$ for $x \in G \backslash C_\varepsilon$. Now if $\dot{x} \in (G/K) \setminus \dot{A}(u)^{-1}(C_\varepsilon)$ we get $\dot{A}(u)\dot{x} \in (G/K) \backslash C_\varepsilon$ and therefore $\left|f\left(\dot{A}(u)\dot{x}\right) - l\right| < \varepsilon$.

But if, in addition, $f \in C(G/K)$ the function f is bounded on G/K and therefore $\mathcal{H}_{\Phi, \dot{A}} f(\dot{x}) \to l$ by the Lebesgue theorem (one can apply the Lebesgue theorem, since G/K is σ-compact).

Thus, proposition 1 shows that Hausdorff operators in a sense of definition 1 gives us a family (for various Φ, $A(u)$, and Ω) of generalized limits at infinity for functions on G/K.

Hausdorff operators on $L^p(G/K)$

In the following we put $L^p(G/K) := L^p(G/K, \lambda)$ $(p \in [1, \infty])$. Formula (2) implies that $\|g\|_{L^p(G)} = \|\dot{g}\|_{L^p(G/K)}$ for every right K-invariant function $g \in L^p(G)$.

In this section we give conditions of boundedness of Hausdorff operators on $L^p(G/K)$. Let (Ω, μ) and $\left(\dot{A}(u)\right)_{u \in \Omega}$ be as in definition 2. For a function Φ on Ω let

$$\|\Phi\|_{p, A} := \int_{\Omega} |\Phi(u)| \left(\operatorname{mod} A(u)\right)^{-1/p} d\mu(u).$$

Theorem 1. *Suppose that the conditions of definition 2 are fulfilled, $p \in [1, \infty]$, and $\|\Phi\|_{p, A} < \infty$. Then $\mathcal{H}_{\Phi, \dot{A}}$ is bounded in $L^p(G/K)$ and*

$$\left\|\mathcal{H}_{\Phi, \dot{A}}\right\|_{\mathcal{L}\left(L^p(G/K)\right)} \leq \|\Phi\|_{p, A}.$$

P r o o f. Let $1 < p < \infty$. Every function $f \in L^p(G/K)$ has the form $f = \dot{g}$ for a unique right K-invariant function $g \in L^p(G)$. Using Minkowski's integral inequality, we have that

$$\left\|\mathcal{H}_{\Phi, \dot{A}} f\right\|_{L^p(G/K)} = \left(\int_{G/K} \left| \int_{\Omega} \Phi(u) f\left(\dot{A}(u)(\dot{x})\right) d\mu(u) \right|^p d\dot{x} \right)^{1/p} \leq$$

$$\leq \int_{\Omega} |\Phi(u)| \left(\int_{G/K} \left| f\left(\dot{A}(u)(\dot{x})\right) \right|^p d\dot{x} \right)^{1/p} d\mu(u).$$

Since the function $x \mapsto g\left(A(u)(x)\right)$ is right K-invariant, as well, formula (2) yields

$$\int_{G/K} \left| f\left(\dot{A}(u)(\dot{x})\right)\right|^p d\dot{x} = \int_G \left| g\left(A(u)(x)\right)\right|^p dx.$$

On the other hand, by [11, chapter VII, subsection 1.4, formula (31)] we have

$$\int_G \left| g\left(A(u)(x)\right)\right|^p dx = \left(\operatorname{mod} A(u)\right)^{-1} \int_G \left| g(x)\right|^p dx.$$

Thus,

$$\left\| \mathcal{H}_{\Phi,\, \dot{A}} f \right\|_{L^p(G/K)} \leq \int_\Omega |\Phi(u)| \left(\operatorname{mod} A(u)\right)^{-1/p} d\mu(u) \left(\int_G |g(x)|^p dx \right)^{1/p} =$$

$$= \|\Phi\|_{p,\, A} \left(\int_{G/K} |f(\dot{x})|^p d\dot{x} \right)^{1/p} = \|\Phi\|_{p,\, A} \|f\|_{L^p(G/K)}.$$

For $p = 1$ the statement of theorem 1 follows from Fubini's theorem and for $p = \infty$ it is obvious.

The following simple example is intended to illustrate preceding constructions.

Example. Let G be the multiplicative group $\mathbb{C}^\times$ of the complex field $\mathbb{C}$ and $K := \left\{ z \in \mathbb{C}^\times : |z| = 1 \right\}$ the circle group. Then G/K can be identified with $(0, \infty)$ via the map $zK = \dot{z} \mapsto r,$ where $r = |z|$. In other words, we use $(0, \infty)$ as a model of G/K, the positive number r representing the circle of radius r. Automorphisms of G have the form $A\left(re^{i\alpha}\right) = r^u e^{i\alpha}$ or $A\left(re^{i\alpha}\right) = r^u e^{-i\alpha}$ ($u \in \mathbb{R}$). Thus $\operatorname{Aut}_K(G) = \operatorname{Aut}(G)$. It follows that $\dot{A}(\dot{z}) = r^u$ ($u \in \mathbb{R}$). So the general form of a Hausdorff operator on $(0, \infty)$ looks as follows ($f : (0, \infty) \to \mathbb{C},\ r > 0$)

$$\mathcal{H}_{\Phi,\, \dot{A}} f(r) = \int_{\mathbb{R}} \Phi(u) f\left(r^u\right) d\mu(u)$$

(we take $\Omega = \mathbb{R}$, and μ is any positive measure on $\mathbb{R}$). Since G is commutative, $\operatorname{mod} A = 1$ for all $A \in \operatorname{Aut}(G)$. So theorem 1 implies that $\mathcal{H}_{\Phi,\, \dot{A}}$ is bounded on $L^p(0, \infty)$ if $\Phi \in L^1(\mu)$ and in this case $\left\|\mathcal{H}_{\Phi,\, \dot{A}}\right\|_{\mathcal{L}(L^p)} \leq \|\Phi\|_{L^1(\mu)}$.

If we take in theorem 1 the space $\mathbb{Z}_+$ (endowed with counting measure) as Ω, we arrive at the following.

Corollary 1. *Let $\Phi(j)$ be a sequence of complex numbers. Consider a discrete Hausdorff operator over G/K*

$$\mathcal{H}_{\Phi,\, \dot{A}} f(\dot{x}) := \sum_{j=0}^{\infty} \Phi(j) f\left(\dot{A}(j)(\dot{x})\right)$$

(*discrete Hausdorff operators were introduced in [12; 13]).*

Then

$$\left\|\mathcal{H}_{\varphi,\, \dot{A}}\right\|_{\mathcal{L}\left(L^p(G/K)\right)} \leq \sum_{j=0}^{\infty} |\Phi(j)| \left(\operatorname{mod} A(j)\right)^{-1/p}.$$

Putting

$$\Phi(u) = \frac{\chi_{\{\operatorname{mod} A(u) \geq 1\}}(u)}{\operatorname{mod} A(u)}$$

in definition 2 (χ_E denotes the indicator function of the set E), we get a *Cesaro operator* over G/K (cf. [8])

$$\mathcal{C}_{A,\, \mu} f(\dot{x}) := \int_{\{\operatorname{mod} A(u) \geq 1\}} \frac{f\left(\dot{A}(u)(\dot{x})\right)}{\operatorname{mod} A(u)} d\mu(u).$$

Corollary 2. *For a Cesaro operator over G/K the following estimate holds*

$$\left\|\mathcal{C}_{A,\, \mu}\right\|_{\mathcal{L}\left(L^p(G/K)\right)} \leq \int_{\{\operatorname{mod} A(u) \geq 1\}} \frac{d\mu(u)}{\left(\operatorname{mod} A(u)\right)^{1 + (1/p)}}.$$

Hausdorff operators on atomic Hardy space $H^1(G/K)$

The goal of this section is to introduce the atomic Hardy space $H^1(G/K)$ and to obtain conditions for boundedness of Hausdorff operators on this space.

In the rest of the paper as in [9] we assume in addition that G is metrizable via a left invariant metric ρ and the following *doubling condition* in a sense of [14] holds.

There exists a constant C such that

$$\nu(B(x, 2r)) \leq C\nu(B(x, r))$$

for each $x \in G$ and $r > 0$. Here $B(x, r)$ denotes the ball of radius r around x. The *doubling constant* is the smallest constant $C \geq 1$ for which the last inequality holds. We denote this constant by C_ν. Then for each $x \in G$, $k \geq 1$ and $r > 0$

$$\nu(B(x, kr)) \leq C_\nu k^s \nu(B(x, r)), \tag{3}$$

where $s = \log_2 C_\nu$ (see, e. g., [15, p. 76]). The number s takes the role of a «dimension» for a doubling metric measure space G.

To introduce the space $H^1(G/K)$ first recall that a function a on G is an *atom* if it satisfies the following conditions (see [14, p. 591]):

(i) the support of a is contained in a ball $B(x, r)$;

(ii) $\|a\|_\infty \leq \dfrac{1}{\nu(B(x, r))}$;

(iii) $\int\limits_G a(x)dx = 0.$

In case $\nu(G) < \infty$ we shall assume $\nu(G) = 1$. Then the constant function having value 1 is also considered to be an atom.

So by atom we mean an $(1, \infty)$-atom.

Definition 3. We define the *Hardy space* $H^1(G/K)$ as a space of such functions $f = \dot{g}$ on G/K that g admits an atomic decomposition of the form

$$g = \sum_{j=1}^{\infty} \alpha_j a_j, \tag{4}$$

where a_j are right K-invariant atoms and $\sum\limits_{j=1}^{\infty} |\alpha_j| < \infty$. In this case,

$$\|f\|_{H^1(G/K)} := \inf \sum_{j=1}^{\infty} |\alpha_j|,$$

infimum is taken over all decompositions above of g.

Thus, a function $f = \dot{g}$ from $H^1(G/K)$ admits an atomic decomposition $f = \sum\limits_{j=1}^{\infty} \alpha_j \dot{a}_j$ such that $\sum\limits_{j=1}^{\infty} |\alpha_j| < \infty$, and $\|f\|_{H^1(G/K)} = \|g\|_{H^1(G)}$.

Remark. Real Hardy spaces over compact connected (not necessary quasi-metric) Abelian groups were defined in [16]. The case of semisimple Lie groups was considered earlier in [17].

Proposition 2. The space $H^1(G/K)$ is Banach. If, in addition, for some h from $H^1(G)$ *the function* $x \mapsto \int\limits_K h(xk)dk$ *is not identically zero, the space* $H^1(G/K)$ *is non-trivial.*

P r o o f. First, we shall show that $H^1(G/K)$ is complete. Note that since $\|a\|_{L^1(G)} \leq 1$ for each atom a, we have $\|g\|_{L^1(G)} \leq \|\dot{g}\|_{H^1(G/K)}$ for each right K-invariant function $g \in H^1(G)$. Then for a function $f = \dot{g}$ we have $\|f\|_{L^1(G/K)} \leq \|f\|_{H^1(G/K)}$. Let a sequence $f_j \in H^1(G/K)$ be such that $\sum\limits_j \|f_j\|_{H^1(G/K)} < \infty$. It is enough to prove that the series $\sum\limits_j f_j$ converges in $H^1(G/K)$. The sequence S_n of partial sums of this series is a Cauchy sequence in $L^1(G/K)$ because for $m < n$

$$\left\|S_n - S_m\right\|_{L^1(G/K)} \le \sum_{j=m+1}^{n} \left\|f_j\right\|_{L^1(G/K)} \le \sum_{j=m+1}^{n} \left\|f_j\right\|_{H^1(G/K)}.$$

So the series $\sum_j f_j$ converges in $L^1(G/K)$ to some function f. On the other hand, each f_j has an atomic decomposition $f_j = \sum_i \alpha_{ij} \dot{a}_{ij}$ such that $\sum_i \left|\alpha_{ij}\right| < 2\left\|f_j\right\|_{H^1(G/K)}$. Then f has an atomic decomposition

$$f = \sum_j \sum_i \alpha_{ij} \dot{a}_{ij},$$

and

$$\sum_j \sum_i \left|\alpha_{ij}\right| < 2\sum_j \left\|f_j\right\|_{H^1(G/K)} < \infty.$$

Thus, $f \in H^1(G/K)$. Moreover,

$$\left\| f - \sum_{j=1}^{n} f_j \right\|_{H^1(G/K)} \le \sum_{j=n+1}^{\infty} \left\|f_j\right\|_{H^1(G/K)} \to \text{as } n \to \infty.$$

Finally, we shall show that the space $H^1(G/K)$ is non-trivial. It is enough to prove that non-trivial right K-invariant atoms exist. To this end, for an atom a on G let's consider the function

$$a'(x) := c\int_K a(xk)\,dk.$$

Then a' is right K-invariant and satisfies (i) for every constant $c > 0$. Indeed, if a is supported in a ball $B = B(e, r_B)$, then $a'(x) = 0$ for $x \notin BK$. Since

$$\rho(e, xk) \le \rho(e, x) + \rho(x, xk) = \rho(e, x) + \rho(e, k) < r_B + \operatorname{dist}(e, K) =: r'_B$$

for $x \in B$, the set BK is contained in a ball $B' = B(e, r'_B)$. Thus, a' is supported in B'. From now on we assume that $c = \nu(B)/\nu(B')$. Then (ii) holds for a' because $\|a'\|_\infty \le c\|a\|_\infty \le c/\nu(B) = 1/\nu(B')$ for such c. The property (iii) for a' follows from the equality (Δ_G denotes the modular function)

$$\int_G a'(x)\,dx = c\int_G \int_K a(xk)\,dk\,dx = c\int_K \Delta_G(k) \int_G a(x)\,dx\,dk = 0.$$

So, a' is a right K-invariant atom. On the other hand, since $h \in H^1(G)$, we have an atomic decomposition $h = \sum_{j=1}^{\infty} \alpha_j a_j$. Since, by assumption, the function

$$x \mapsto \int_K h(xk)\,dk = \sum_{j=1}^{\infty} \alpha_j \int_K a_j(xk)\,dk$$

is non-trivial, the right K-invariant atom $\left(a_j\right)'$ is non-trivial for some j, as well. This completes the proof.

In the proof of theorem 2 the next lemmas play an important role.

Lemma 1 [8]. *Let (Ω, q, μ) be σ-compact quasi-metric space with quasi-metric q and positive Radon measure μ, (X, m) be a measure space and $\mathcal{F}(X)$ be some Banach space of m-measurable functions on X. Assume that the convergence of a sequence strongly in $\mathcal{F}(X)$ yields the convergence of some subsequence to the same function for m a. e. $x \in X$. Let $F(u, x)$ be a function such that $F(u, \cdot) \in \mathcal{F}(X)$ for μ a. e. $u \in \Omega$ and the map $u \mapsto F(u, \cdot) : \Omega \to \mathcal{F}(X)$ is Bochner integrable with respect to μ. Then for m a. e. $x \in X$ we have (below (B) means a Bochner integral)*

$$\left((B)\int_\Omega F(u, \cdot)\,d\mu(u)\right)(x) = \int_\Omega F(u, x)\,d\mu(u).$$

Lemma 2 [9]. *Every automorphism* $A \in \mathrm{Aut}(G)$ *is Lipschitz. Moreover, one can choose the Lipschitz constant to be*

$$L_A = \kappa_\rho \operatorname{mod} A,$$

where the constant κ_ρ *depends on the metric* ρ *only.*

Now we are in position to prove the following.

Theorem 2. *Under the assumptions of definition 2 let* (Ω, q, μ) *be* σ*-compact quasi-metric space with positive Radon measure* μ *and* $\Phi \in L^1(k^s\mu)$ *where* $k(u) := \kappa_\rho / \operatorname{mod} A(u)$. *Then the operator* $\mathcal{H}_{\Phi, \dot{A}}$ *is bounded on the space* $H^1(G/K)$ *and*

$$\left\|\mathcal{H}_{\Phi, \dot{A}}\right\|_{\mathcal{L}\left(H^1(G/K)\right)} \leq C_\nu \|\Phi\|_{L^1\left(k^s\mu\right)}.$$

P r o o f. We proceed as in the proof of the main theorem in [8; 9]. If we set $X = G/K$ and $m = \lambda$ the pair (X, m) satisfies the conditions of lemma 1 with $H^1(G/K)$ in place of $\mathcal{F}(X)$. Indeed, let $f_n = \dot{g}_n \in H^1(G/K)$, $f = \dot{g} \in H^1(G/K)$, and $\|f_n - f\|_{H^1(G/K)} \to 0$ $(n \to \infty)$. Since

$$\|f_n - f\|_{L^1(G/K)} = \int\limits_{G/K} \left|\pi_K(g_n - g)\right| d\lambda =$$
$$= \int\limits_G \left|g_n(x) - g(x)\right| dx \leq \|g_n - g\|_{H^1(G)} = \|f_n - f\|_{H^1(G/K)} \to 0,$$

there is a subsequence of f_n that converges to f λ a. e. Then definition 3 and lemma 1 imply for $f \in H^1(G/K)$ that

$$\mathcal{H}_{\Phi, \dot{A}} f = \int_\Omega \Phi(u) f \circ \dot{A}(u) d\mu(u)$$

(the Bochner integral; $\circ$ stands for a composition sign). Therefore (below $f = \dot{g}$)

$$\left\|\mathcal{H}_{\Phi, \dot{A}} f\right\|_{H^1(G/K)} \leq \int\limits_\Omega |\Phi(u)| \left\|f \circ \dot{A}(u)\right\|_{H^1(G/K)} d\mu(u) =$$
$$= \int\limits_\Omega |\Phi(u)| \left\|g \circ A(u)\right\|_{H^1(G)} d\mu(u). \tag{5}$$

If g has representation (4) then

$$g \circ A(u) = \sum_{j=1}^{\infty} \alpha_j a_j \circ A(u). \tag{6}$$

We claim that $b_{j,u} := C_\nu^{-1} k(u)^{-s} a_j \circ A(u)$ is an atom, too. Indeed, lemma 2 implies that

$$A(u)^{-1}(B(x, r)) \subseteq B(x', k(u)r),$$

where $x' = A(u)^{-1}(x)$. If a_j is supported in $B(x_j, r_j)$ then $b_{j,u}$ is supported in $B(x_{j'}, k(u)r_j)$. So the condition (i) holds for $b_{j,u}$. Next, by (3) we have

$$\nu\left(B\left(x_{j'}, k(u)r_j\right)\right) \leq C_\nu k(u)^s \nu\left(B\left(x_j, r_j\right)\right).$$

This implies that

$$\left\|a_j \circ A(u)\right\|_\infty = \left\|a_j\right\|_\infty \leq \frac{1}{\nu\left(B\left(x_j, r_j\right)\right)} \leq C_\nu k(u)^s \frac{1}{\nu\left(B\left(x_{j'}, k(u)r_j\right)\right)}.$$

So, the condition (ii) is also fulfilled for $b_{j,u}$. The validity of (iii) follows from [11, chapter VII, subsection 1.4, formula (31)].

Finally, since formula (6) can be rewritten as

$$g \circ A(u) = \sum_{j=1}^{\infty} \left(C_\nu k(u)^s \alpha_j\right) b_{ju},$$

we get

$$\left\|g \circ A(u)\right\|_{H^1(G)} \le C_\nu k(u)^s \sum_{j=1}^{\infty} \left|\alpha_j\right|,$$

which in turn implies that

$$\left\|g \circ A(u)\right\|_{H^1(G)} \le C_\nu k(u)^s \left\|g\right\|_{H^1(G)} = C_\nu k(u)^s \left\|f\right\|_{H^1(G/K)}.$$

Thus, the statement of the theorem follows from formula (5).

Corollary 3. *Let the assumptions of theorem 2 holds. Then we have for a discrete Hausdorff operator over G/K (see corollary 1)*

$$\left\|\mathcal{H}_{\Phi,\dot{A}}\right\|_{\mathcal{L}\left(H^1(G/K)\right)} \le C_\nu \kappa_\rho^s \sum_{j=0}^{\infty} \frac{\left|\Phi(j)\right|}{\left(\operatorname{mod} A(j)\right)^s}.$$

Corollary 4. *Let the assumptions of theorem 2 holds. Then we have for a Cesaro operator over G/K*

$$\left\|\mathcal{C}_{A,\mu}\right\|_{\mathcal{L}\left(H^1(G/K)\right)} \le C_\nu \kappa_\rho^s \int\limits_{\{\operatorname{mod} A(u) \ge 1\}} \frac{d\mu(u)}{\left(\operatorname{mod} A(u)\right)^{1+s}}.$$

(Recall, that for this operator $\Phi = \chi_{\{\operatorname{mod} A(u) \ge 1\}} / \operatorname{mod} A$, see corollary 2.)

Concluding remarks

It would be of interest to apply theorems 1 and 2 to classical homogeneous spaces such as Euclidean plane $\mathbb{R}^2 = M(2)/O(2)$, sphere $S^2 = O(3)/O(2)$, non-Euclidean plane $\mathbb{H} = SU(1,1)/SO(2)$ [18, section 4], to other Riemannian symmetric spaces etc. Would also be of interest to generalize theorems 1 and 2 to the case when the group K is non-compact.

References

1. Hardy GH. *Divergent Series*. Oxford: Clarendon Press; 1949. 396 p.

2. Georgakis C. The Hausdorff mean of a Fourier – Stieltjes transform. *Proceedings of the American Mathematical Society*. 1992; 116:465–471. DOI: 10.1090/S0002-9939-1992-1096210-9.

3. Liflyand E, Moricz F. The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$. *Proceedings of the American Mathematical Society*. 2000;128:1391–1396. DOI: 10.1090/S0002-9939-99-05159-X.

4. Brown G, Moricz F. Multivariate Hausdorff operators on the spaces $L^p(\mathbb{R}^n)$. *Journal of Mathematical Analysis and Applications*. 2002;271(2):443–454. DOI: 10.1016/S0022-247X(02)00128-2.

5. Lerner A, Liflyand E. Multidimensional Hausdorff operators on the real Hardy space. *Journal of the Australian Mathematical Society*. 2007;83(1):79–86. DOI: 10.1017/S1446788700036399.

6. Liflyand E. Hausdorff operators on Hardy spaces. *Eurasian Mathematical Journal*. 2013;4(4):101–141.

7. Chen J, Fan D, Wang S. Hausdorff operators on Euclidean space. *Applied Mathematics. A Journal of Chinese Universities. Series B*. 2013;28(4):548–564. DOI: 10.1007/s11766-013-3228-1.

8. Mirotin AR. Boundedness of Hausdorff operators on Hardy spaces H^1 over locally compact groups. *Journal of Mathematical Analysis and Applications*. 2019;473(1):519–533. DOI: 10.1016/j.jmaa.2018.12.065.

9. Mirotin AR. Addendum to «Boundedness of Hausdorff operators on real Hardy spaces H^1 over locally compact groups» [J. Math. Anal. Appl. 473(2019)519–533]. *Journal of Mathematical Analysis and Applications*. 2019;479(1):872–874. DOI: 10.1016/j.jmaa.2019.06.055.

10. Hewitt E, Ross K. *Abstract Harminic Analysis. Volume 1*. Berlin: Springer Verlag; 1963. 519 p. (Grundlehren der mathematischen Wissenschaften; volume 115). DOI: 10.1017/S0013091500025761.

11. Bourbaki N. *Elements de mathematique. Integration. Chapitres 7 et 8. Volume 29*. Paris: Hermann; 1963. 222 p.

12. Mirotin AR. The structure of normal Hausdorff operators on Lebesgue spaces. *Forum Mathematicum*. 2020;32(1):111–119. DOI: 10.1515/forum-2019-0097.

13. Mirotin AR. On the structure of normal Hausdorff operators on Lebesgue spaces. *Functional Analysis and Its Applications*. 2019;53(4):261–269. DOI: 10.1134/S0016266319040038.

14. Coifman RR, Weiss G. Extensions of Hardy spaces and their use in analysis. *Bulletin of the American Mathematical Society*. 1977;83(4):569–645. DOI: 10.1090/S0002-9904-1977-14325-5.

15. Heinonen J, Koskela P, Shanmugalingam N, Tyson JT. *Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients*. Cambridge: Cambridge University Press; 2015. 434 p. DOI: 10.1017/CBO9781316135914.

16. Mirotin AR. On the general form of linear functionals on the Hardy spaces H^1 over compact Abelian groups and some of its applications. *Indagationes Mathematicae*. 2017;28(2):451–462. DOI: 10.1016/j.indag.2016.11.023.

17. Kawazoe T. Atomic Hardy spaces on semisimple Lie groups. *Japanese Journal of Mathematics*. 1985;11(2):293–343. DOI: 10.4099/math1924.11.293.

18. Helgason S. *Topics in harmonic analysis on homogeneous spaces*. Boston: Birkhauser; 1981. 142 p. (Progress in Mathematics; book 13). DOI: 10.1002/zamm.19820620921.

Received by editorial board 11.06.2020.