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О ХОЛЛОВЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

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ON HALL SUBGROUPS OF FINITE GROUPS

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Пусть H – подгруппа конечной группы G. Будем говорить, что подгруппа H τ-квазинормальна в G, если H перестановочна с каждой силовской подгруппой Q из G, такой что $(|H|,|Q|)=1$ и $(|H|,|Q^G|)\neq 1$. В данной работе получено обобщение теоремы Шура-Цассенхауза в терминах τ-квазинормальных подгрупп.

Ключевые слова: *τ-квазинормальная подгруппа, силовская подгруппа, холлова подгруппа, разрешимая группа.*

Let G be a finite group and H a subgroup of G. Then H is said to be τ-quasinormal in G if H permutes with all Sylow subgroups Q of G such that $(|H|,|Q|)=1$ and $(|H|,|Q^G|)\neq 1$. A generalization of Schur-Zassenhaus Theorem in terms of τ-quasinormal subgroups is obtained.

Keywords: *τ-quasinormal subgroup, Sylow subgroup, Hall subgroup, soluble group.*

1 Definitions and preliminary results

Throughout this paper, all groups are finite.

Let H be a subgroup of a group G. Then $\pi(G)$ denotes the set of all primes dividing $|G|$, H^G denotes the normal closure of H in G, that is, the intersection of all normal subgroups of G containing H. Recall that a subgroup A of G is said to permute with a subgroup B if $AB=BA$. A subgroup H of G is said to be $\pi(G)$*-permutable* or $\pi(G)$*-quasinormal* in G (O. Kegel, [9]) if H permutes with all Sylow subgroups of G. A subgroup H of a group G is said to be τ*-quasinormal* in G [10] if H permutes with all Sylow subgroups Q of G such that $(|H|,|Q|)=1$ and $(|H|,|Q^G|)\neq 1$. It is clear that every $\pi(G)$-quasinormal subgroup is τ-quasinormal. The Example 1.2 in [10] shows that the converse does not hold in general.

By Sylow's Theorem a group G possesses a Sylow p-subgroup for any prime p and any two Sylow p-subgroups of G are conjugate in G. For an arbitrary set of primes π a group G may or may not possess a Hall π-subgroup and, if it does, it may or may not be true that any two of them are conjugate in G. The Schur-Zassenhaus Theorem gives an important sufficient condition for the existence and conjugacy of Hall subgroups: *If a group G has a normal Hall π-subgroup A, then G is an $E_{\pi'}$-group* (*i. e. G has a Hall π'-subgroup*). *In addition, if either A or G/A is soluble, then G is a $C_{\pi'}$-group* (*i. e. any two Hall π'-subgroups of G are conjugate in G*). Naturally, one can ask whether or not the conclusion of the Schur-Zassenhaus Theorem holds if the Hall subgroup of G is not normal. In other words, can we weaken the condition of normality for the Hall subgroup of G so that the Schur-Zassenhaus Theorem still holds? The results of [5]–[7] give the positive answer to this question. In this paper, we prove the following generalization of these results.

***Theorem* 1.1.** *Let $G=AT$, where A is a Hall π-subgroup of a group G. If A is τ-quasinormal in G and either A is soluble or every π'-subgroup of T is soluble, then T contains a complement of A in G and any two complements of A in G are conjugate.*

Notice that the well known Feit-Thompson Theorem 1.1 of groups of odd order is not used in the proof of the Theorem.

The reader is referred to [1], [3] and [4] for all unexplained terminology and definitions if necessary.

2 Proof of the main result

Before continuing, we will need to know a few facts about τ-quasinormal subgroups.

***Lemma* 2.1.** *Let G be a group, $H\leq K\leq G$ and $L\leq G$.*

(1) *If H is τ-quasinormal in G, then H is τ-quasinormal in K.*

(2) *Suppose that H is normal in G. Then EH/H is τ-quasinormal in G/H for every τ-quasinormal subgroup E in G satisfying $(|H|,|E|)=1$.*

(3) *Suppose that H is τ-quasinormal in G. If $HL=LH$ and $\pi(H\cap L)=\pi(H)\cap\pi(L)$, then $H\cap L$ is τ-quasinormal in L.*

Proof. (1) and (2) See [10].

(3) Let Q be a Sylow *q*-subgroup of L such that $q \notin \pi(H \cap L)$ and $(|H \cap L|, |Q^L|) \neq 1$. Then $q \notin \pi(H)$, so Q is a Sylow q-subgroup of HL. Besides, $Q^L \leq Q^{HL} \cap L \leq Q^{HL}$. Then

$$(|H|, |Q^{HL}|) \neq 1.$$

Since H is τ-quasinormal in G, we have that H is τ-quasinormal in HL by (1). Hence $HQ = QH$. Therefore, $Q(H \cap L) = QH \cap L = (H \cap L)Q$.

Proof of Theorem 1.1. Assume that the theorem is false and let G be a counterexample of minimal order. Then a Hall π'-subgroup of G is not a Sylow subgroup of G. We proceed the proof by proving the following claims.

(1) $O_{\pi'}(G) = 1$.

Suppose that $Y = O_{\pi'}(G) \neq 1$. We show that the hypothesis still holds on G/Y. Clearly, $Y \leq T$ and $G/Y = (AY/Y)(T/Y)$, where $AY/Y \simeq A$ is a Hall π-subgroup of G/Y. In view of Lemma 2.1(2), AY/Y is τ-quasinormal in G/Y. If A is soluble, then AY/Y is soluble. If every π'-subgroup of T is soluble, then, clearly, every π'-subgroup of T/Y is soluble. Thus the hypothesis still holds for $(AY/Y, T/Y)$ in G/Y. Since $|G/Y| < |G|$, G/Y is a $C_{\pi'}$-group and T/Y contains a complement V/Y of AY/Y in G/Y by the choice of G. It is clear that V/Y is a Hall π'-subgroup of G/Y. Hence V is a Hall π'-subgroup of G, so $G = AV$. Now let T_1 and T_2 be Hall π'-subgroups of G. Then $T_1/Y = (T_2/Y)^{xY} = T_2^x/Y$ for some $x \in G$, so T_1 and T_2 are conjugate in G. Therefore the conclusion of the theorem is true for G, which contradicts to the choice of G. Hence we have (1).

(2) *A permutes with every Sylow q-subgroup of G, where $q \in \pi'$.*

Let Q be any Sylow *q*-subgroup of G for some prime $q \in \pi'$. In view of (1), $(|A|, |Q^G|) \neq 1$. Then $AQ = QA$ by hypothesis.

(3) $O_{\pi}(G) = 1$.

Suppose that $R = O_{\pi}(G) \neq 1$. We show that the hypothesis still holds on G/R. Clearly, $R \leq A$ and $G/R = (A/R)(TR/R)$, where A/R is a Hall π-subgroup of G/R. If A is soluble, then A/R is soluble. Suppose that every π'-subgroup of T is soluble. Let V/R be any π'-subgroup of TR/R. Then $V = V \cap TR = (V \cap T)R$. Hence

$$V/R = (V \cap T)R/R \simeq (V \cap T)/(R \cap V \cap T)$$

is a π'-group. Since R is a π-group, it follows from the Schur-Zassenhaus Theorem that

$$V \cap T = (R \cap V \cap T) \rtimes E$$

for some Hall π'-subgroup E of $V \cap T$. Since by hypothesis E is soluble, V/R is soluble. Thus every π'-subgroup of TR/R is soluble. Now let Q/R be a Sylow *q*-subgroup of G/R, where $q \in \pi'$. Then for some Sylow *q*-subgroup G_q of G we have $Q = G_qR$. In view of (2), $AG_q = G_qA$. Then

$$(A/R)(Q/R) = (A/R)(G_qR/R) = (Q/R)(A/R),$$

so A/R permutes with every Sylow q-subgroup of G/R, where $q \in \pi'$. Hence A/R is τ-quasinormal in G/R. Therefore the hypothesis still holds for $(A/R, TR/R)$ in G/R. Since $|G/R| < |G|$, G/R is a $C_{\pi'}$-group and TR/R contains a complement $V/R \simeq (V \cap T)/(R \cap V \cap T)$ of A/R in G/R by the choice of G. Then $V \cap T = (R \cap V \cap T) \rtimes E$ for some Hall π'-subgroup E of $V \cap T$. It is clear that V/R is a Hall π'-subgroup of G/R. Hence E is a Hall π'-subgroup of G, so $G = AE$. Now let T_1 and T_2 be Hall π'-subgroups of G. Then $T_1R/R = T_2^xR/R$ for some $x \in G$. Therefore by the Schur-Zassenhaus Theorem, T_1 and T_2 are conjugate in T_1R. Hence the conclusion of the theorem is true for G, which contradicts to the choice of G. Thus we have (3).

(4) G *is not a simple group.*

Assume that G is a simple non-abelian group. Let Q be any Sylow *q*-subgroup of G for some prime $q \in \pi'$. Then by (2), $AQ^x = Q^xA$ for all $x \in G$. Besides, $AQ^x \neq G$ by the choice of G. Hence G is not simple by [8, Theorem 3]. This contradiction completes the proof of (4).

(5) *If $D \neq 1$ is a proper normal subgroup of G, then D is a $C_{\pi'}$-group. If, in addition, $T = G$, then $A \leq D$.*

First we show that D is a $C_{\pi'}$-group. In view of (1) and (3), $1 \neq A \cap D \neq D$. It is clear that $A \cap D$ is a Hall π-subgroup of D. If A is soluble, then $A \cap D$ is soluble. Suppose that every π'-subgroup of T is soluble. Since every π'-subgroup of D is contained in $T \cap D$, then every π'-subgroup of D is soluble. Besides, by Lemma 2.1(3), $A \cap D$ is τ-quasinormal in D. Thus D is a $C_{\pi'}$-group by the choice of G. Now suppose that $T = G$ and $A \not\subseteq D$. Let $N = N_G(A \cap D)$. In view of (3), $N \neq G$. Since D is normal in G and $G = DN$ by the Frattini Argument, we have that $A < N$. Then by Lemma 2.1 (1), A is τ-quasinormal in N. Besides, since every π'-subgroup of N is a π'-subgroup of $T = G$, we have that the hypothesis still holds for (A, N) in N. Since $|N| < |G|$, N is a $C_{\pi'}$-group and N contains a complement $N_{\pi'}$ of A in N by the choice of G. It is clear that $DN_{\pi'} \neq G$. Since $ADN_{\pi'} = DAN_{\pi'} = DN = DN_{\pi'}A$ and

$$A \cap DN_{\pi'} = (A \cap D)(A \cap N_{\pi'}) = A \cap D$$

is a Hall π-subgroup of $DN_{\pi'}$, then $A \cap D$ is τ-quasinormal in $DN_{\pi'}$ by Lemma 2.1(3). Hence $DN_{\pi'}$ is a $C_{\pi'}$-group by the choice of G, in particular, $DN_{\pi'}$ has a Hall π'-subgroup, which, evidently, is a Hall π'-subgroup of G. Now let T_1 and T_2 be Hall π'-subgroups of G. Then $D_1 = T_1 \cap D$ and $D_2 = T_2 \cap D$ are Hall π'-subgroups of D. Since D is a $C_{\pi'}$-group, D_1 and D_2 are conjugate in D. Hence $N_G(D_1) = N_G(D_2)^x$ for some $x \in G$. Since $T_1 \leq N_G(D_1)$ and $T_2 \leq N_G(D_2)$, T_1 is conjugate with some Hall π'-subgroup of $N_G(D_2)$. Hence T_1 and T_2 are conjugate in G, so the conclusion of the theorem is true for G, which contradicts to the choice of G. Therefore we have (5).

(6) *T has a Hall π'-subgroup.*

Suppose that this is false. Then $A \cap T \neq 1$. Since A is a Hall π-subgroup of G, $A \cap T$ is a Hall π-subgroup of T. By Lemma 2.1(3), $A \cap T$ is τ-quasinormal in T. Hence the hypothesis still holds for $(A \cap T, T)$ in T. If $T \neq G$, then T is a $C_{\pi'}$-group by the choice of G, in particular, T has a Hall π'-subgroup, which, evidently, is a Hall π'-subgroup of G, a contradiction. Now suppose that $T = G$. Then in view of (4) and (5), G has a proper normal $C_{\pi'}$-subgroup D such that $A < D$. Let $D_{\pi'}$ be a Hall π'-subgroup of D and $N = N_G(D_{\pi'})$. Then $D = AD_{\pi'}$ and by the Frattini Argument, $G = DN = AN$. If $A \cap N = 1$, then N is a Hall π'-subgroup of $T = G$, a contradiction. Suppose that $A \cap N \neq 1$. Since A is a Hall π-subgroup of G, $A \cap N$ is a Hall π-subgroup of N. In view of Lemma 2.1(3), $A \cap N$ is τ-quasinormal in N. By (1), $N \neq G$. Hence N is a $C_{\pi'}$-group by the choice of G, in particular, N has a Hall π'-subgroup, which, evidently, is a Hall π'-subgroup of $G = T$, a contradiction. Thus we have (6).

In view of (6), we may assume without loss that T is a Hall π'-subgroup of G. Hence similarly as in the proof of (1) and (3) we obtain that

(7) *If $D \neq 1$ is a proper normal subgroup of G, then G/D is a $C_{\pi'}$-group.*

Final contradiction.

In view of (4), G has a proper normal subgroup $D \neq 1$. By (5) and (7) both groups D and G/D are $C_{\pi'}$-groups. Hence G is a $C_{\pi'}$-group by [2]. This contradiction completes the proof of the theorem.

Note that the $G = A_5$ shows that under conditions of Theorem 1.1 the subgroup A is not necessarily normal in G and G is not necessarily π-soluble or π'-soluble, where π is the set of all primes dividing $|A|$.

***Corollary* 2.1.** *If every Sylow subgroup of a group G is τ-quasinormal in G, then G is soluble.*

Proof. In view of Theorem 1.1, every Sylow subgroup of a group G has a complement in G. Then by [3], G is soluble.

The example of the symmetric group $G = S_4$ shows that under conditions of Corollary 2.1, the group G is not necessarily supersoluble, p-closed or p-nilpotent for any prime $p \in \pi(G)$.

***Corollary* 2.2.** *If every Hall subgroup of a group G is τ-quasinormal in G, then G is soluble.*

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