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О σ-СВОЙСТВАХ КОНЕЧНЫХ ГРУПП II

А.Н. Скиба

Гомельский государственный университет им. Ф. Скорины, *Гомель*, *Беларусь*

ON σ-PROPERTIES OF FINITE GROUPS II

A.N. Skiba

F. Scorina Gomel State University, *Gomel*, *Belarus*

Пусть G – конечная группа, $\sigma = \{\sigma_i \mid i \in I\}$ – некоторое разбиение множества всех простых чисел $\mathbb{P}$ и Π – подмножество множества σ. Множество $\mathcal{H}$ подгрупп из G называется *полным холловым* Π *-множеством* в G, если $\mathcal{H}$ содержит в точности одну холлову σ_i -подгруппу из G для каждого такого $\sigma_i \in \Pi$, что $\sigma_i \cap \pi(G) \neq \varnothing$. Мы также говорим, что G является: Π *-полной*, если G обладает *полным холловым* Π *-множеством*; Π *-полной группой силовского типа*, если для всякого $\sigma_i \in \Pi$ каждая подгруппа E группы G является D_{σ_i} -группой, т. е. E содержит холлову σ_i -подгруппу H и каждая σ_i -подгруппа из E содержится в некоторой сопряженной с H подгруппой H^x $(x \in E)$. В данной работе мы исследуем свойства конечных Π -полных групп. Работа продолжает исследования статьи [1].

Ключевые слова: *конечная группа*, Π *-полная группа*, *σ-разрешимая группа*, *σ-нильпотентная группа*, *σ-квазинильпотентная группа.*

Let G be a finite group, $\sigma = \{\sigma_i \mid i \in I\}$ some partition of the set $\mathbb{P}$ of all primes and Π a subset of the set σ. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* Π *-set* of G if $\mathcal{H}$ contains exact one Hall σ_i -subgroup of G for every $\sigma_i \in \Pi$ such that $\sigma_i \cap \pi(G) \neq \varnothing$. We say also that G is: Π *-full* if G possess a *complete Hall* Π *-set*; a Π *-full group of Sylow type* if for each $\sigma_i \in \Pi$, every subgroup E of G is a D_{σ_i} -group, that is, E has a Hall σ_i -subgroup H and every σ_i -subgroup of E is contained in some conjugate of H^x $(x \in E)$. In this paper we study properties of finite Π -full groups. The work continues the research of the paper [1].

Keywords: *finite group*, Π *-full group*, *σ-soluble group*, *σ-nilpotent group*, *σ-quasinilpotent group.*

1 Basic concepts

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq P$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order $|G|$ of G.

In what follows, $\sigma = \{\sigma_i \mid i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$; Π is always supposed to be a subset of the set σ and $\Pi' = \sigma \setminus \Pi$.

We put $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$, $\sigma(G) = \sigma(|G|)$, and we say that G is: σ-*primary* if either $G = 1$ or $|\sigma(G)| = 1$; σ-*biprimary* if $|\sigma(G)| = 2$; a Π *-group* if $\sigma(G) \subseteq \Pi$.

A chief factor H/K of G is said to be σ-*central* (in G) if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is σ-primary, otherwise it is called σ-*eccentric* (in G).

***Definition* 1.1.** We say that G is: (i) σ-*soluble* [2] if every chief factor of G is σ-primary; (ii) σ-*nilpotent* if every chief factor of G is σ-central.

Example 1.2. (i) Every σ-nilpotent group is also σ-soluble, and G is σ-soluble if and only if it is σ_i -separable for all $i \in I$; G is soluble (respectively nilpotent) if and only if it is σ-soluble (respectively σ-nilpotent), where $\sigma = \{\{2\}, \{3\}, \{5\}, ...\}$ is the *smallest* partition of $\mathbb{P}$, that is, for any $i \in I$, σ_i is a one-element set.

(ii) G is π -separable if and only if it is σ-soluble, where $\sigma = \{\pi, \pi'\}$.

(iii) Let $\pi = \{p_1, ..., p_t\}$. Then G is π -soluble if and only if it is σ-soluble, where $\sigma = \{\{p_1\}, ..., \{p_t\}, \pi'\}$.

(iv) In view of Theorem 4.1 in [1], $G \neq 1$ is σ-nilpotent if and only if $G = O_{\sigma_1}(G) \times \cdots \times O_{\sigma_t}(G)$, where $\{\sigma_1, ..., \sigma_t\} = \sigma(G)$.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* Π *-set* of G if $\mathcal{H}$ contains exact one Hall σ_i -subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$.

***Definition* 1.3.** We say that G is:

(i) Π *-full* if G possess a *complete Hall* Π *-set*;

(ii) a D_Π *-group* if for each $\sigma_i \in \Pi$, G is a D_{σ_i} -group.

(iii) a Π*-full group of Sylow type* if every subgroup of *G* is a D_Π-group.

Example **1.4.** (i) If *G* is π-soluble, then *G* is a Π-full group of Sylow type for each $\Pi \subseteq \sigma$ such that $\cup_{\sigma_i \in \Pi} \sigma_i \subseteq \pi$. In particular, if *G* is soluble, then *G* is a σ-full group of Sylow type for every partition σ of $\mathbb{P}$.

(ii) In more general case, we say that *G* is Π*-soluble* if every chief factor of *G* is either a Π'-group or a σ_i-group for some $\sigma_i \in \Pi$. If $\Pi \cap \sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$, $\pi = \sigma_1 \cup \cdots \cup \sigma_t$ and $\sigma^* = \{\sigma_1, \ldots, \sigma_t, \pi'\}$, then *G* is Π-soluble if and only if *G* is σ^*-soluble. Therefore, in view of Theorem B in [3], a Π-soluble group is a Π-full group of Sylow type.

(iii) Let $G = A \times B$, where $A = Ly$ is the Lyons group and B is a group of prime order $p > 67$. Let $\Pi = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{11, 67\}$ and $\sigma_2 = \{2, p\}$. Then *G* is a Π-full group of Sylow type. It is also clear that *G* is not Π-soluble.

(iv) In view of Example 1.2 (iv), every σ-nilpotent group $G \neq 1$ is σ-full, and if $\sigma(G) = \{\sigma_1, \ldots, \sigma_t\}$, then $\{O_{\sigma_1}(G), \ldots, O_{\sigma_t}(G)\}$ is the unique complete Hall σ-set of *G*.

Recall that a group *G* is said to be *quasinilpotent* if for every its chief factor H/K and every $x \in G$, x induces an inner automorphism on H/K [4, X, Definition 13.2]. Note that since for every central chief factor H/K of *G*, an element of *G* induces the trivial automorphism on H/K, one can say that a group *G* is quasinilpotent if for every its *eccentric* chief factor H/K and for every $x \in G$, x induces an inner automorphism on H/K. This elementary observation allows us to consider the following analogue of quasinilpotency:

Definition **1.5.** We say that *G* is *σ-quasinilpotent* if for every σ-eccentric chief factor H/K of *G*, every automorphism of H/K induced by an element of *G* is inner.

Example **1.6.** (i) *G* is quasinilpotent if and only if it is σ-quasinilpotent, where σ is the smallest partition of $\mathbb{P}$.

(ii) Let $G = (A_5 \wr A_5) \times (A_7 \times A_{11})$ and $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$. Then *G* is σ-quasinilpotent but *G* is neither σ-nilpotent nor quasinilpotent.

We use $G^{\mathfrak{N}_\sigma}$ to denote the *σ-nilpotent residual* of *G*, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Definition **1.7.** (i) The product of all normal respectively σ-soluble, σ-nilpotent, σ-quasinilpotent subgroups of *G* is said to be respectively the *σ-radical*, the *σ-Fitting subgroup*, the *generalized σ-Fitting subgroup* of *G* and we denote it respectively by $R_\sigma(G)$, $F_\sigma(G)$, $F^*_\sigma(G)$.

(ii) We use $E_\sigma(G)$ to denote the σ-nilpotent residual of $F^*_\sigma(G)$, and we say that $E_\sigma(G)$ is the *σ-layer of G*.

Remark **1.8.** It is clear that $F(G) \leq F_\sigma(G)$ and $F^*(G) \leq F^*_\sigma(G)$. Moreover, if σ is the *smallest* partition of $\mathbb{P}$, then $F_\sigma(G) = F(G)$ and $F^*_\sigma(G) = F^*(G)$ is the generalized Fitting subgroup of *G*. Note also that, in view of Example 1.2 (iv), $F_\sigma(G) = O_{\sigma_1}(G) \times \cdots \times O_{\sigma_t}(G)$, where $\{\sigma_1, \ldots, \sigma_t\} = \sigma(G)$.

Example **1.9.** Let $G = (A_5 \times A_7) \wr \langle x \rangle = K \rtimes \langle x \rangle$, where $|x| = p > 5$ is a prime and K is the base group of the regular wreath product G. Let $R = A_5^\natural$ and $L = A_7^\natural$ (we use here the terminology in [5, Ch. A]). Finally, let $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$. Then $K = R \times L$ and so, in view of Example 1.2 (iv), $F_\sigma(G) = R$. It is clear also that $K \leq F^*_\sigma(G)$ and the automorphism of R induced by x is not inner. Hence $F^*_\sigma(G) = K$. It is also clear that $E_\sigma(G) = L$ and $E(G) = K$.

In Sections 2–4 we study properties and some applications of Π-full, σ-soluble, σ-nilpotent, and σ-quasinilpotent groups and, in particular, the relationship between the subgroups $F_\sigma(G)$, $F^*_\sigma(G)$ and $E_\sigma(G)$. In Section 5 we analyze some applications of the results in Sections 2–4 in the theory of permutable subgroups. Finally, in Section 6 we discuss some open questions.

2 Π-soluble groups

We use $\mathfrak{S}_\Pi$ to denote the class of all Π-soluble groups.

The direct calculations show that the following properties of Π-soluble groups are true.

Proposition **2.1.** (i) *The class* $\mathfrak{S}_\Pi$ *is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, any extension of the* Π*-soluble group by a* Π*-soluble group is a* Π*-soluble group as well.*

(ii) $\mathfrak{S}_\Pi \subseteq \mathfrak{S}_{\Pi^*}$ *for any partition* $\sigma^* = \{\sigma^*_j \mid j \in J\}$ *of* $\mathbb{P}$ *such that* $J \subseteq I$ *and* $\sigma_j \subseteq \sigma^*_j$ *for all* $j \in J$ *and for* $\Pi^* \subseteq \sigma^*$ *such that*

$$\cup_{\sigma^*_j \in \Pi^*} \sigma^*_j = \cup_{\sigma_i \in \Pi} \sigma_i .$$

Proposition **2.2.** *Let G be* Π*-soluble.*

(i) *If M is a maximal subgroup of G such that* $\sigma(|G:M|) \cap \Pi \neq \varnothing$, *then* $|G:M|$ *is* σ*-primary.*

(ii) *For every* $\sigma_i \in \sigma(G) \cap \Pi$, *G has a maximal subgroup* M *such that* $|G:M|$ *is a* σ_i*-number.*

Let A, B and R be subgroups of G. Then A is said to *R-permute* with B [6] if for some $x \in R$ we have $AB^x = B^x A$.

A subgroup H of G is said to be: a *Hall* Π*-subgroup* of G [2] if H is a Π-subgroup of G and $|G:H|$ is a Π'-number; a σ*-Hall subgroup* of G if H is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.

It is clear that every σ-Hall subgroup is also a Hall subgroup of the group. In the group $G = S_3 \times C_5$, S_3 is a Hall subgroup of G but it is not a σ-Hall subgroup of G, where $\sigma = \{\{3,5\},\{3,5\}'\}$.

If G has a complete Hall set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of type σ such that $H_iH_j = H_jH_i$ for all i, j, then $\mathcal{H}$ is said to be a σ*-basis* [3] of G.

By the classical Hall theorem, G is soluble if and only if it has a Sylow basis. The direct analogue of this result for σ-soluble groups is not true in general. Indeed, let $\sigma = \{\{2,3\},\{2,3\}'\}$. Then the alternating group A_5 of degree 5 has a σ-basis and it is not σ-soluble. Nevertheless, the following generalizations of the Hall result are true.

***Theorem* 2.3** (Skiba [3]). *Let $R = R_\sigma(G)$ be the σ-radical of G. Then any two of the following conditions are equivalent:*

(i) *G is σ-soluble.*

(ii) *For any* Π, *G has a Hall* Π*-subgroup and every σ-Hall subgroup of G R-permutes with every Sylow subgroup of G.*

(iii) *G has a σ-basis $\{H_1, \ldots, H_t\}$ such that for each $i \neq j$ every Sylow subgroup of H_i R-permutes with every Sylow subgroup of H_j.*

***Theorem* 2.4** (Skiba [3]). *Let $R = R_\sigma(G)$ be the σ-radical of G. Then G is σ-soluble if and only if for any* Π *the following hold: G has a Hall* Π*-subgroup E, every* Π*-subgroup of G is contained in some conjugate of E and E R-permutes with every Sylow subgroup of G.*

Recall that $G^{\mathfrak{N}}$ is the nilpotent residual of G, that is, the smallest normal subgroup of G with nilpotent quotient.

As one of the steps in the proof of Theorem 2.3, the following useful fact can be used.

***Proposition* 2.5** (Skiba [7]). *Suppose that $G = A_1A_2 = A_2A_3 = A_1A_3$, where A_1, A_2 and A_3 are σ-soluble subgroups of G. If the three indices $|G:N_G(A_1^{\mathfrak{N}})|$, $|G:N_G(A_2^{\mathfrak{N}})|$, $|G:N_G(A_3^{\mathfrak{N}})|$ are pairwise σ-coprime, then G is σ-soluble.*

From Theorems 2.3 and 2.4 we get the following characterizations of the π-separable groups.

***Corollary* 2.6.** *Let R be the product of all normal π-separable subgroups of G. Then G is π-separable if and only if $G = AB$, where A and B are a Hall π-subgroup and a Hall π'-subgroup of G, respectively, and every Sylow subgroup of A R-permutes with every Sylow subgroup of B.*

***Corollary* 2.7.** *Let R be the product of all normal π-separable subgroups of G. Then G is π-separable if and only if $G = AB$, where A and B are a Hall π-subgroup and a Hall π'-subgroup of G, respectively, and every Sylow subgroup of G R-permutes with A and with B.*

Now we give a characterization of σ-soluble groups in the terms of the normalizers of Sylow subgroups.

***Theorem* 2.8** (Skiba [3]). *Let G be a σ-full group and $\mathcal{H} = \{H_1, \ldots, H_t\}$ a complete Hall σ-set of G. Then any two of the following conditions are equivalent:*

(i) *G is σ-soluble.*

(ii) *Every σ-biprimary subgroup of G is σ-soluble and for every chief factor H/K of G and every $A \in \mathcal{H}$ the number $|G:N_G((A \cap H)K)|$ is σ-primary.*

(iii) *Every σ-biprimary subgroup of G is σ-soluble and for any $k \in \{1, \ldots, t\}$ there is a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ of G such that the number $|G:N_G((H_k \cap G_i)G_{i-1})|$ is σ-primary for all $i = 1, \ldots, n$.*

***Definition* 2.9** (sf. [8]). If G has a complete Hall Π-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, where H_i is nilpotent (respectively supersoluble) for all $i = 1, \ldots, t$, then we say that $\mathcal{H}$ is a *Wielandt* Π*-set* (respectively a *generalized Wielandt* Π*-set*) of G.

***Example* 2.10.** (i) If σ is the smallest partition of $\mathbb{P}$, then every complete Hall σ-set of G is clearly a Wielandt σ-set of G.

(ii) Let $\sigma = \{\sigma_i \mid i \in I\}$ is such that $\sigma_1 = \{5, 11\}$ and σ_i is a one-element set for all $i \neq 1$. Then the group $PSL(2,11)$ possess a generalized complete Wielandt σ-set, and it does not possess a complete Wielandt Π-set for every Π containing σ_1.

***Corollary* 2.11.** *Suppose that G has a complete Wielandt set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of type σ. Then any two of the following conditions are equivalent:*

(i) *G is soluble.*

(ii) *For every chief factor H/K of G and every $A \in H$ the number $|G:N_G((A \cap H)K)|$ is σ-primary.*

(iii) *For any $k \in \{1, \ldots, t\}$ there is a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ of G such that the number $|G:N_G((H_k \cap G_i)G_{i-1})|$ is σ-primary for all $i = 1, \ldots, n$.*

We say that an integer n is *primary* if $n = p^m$ is a power of some prime p.

***Corollary* 2.12** (Guo and Skiba [8]). *Let $S = \{P_1, \ldots, P_t\}$ be a complete Sylow set of G. Then any two of the following conditions are equivalent:*

(i) *G is soluble.*

(ii) *For every chief factor H/K of G and every $P \in S$ the number $|G:N_G((P \cap H)K)|$ is primary.*

(iii) *For any* $k \in \{1,...,t\}$ *there is a normal series* $1 = G_0 \le G_1 \le \cdots \le G_n = G$ *of G such that the number* $|G : N_G((P_k \cap G_i)G_{i-1})|$ *is primary for all* $i = 1,...,n$.

***Corollary* 2.13.** *If for every Sylow subgroup P of G and for every chief factor* H/K *of G,* $|(G/K) : N_{G/K}((P \cap H)K/K)|$ *is a prime power, then G is soluble.*

From Corollary 2.13 we get the following known result.

Corollary 2.14 (See Zhang [9] or Guo [10]). *If for every Sylow subgroup P of G the number* $|G : N_G(P)|$ *is a prime power, then G is soluble.*

The σ-system normalizers of σ-soluble groups. If $\mathcal{H} = \{H_1,...,H_t\}$ is a σ-basis of *G* and $\mathcal{H}^* = \{Q_1,...,Q_t\}$, where $Q_i = H_1 ... H_{i-1}H_{i+1}...H_t$, then we say that $\mathcal{H}^*$ is a *Hall σ-system* of *G* (corresponding $\mathcal{H}$).

Now, let $\sigma(G) = \{\sigma_1,...,\sigma_t\}$ and Q_i be a Hall σ_i-subgroup of *G* (we say that Q_i is a σ_i*-complement* of *G*). Then $H_i = \cap_{j \ne i} Q_j$ is a Hall σ_i-subgroup of *G* and $\mathcal{H} = \{H_1,...,H_t\}$ is a σ-basis of *G* such that $\mathcal{H}^* = \{Q_1,...,Q_t\}$ is a Hall σ-system of *G* corresponding $\mathcal{H}$ (see [11, VI, Section 2]).

***Theorem* 2.15.** *If G is σ-soluble, then any two σ-basis of G are conjugate, as are any two Hall σ-systems.*

Proof. See the proof of Theorem 2.4 in [11, VI].

***Definition* 2.16.** Let $\mathcal{H} = \{H_1,...,H_t\}$ is a σ-basis of *G* and $\mathcal{H}^* = \{Q_1,...,Q_t\}$ is a Hall σ-system of *G* corresponding $\mathcal{H}$. Then

$$N = N_G(H_1) \cap \cdots \cap N_G(H_t) = N_G(Q_1) \cap \cdots \cap N_G(Q_t)$$

(see Section 11 in [11, VI]). We say that *N* is a *σ-system normalizer* of *G* (corresponding $\mathcal{H}$).

***Example* 2.17.** Let $p < q < r$ be primes, where p divides $q-1$. Let $A = Q \rtimes P$ be a non-abelian group of order *pq* and *B* a group of order *r*. Let $G = A \wr B = K \rtimes B$, where K is the base group of the regular wreath product *G*. Let $R = Q^\natural$ (we again use here the terminology in [5, A]) and $Z = \{(a,...,a) \in K \mid a \in A\}$. Then R is a minimal normal subgroup of *G* by [5, A, 18.5]. It is clear that $C_G(R) = R$ and $[Z, B] = 1$. Let $\sigma\{\{p,q\},\{p,q\}'\}$. Then $\{K, B\}$ is a σ-basis of *G*, $D = N_G(B)$ is a σ-system normalizer of *G* and $R/1$ is a σ-eccentric chief factor of *G*. Hence *D* does not cover $R/1$ by Theorem 2.19 below. It is also clear that $R \cap D \ne 1$. Hence *D* does not avoid $R/1$. Therefore in view of Theorem 3.2 in [5, V] and Corollary 3.4 below, a σ-system normalizer of a soluble group *G* in general is not a system $\mathfrak{N}_\sigma$-normalizer of *G*, where $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups, in the sense of Definition 1.2 in [5, V].

Nevertheless, the following result shows that the σ-system normalizers of a σ-soluble group partially inherits the properties of the system normalizers of a soluble group.

***Theorem* 2.18.** *Let G be σ-soluble and D a σ-system normalizer of G.*

(i) *Any σ-system normalizer of G is σ-nilpotent and any two are conjugate.*

(ii) *D covers every σ-central chief factor of G and it does not cover every σ-eccentric chief factor of G; D avoids every σ-eccentric chief p-factor* H/K *of G such that* $p \in \sigma_i$, *a Hall* σ_i*-subgroup of G is nilpotent and G is p-soluble.*

(iii) $D^G = G$ *and* $D_G = Z_\sigma(G)$.

Proof. (i) See the proof of Theorem 11.2 in [11, VI].

(ii) Let H/K be a chief factor of *G* and $C = C_G(H/K)$. Since *G* is σ-soluble by hypothesis, H/K is a σ_i-group for some $\sigma_i \in \sigma(G)$. Let the σ-system normalizer *D* of *G* arises from a σ-basis $\mathcal{H} = \{H_1,...,H_t\}$ of *G*. Without loss of generality we can assume that $i = 1$. Let $\pi = \sigma_1$ and $S = H_2 ... H_t$ of *G*. Then S is a σ_1-complement of G.

First assume that H/K is σ-central in *G*, that is, $(H/K) \rtimes (G/C)$ is a σ_1-group. Then G/C is a π'-group. Hence $S \le C$, which implies that $SK/K \le N_{G/K}(H/K)$. Hence

$$SH/K = (SK/K)(H/K) = (SK/K) \times (H/K).$$

Then SK is normal in SH and $|SH : SK|$ is a π-number. Applying the Frattini argument to the σ_1-complement S of SH, we have $SH = SKN_{SH}(S) \le N_G(S)K$. Therefore the normal π-subgroup H/K of G/K is contained in every Hall σ_1-subgroup of NK/K, where $N = N_G(S)$. Let $H_0 = H_1 \cap N$. Since $G = H_1S = H_1N$, $N = (H_1 \cap N)S = H_0S$. Therefore $|H_0| = |N : S|$, so H_0 is a Hall σ_1-subgroup of N. Now, let $\mathcal{H}^* = \{Q_1,...,Q_t\} = \{S, Q_2,...,Q_t\}$ be a Hall σ-system of *G* corresponding $\mathcal{H}$. Then $H_1 \le Q_2 \cap \cdots \cap Q_t$ and hence $H_0 \le N_G(Q_2) \cap \cdots \cap N_G(Q_t)$, so $H_0 \le D$ since $H_0 \le H_1$. Hence H_0K/K is a Hall σ_1-subgroup of NK/K, and so we have $H/K \le H_0K/K \le DK/K$. Hence $H = H \cap DK = K(H \cap D)$, so *D* covers H/K.

Now, suppose that H/K is σ-eccentric in *G*. Then H/K is σ-eccentric in G/K. Without loss of generality we can assume that $\sigma(G) = \sigma(G/K)$. Then $\mathcal{H}K/K = \{H_1K/K,...,H_tK/K\}$ is a σ-basis of G/K. Moreover, if D^*/K is a σ-system normalizer of G/K corresponding $\mathcal{H}K/K$, then $DK/K \le D^*/K$. If $K \ne 1$, then D^*/K does not cover (D^*/K avoids, respectively) H/K by induction and so DK/K does not cover (avoids, respectively) H/K. But then D does not cover (avoids, respectively) H/K.

Now assume that $K = 1$. Suppose that $H \le D$. Then $HH_i = H \times H_i$ for all $i > 1$, so G/C is a σ_1-group and hence H/K is σ-central in G. This contradiction completes the proof of the first assertion of (ii). Finally, suppose that H is a p-group, where $p \in \sigma_1$, a Hall σ_1-subgroup of G is nilpotent and G is p-soluble. Suppose that $D \cap H \ne 1$. Then $N_G(S) \cap R \ne 1$ and hence by Lemma 1.4 in [5, Ch. 5] we have $M \le C_G(H)$, which implies that G/C is a σ_1-group and hence $H/1$ is σ-central in G. This contradiction completes the proof of Assertion (ii).

(iii) Assume that $D^G < G$. Then, since G/D^G is σ-soluble, $G^{\mathfrak{N}_\sigma} D^G / D^G = (G/D^G)^{\mathfrak{N}_\sigma} < G/D^G$ and hence $G^{\mathfrak{N}_\sigma} D < G$, contrary to Assertion (ii). Hence $D^G = G$. The second assertion of the result is a corollary of Assertion (ii) and Proposition 3.5 (i) below. The theorem is proved.

***Corollary* 2.19.** *Assume that G has a σ-basis which is a Wielandt σ-set of G. Then a σ-system normalizer of G covers the σ-central chief factors of G and avoids the σ-eccentric chief factor of G.*

***Corollary* 2.20** (P. Hall). *A system normalizer of a soluble group G covers the central chief factors of G and avoids the eccentric chief factor of G.*

3 General properties of the σ-nilpotent and σ-quasinilpotent groups

Recall that a subgroup A of G is *σ-subnormal* in G [2] if there is a subgroup chain $A = A_0 \le \le A_1 \le \dots \le A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, n$.

The following theorem collects the main properties of σ-subnormal subgroups.

***Theorem* 3.1** (Skiba [2]). *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.*

(3) *If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K \rangle$ are σ-subnormal in G.*

(4) *AN/N is σ-subnormal in G/N.*

(5) *If $N \le K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

(6) *If $K \le A$ and A is σ-nilpotent, then K is σ-subnormal in G.*

(7) *If $H \ne 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \ne 1$ is a Hall Π-subgroup of A.*

(8) *If $|G:A|$ is a Π-number, then $O^\Pi(A) = O^\Pi(G)$.*

(9) *If N is a Π-group of G, then $N \le N_G(O^\Pi(A))$.*

(10) *If A is a σ-Hall subgroup of G, then A is normal in G.*

(11) *If G is a σ-group and A is σ-nilpotent, then A is contained in $F_\sigma(G)$.*

In this theorem $O^\Pi(G)$ denotes the subgroup of G generated by all its Π'-subgroups. Instead of $O^{\{\sigma_i\}}(G)$ we write $O^{\sigma_i}(G)$.

Before continuing, let's consider the following elementary example.

***Example* 3.2.** Let p, q, r be different primes, where q divide $p-1$. Let $P \rtimes Q$ be a non-abelian group of order pq and R a group of order r. Let $G = (P \rtimes Q) \wr R$ be the regular wreath product of the group $P \rtimes Q$ with R and $H = Q^\natural$. If $\sigma = \{\{p, q\}, \{p, q\}'\}$, then the subgroup H is σ-subnormal in G by Theorem 3.1(6) but it is not subnormal in G.

The following result indicates the importance of the concept of σ-subnormality.

***Theorem* 3.3.** *Any two of the following conditions are equivalent:*

(i) *G is σ-nilpotent.*

(ii) *Every chief factor of G is σ-central.*

(iii) *G has a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that $G = H_1 \times \dots \times H_t$.*

(iv) *G has a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that every member of H is σ-subnormal in G.*

(v) *Every subgroup of G is σ-subnormal in G.*

(vi) *Every maximal subgroup of G is σ-subnormal in G.*

Proof. See the proof of Theorem 4.1 in [1].

We use $\mathfrak{N}_\sigma$ and $\mathfrak{N}_\sigma^*$ to denote the classes of all σ-nilpotent groups and of all σ-quasinilpotent groups, respectively.

***Corollary* 3.4.** *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/\Phi(G) \cap E$ is σ-nilpotent, then E is σ-nilpotent.*

A normal subgroup E of G is said to be *σ-hypercentral* (in G) if either $E = 1$ or every chief factor of G below E is σ-central (in G). We use $Z_\sigma(G)$ to denote the product of all normal σ-hypercentral subgroups of G. It is not difficult to show (see Proposition 3.5 (i) below) that $Z_\sigma(G)$ is also σ-hypercentral in G. We call the subgroup $Z_\sigma(G)$ the *σ-hypercentre* of G.

The next proposition collects the main properties of the σ-hypercentre.

***Proposition* 3.5** (Skiba [7]). *Let G be a σ-full group and $Z = Z_\sigma(G)$. Let A, B and N be subgroups of G, where N is normal in G.*

(i) *Every chief factor of G below Z is σ-central in G.*

(ii) $ZN/N \le Z_\sigma(G/N)$.

(iii) $Z_\sigma(A)N/N \le Z_\sigma(AN/N)$.

(iv) *For every subgroup H of G we have* $Z_\sigma(H) \cap A \le Z_\sigma(H \cap A)$.

(v) $G/C_G(Z)$ *and* Z *are σ-nilpotent.*

(vi) *If* G/Z *is σ-nilpotent, then G is also σ-nilpotent.*

(vii) *If* $N \le Z$, *then* $Z/N = Z_\sigma(G/N)$.

(viii) *If A is σ-nilpotent, then* ZA *is also σ-nilpotent.*

(ix) *If* $G = A \times B$, *then* $Z = Z_\sigma(A) \times Z_\sigma(B)$. *Moreover, if a subgroup U of G is subdirectly contained in* G, *then* $Z_\sigma(U) = U \cap Z_\sigma(G)$.

(x) *If* $N \le Z$, *then* A *is σ-subnormal in* NA.

(xi) *If* $N \le Z$, *then* A *is σ-subnormal in G if and only if* NA/N *is σ-subnormal in* G/N.

Corollary **3.6.** $[G^{\mathfrak{N}_\sigma}, Z_\sigma(G)] = 1$.

A subgroup *H* of *G* is said to be a *maximal σ-nilpotent subgroup* of *G* if *H* is σ-nilpotent subgroup but every subgroup E of *G* such that $H < E$ is not σ-nilpotent.

We have already known (see Theorem 2.18 (iii)) that if *G* is σ-soluble, then the σ-hypercentre $Z_\sigma(G)$ of *G* coincides with the intersection of all conjugates of *H*, where *H* is a σ-system normalizer of *G*. In the general case, we have

Theorem **3.7** (Skiba [7]). $Z_\sigma(G)$ *coincides with the intersection of all maximal σ-nilpotent subgroups of G.*

Corollary **3.8** (Baer). *The hypercentre* $Z_\infty(G)$ *of G coincides with the intersection of all maximal nilpotent subgroups of G.*

Lemma **3.9.** (i) *If G is σ-quasinilpotent group and N a normal subgroup of G, then* N *and* G/N *are σ-quasinilpotent.*

(ii) *If* G/N *and* G/L *are σ-quasinilpotent, then* $G/(N \cap L)$ *is σ-quasinilpotent.*

Proof. (i), (ii) See the proof of Lemma 13.2 in [4, X].

Lemma **3.10.** *Let* H/K *be a chief factor of* G. *Then every automorphism of* H/K *induced by an element of G is inner if and only if*

$$G = (H/K)C_G(H/K).$$

Proof. See the proof of Lemma 13.4 in [4].

Definition **3.11.** We say that G is: *σ-perfect* if $G^{\mathfrak{N}_\sigma} = G$; *σ-semisimple* if either $G = 1$ or $G = A_1 \times \cdots \times A_t$ is the direct product of non-abelian simple non-σ-primary groups $A_1, \ldots, A_t$.

Note that if $\sigma = \{\{2,3,5\}, \{2,3,5\}'\}$ and $G = A_7 \times A_{11}$, then *G* is σ-semisimple and σ-perfect.

Lemma **3.12.** *Let N be a normal* σ_i*-subgroup of G. Then* $N \le Z_\sigma(G)$ *if and only if* $O^{\sigma_i}(G) \le C_G(N)$.

Proof. If $O^{\sigma_i}(G) \le C_G(N)$, then for every chief factor H/K of G below N both groups H/K and $G/C_G(H/K)$ are σ_i-group since $G/O^{\sigma_i}(G)$ is a σ_i-group, so $N \le Z_\sigma(G)$.

Now assume that $N \le Z_\sigma(G)$. Let $1 = Z_0 < < Z_1 < \ldots < Z_t = N$ be a chief series of *G* below *N* and $C_i = C_G(Z_i/Z_{i-1})$. Let $C = C_1 \cap \cdots C_t$. Then G/C is a σ_i-group. On the other hand, $C/C_G(N) \simeq \simeq A \le Aut(N)$ stabilizes the series $1 = Z_0 < Z_1 < < \ldots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$-group by Theorem 0.1 in [12]. Hence $G/C_G(N)$ is a σ_i-group, so $O^{\sigma_i}(G) \le C_G(N)$. The lemma is proved.

Theorem **3.13.** *Given group G the following are equivalent:*

(i) *G is σ-quasinilpotent.*

(ii) $G/Z_\sigma(G)$ *is σ-semisimple.*

(iii) $G/F_\sigma(G)$ *is σ-semisimple and*

$$G = F_\sigma(G)C_G(F_\sigma(G)).$$

Proof. Let $Z = Z_\sigma(G)$. (i) $\Rightarrow$ (ii) Assume that this is false and let *G* be a counterexample of minimal order. Then the hypothesis holds for G/R by Lemma 3.9 (i). On the other hand, $Z_\sigma(G/Z) = 1$ by Proposition 3.5 (vii). Hence in the case when $Z \ne 1$, $G/Z_\sigma(G)$ is σ-semisimple by the choice of G.

Now assume that $Z = 1$ and let *R* be any minimal normal subgroup of *G*. Then $R/1$ is σ-eccentric since $Z(G) \le Z = 1$. Hence R is non-abelian and $G = R \times C_G(R)$ by Lemma 3.10. Therefore

$$Z_\sigma(R) \times Z_\sigma(C_G(R)) = Z_\sigma(G) = 1$$

by Proposition 3.5 (ix). Hence the choice of *G* implies that R and $C_G(R)$ are σ-semisimple, so *G* is σ-semisimple, a contradiction. Hence G/Z is σ-semisimple.

(ii) $\Rightarrow$ (iiii) First note that $Z \le F_\sigma(G)$ by Proposition 3.5 (v), so $Z = F_\sigma(G)$ since G/Z is σ-semisimple by hypothesis. But then $G/C_G(F_\sigma(G))$ is σ-nilpotent by Proposition 3.5(v). Hence $G = = F_\sigma(G)C_G(F_\sigma(G))$ since $G/F_\sigma(G) = G/Z$ is σ-semisimple.

(iii) $\Rightarrow$ (i) Let H/K be a chief factor of G. If $F_\sigma(G) \le K$, then every automorphism of H/K induced by an element of *G* is inner by Lemma 3.10 since $G/F_\sigma(G)$ is σ-semisimple by hypothesis. Now suppose that $H \le F_\sigma(G)$. Then

$$C_G(H/K) = C_G(H/K) \cap F_\sigma(G)C_G(F_\sigma(G)) = \\ = C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K),$$

so

$$G/C_G(H/K) = \\ = F_\sigma(G)C_G(F_\sigma(G))/C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) \simeq$$

$$F_\sigma(G)/F_\sigma(G)\cap C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K)=$$
$$=F_\sigma(G)/C_{F_\sigma(G)}(H/K)Z(F_\sigma(G))\simeq$$
$$\simeq(F_\sigma(G)/C_{F_\sigma(G)}(H/K))/(C_{F_\sigma(G)}(H/K)\times$$
$$\times Z(F_\sigma(G))/C_{F_\sigma(G)}(H/K))$$

is σ-primary by Lemma 3.12. Therefore H/K is σ-central in G. Now applying the Jordan-Hölder theorem for the chief series [5] we get that for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. The theorem is proved.

***Corollary* 3.14.** *Let G be σ-quasinilpotent.*

(i) *If G is σ-perfect, then* $Z_\sigma(G)=Z(G)$.

(ii) *If H is a normal σ-soluble subgroup of* G, *then* $H\le Z_\sigma(G)$.

Proof. (i) This assertion follows from Proposition 3.5 (v) and Theorem 3.13.

(ii) This directly follows from Theorem 3.13.

***Corollary* 3.15.** *If a σ-quasinilpotent group* $G\ne 1$ *is σ-soluble, then* $G=O_{\sigma_1}(G)\times\cdots\times O_{\sigma_t}(G)$, *where* $\{\sigma_1,\dots,\sigma_t\}=\sigma(G)$.

***Corollary* 3.16.** *Let* $\pi=\cup_{\sigma_i\in\Pi}\sigma_i$. *If a σ-quasinilpotent group* $G\ne 1$ *is* π*-separable*, *then*

$$G=O_\pi(G)\times O_{\pi'}(G).$$

***Corollary* 3.17.** *If a quasinilpotent group G is* π*-separable, then* $G=O_\pi(G)\times O_{\pi'}(G)$.

A *formation* is a class $\mathfrak{F}$ of groups with the following properties:

(i) Every homomorphic image of an $\mathfrak{F}$-group is an $\mathfrak{F}$-group.

(ii) If G/M and G/N are $\mathfrak{F}$-groups, then also $G/(M\cap N)$ belongs to $\mathfrak{F}$.

The formation $\mathfrak{F}$ is said to be: (*solubly*) *saturated* if $G\in\mathfrak{F}$ whenever $G/\Phi(N)\in\mathfrak{F}$ for some (soluble) normal subgroup N of G; (*normally*) *hereditary* if $H\in\mathfrak{F}$ whenever $G\in\mathfrak{F}$ and H is a (normal) subgroup of G.

A class $\mathfrak{F}$ of groups is called a *Fitting class* if it is closed under taking normal subgroups and products of normal subgroups.

From Corollary 3.4 we get at once the following fact.

***Theorem* 3.18** *The class* $\mathfrak{N}_\sigma$ *is a hereditary saturated formation. Moreover*, $\mathfrak{N}_\sigma$ *is a Fitting class.*

We write $Com(G)$ to denote the class of all groups L such that L is isomorphic to some abelian composition factor of G; $R(G)$ denotes the largest normal soluble subgroup of G.

For a formation function of the form

$$f:\mathbb{P}\cup\{0\}\to\{\text{formations of groups}\}\quad(3.1)$$

we put, following [13],

$$CLF(f)=\{G \text{ is a group}|G/R(G)\in f(0)$$

and

$$G/C^p(G)\in f(p) \text{ for any prime } p\in\pi(Com(G))\}.$$

If $\mathfrak{F}=CLF(f)$ for some formation function *f*, then we say that *f* is a *composition* satellite of the formation $\mathfrak{F}$.

From [14, I, 3.2] and Baer's Theorem [5, IV, 3.17], the following result follows.

***Lemma* 3.19.** (i) *For any function f of the form* (3.1), *the class* $CLF(f)$ *is a solubly saturated formation.*

(ii) *For any non-empty solubly saturated formation* $\mathfrak{F}$, *there is a unique function* F *of the form* (3.1) *such that* $\mathfrak{F}=CLF(F)$, $F(p)=\mathfrak{G}_pF(p)\subseteq\mathfrak{F}$ *for all primes* p, *and* $F(0)=\mathfrak{F}$.

The function *F* in Lemma 3.19 (ii) is called the *canonical* composition satellite of $\mathfrak{F}$.

Now, being based on Theorem 3.13 and Lemma 3.19, we prove the following useful fact.

***Theorem* 3.20.** *The class* $\mathfrak{N}_\sigma^*$ *is a normally hereditary solubly saturated formation. Moreover*, $\mathfrak{N}_\sigma^*$ *is a Fitting class.*

Proof. In order to prove the first assertion of the theorem, it is enough to prove, in view of Lemma 3.9, that $\mathfrak{N}_\sigma^*$ is a solubly saturated formation. Let $\mathfrak{M}=CLF(f)$, where $f(p)=\mathfrak{G}_{\sigma_i}$ is the class of all σ_i-groups for all $p\in\sigma_i$, and $f(0)=\mathfrak{N}_\sigma^*$. We show that $\mathfrak{M}=\mathfrak{N}_\sigma^*$. Let G be a group of minimal order in $\mathfrak{M}\setminus\mathfrak{N}_\sigma^*$ and *R* a minimal normal subgroup of *G*. Then, in view of Lemma 3.9, *R* is the unique minimal normal subgroup of *G* and G/R is σ-quasinilpotent. Therefore, in view of Theorem 3.13, *R* is not σ-central in *G*. Hence *R* is non-abelian. But then $R(G)=1$ and so $G\simeq G/R(G)\in f(0)=\mathfrak{N}_\sigma^*$, a contradiction. Thus $\mathfrak{M}\subseteq\mathfrak{N}_\sigma^*$. Now, assume that $\mathfrak{N}_\sigma^*\not\subseteq\mathfrak{M}$ and *G* be a group of minimal order in $\mathfrak{N}_\sigma^*\setminus\mathfrak{M}$ with a minimal normal subgroup *R*. Then $R=G^{\mathfrak{M}}$ is the unique minimal normal subgroup of *G*. If *R* is non-abelian, then $R(G)=1$ and therefore $G\simeq G/R(G)\in f(0)=\mathfrak{N}_\sigma^*$. Moreover, in this case we have $R\le C^p(G)$ and

$$G/C^p(G)\simeq(G/R)/(C^p(G)/R)=$$
$$=(G/R)/C^p(G/R)\in f(p)$$

for all $p\in\pi(Com(G))$ since $G/R\in\mathfrak{M}$ and so $G\in\mathfrak{M}$, a contradiction. Hence *R* is a *p*-group for some prime $p\in\sigma_i$. But $G\in\mathfrak{N}_\sigma^*$, so $R\rtimes(G/C_G(R))$ is a σ_i-group. But then $G/C^p(G)$ is a σ_i-group and so $G\in\mathfrak{M}$. Hence $\mathfrak{M}=\mathfrak{N}_\sigma^*$. Therefore $\mathfrak{N}_\sigma^*$ is a solubly saturated formation by Lemma 3.19.

Since the class $\mathfrak{N}_\sigma^*$ is normally hereditary by Lemma 3.9, in order to prove the second assertion of the theorem it is enough to show that if $G = AB$, where A and B are normal σ-quasinilpotent subgroups of G, then G is σ-quasinilpotent. Let R be a minimal normal subgroup of G and $C = C_G(R)$. By Lemma 2.9 (i), the hypothesis holds for G/R, so the choice of G implies that G/R is σ-quasinilpotent. Therefore in view of Lemma 2.9 (ii), R is the unique minimal normal subgroup of G.

Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. If $A \cap B = 1$, then $Z_\sigma(G) = Z_1 \times Z_2$ by Proposition 3.5 (ix). On the other hand, A/Z_1 and B/Z_2 are σ-semisimple by Theorem 3.13, so

$$G/Z = (A \times B)/(Z_1 \times Z_2) \simeq (A/Z_1) \times (B/Z_2)$$

is σ-semisimple. Hence G is σ-quasinilpotent by Theorem 3.13.

Now suppose that $A \cap B \neq 1$. Then $R \leq A \cap B$. First assume that R is σ-primary, say R is a σ_i-group. Then by Proposition 3.5, $R \leq F(A) \cap F(B) \leq Z_1 \cap Z_2$. Then $AC/C \simeq A/A \cap C$ and $BC/C \simeq B/B \cap C$ are σ_i-groups and hence $G/C = (AC/C)(BC/C)$ is a σ_i-group. Hence R is σ-central in G. Therefore $R \leq Z_\sigma(G)$ and so $Z_\sigma(G/R) = Z_\sigma(G)/R$ by Proposition 3.5 (vii). Thus G is σ-quasinilpotent by Theorem 3.13.

Therefore R is not σ-primary. Hence R is non-abelian, so $C = 1$. Then $R = R_1 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ are minimal normal subgroups of A. Let $C_i = C_A(R_i)$ $(i = 1, \ldots, t)$. Then $C = 1 = C_1 \cap \cdots \cap C_t$. Since A is σ-quasinilpotent by hypothesis, $A = R_i C_i$ for all $i = 1, \ldots, t$ by Lemma 3.9. Hence

$$R = RC = R_1 \ldots R_t (C_t \cap \cdots \cap C_1) =$$
$$= R_1 \ldots R_{t-1} (R_t C_t \cap C_{t-1} \cap \cdots \cap C_1) =$$
$$= R_1 \ldots R_{t-1} (A \cap C_{t-1} \cap \cdots \cap C_1) = \cdots = R_1 C_1 = A.$$

Similarly we can get that $B = R$, so $G = R$ is σ-semisimple. Hence G is σ-quasinilpotent. The theorem is proved.

4 The subgroups $F_\sigma(G)$, $F^*_\sigma(G)$, and $E_\sigma(G)$

We use the symbol $\Phi^*(G)$ to denote the subgroup $\Phi(R(G))$ [15].

The following result collect basic properties of the $F_\sigma(G)$, $F^*_\sigma(G)$ and $E_\sigma(G)$, and describes the main relations between them.

***Theorem* 4.1** (Skiba [7]). *Let G be a σ-full group. Let $F_\sigma = F_\sigma(G)$, $F^*_\sigma = F^*_\sigma(G)$ and $E_\sigma = E_\sigma(G)$.*

(i) *F^*_σ is σ-quasinilpotent and $F_\sigma = Z_\sigma(F^*_\sigma)$. Hence F^*_σ / F_σ is σ-semisimple and F^*_σ / F_σ is the product of all minimal normal subgroups of G/F_σ contained in $F_\sigma C_G(F_\sigma)/F_\sigma$. Also $F_\sigma / Z_\sigma(G) = F_\sigma(G/Z_\sigma(G))$ and $F^*_\sigma / Z_\sigma(G) = F^*_\sigma(G/Z_\sigma(G))$.*

(ii) *$F^*_\sigma = E_\sigma F_\sigma = C_{F^*_\sigma}(F_\sigma) F_\sigma$ and $F_\sigma = C_{F^*_\sigma}(E_\sigma)$. Also E_σ is a σ-perfect characteristic subgroup of F^*_σ and $E_\sigma / Z(E_\sigma)$ is σ-semisimple. Hence $E_\sigma(G) = E_\sigma(E_\sigma(G))$.*

(iii) *A σ-subnormal subgroup H of G is contained in F^*_σ (respectively in F_σ) if and only if it is σ-quasinilpotent (respectively σ-nilpotent). Moreover, if H also is σ-quasinilpotent σ-perfect, then $H \leq E_\sigma$.*

(iv) $C_G(F^*_\sigma) \leq Z(F^*_\sigma)$.

(v) $F_\sigma(G/\Phi(G)) = F_\sigma / \Phi(G)$ *and*

$$F^*_\sigma(G/\Phi^*(G)) = F^*_\sigma / \Phi^*(G).$$

***Corollary* 4.2.** *If G is σ-full, then for every σ-subnormal subgroup V of G we have $F_\sigma(G) \cap V = F_\sigma(V)$ and $F^*_\sigma(G) \cap V = F^*_\sigma(V)$.*

It is clear that if $R \leq E \leq G$, where R is a non-abelian minimal normal subgroup of G and E is normal in G, then R is the product of some minimal normal subgroups of E [5, A, 4.13]. Hence we get from Theorem 4.1 (i) the following

***Corollary* 4.3.** *If G is σ-full, then $F^*_\sigma(G)/F_\sigma(G)$ is the group generated by all minimal normal subgroup of*

$$C_G(F_\sigma(G))F_\sigma(G)/F_\sigma(G).$$

From Theorem 4.1 (iv) we get

***Corollary* 4.4** (Skiba [1]). *If G is σ-soluble, then $C_G(F_\sigma(G)) \leq F_\sigma(G)$.*

Note that in view of Example 1.2 (ii) in the special case, when $\sigma = \{\pi, \pi'\}$, we get from Corollary 4.3 the following fact.

***Corollary* 4.5.** *If G is π-separable, then*

$$C_G(O_\pi(G) \times O_{\pi'}(G)) \leq O_\pi(G) \times O_{\pi'}(G).$$

***Theorem* 4.6.** *Let G be a σ-full group and H a σ-soluble subgroup of G. If $E_\sigma(G) \leq N_G(H)$, then $E_\sigma(G) \leq N_G(H)$.*

Proof. Since $E_\sigma(G) \leq N_G(H)$, $[E_\sigma(G), H = 1] \leq E_\sigma(G) \cap H$ and $E_\sigma(G) \cap H$ is a σ-soluble normal of $E_\sigma(G)$. Hence $E_\sigma(G) \cap H \leq Z(E_\sigma(G))$ since $E_\sigma(G)/Z(E_\sigma(G))$ is σ-semisimple by Theorem 4.1 (ii). Hence $[E_\sigma(G), H, E_\sigma(G)] = 1$, so $[E_\sigma(G), H] = [E_\sigma(G), E_\sigma(G), H] = 1$ by the lemma on three subgroups [11, III, 1.10]. The theorem is proved.

***Definition* 4.7.** A *σ-component of* $E_\sigma(G)$ (sf. [4, Definition 13.17]) is a σ-perfect normal subgroup H of $E_\sigma(G)$ such that that $H/Z(H)$ is simple.

Theorem 4.1 makes possible to prove the following two results.

***Theorem* 4.8** (Skiba [7]). *Suppose that G is σ-full and let $Z = Z(E_\sigma(G))$.*

(i) *$E_\sigma(G)$ is the product of its σ-components but is not the product of any proper subset of them.*

(ii) *If* H *is a* σ*-component of* $E_\sigma(G)$, *then* HZ/Z *is a simple direct factor of* $E_\sigma(G)$ *and* $Z(H) = H \cap Z$.

(iii) *If* H_1 *and* H_2 *are distinct* σ*-components of* $E_\sigma(G)$, *then* $[H_1, H_2] = 1$.

(iv) *If R is a* σ*-subnormal subgroup of* $E_\sigma(G)$, *then R is the product of* $R \cap Z$ *and certain* σ*-component of* $E_\sigma(G)$. *In particular*, *R is normal in* $E_\sigma(G)$. *Also* $Z(E_\sigma(G)) = ZR/R$ *and* $E_\sigma(G) = RC_{E_\sigma(G)}(R)$.

(v) *If H is a* σ*-component of* $E_\sigma(G)$ *and* $A \le G$, *then either* $H \le [H, A]$ *or* $[H, A] = 1$. *If*, *further*, $H \le N_G(A)$, *then either* $H \le E_\sigma(A)$ *or* $[H, A] = 1$.

Theorem **4.9** (Skiba [7]). *Let G be a* σ*-full group and a Hall* σ_1*-subgroup of G is nilpotent. Suppose that* S *is a* σ_1*-subgroup of G. Then*

$$O^{\sigma_1}(F^*_\sigma(N_G(S))) = O^{\sigma_1}(F^*_\sigma(C_G(S))) \le C_G(O_\sigma(G)).$$

Corollary **4.10** (Bender [16]). *If S is a p-subgroup of G*, *then*

$$O^p(F^*(N_G(S))) = O^p(F^*(C_G(S))) \le C_G(O_\sigma(G)).$$

***Some other applications of Theorem* 4.1.** Theorem 4.1 not only covers a large number of known results, but it also allows you to establish a link between some of these results. Note for example that the following known results are special cases of Corollary 4.3.

Corollary **4.11** (See [17, Ch. 6, 1.3]). *If G is soluble*, *then* $C_G(F(G)) \le F(G)$.

Corollary **4.12** (See [17, Ch. 6, 3.2]). *If G is* π*-separable*, *then the following inclusion holds:*

$$C_{G/O_{\pi'}(G)}(O_\pi(G/O_{\pi'}(G)) \le O_\pi(G/O_{\pi'}(G).$$

In view of Example 1.2 (iii) and Remark 1.6, we get from Corollary 4.3 also the following

Corollary **4.13** (Monakhov and Shpyrko [18]). *Let G be a* π*-soluble group.*

(1) $C_G(O_\pi(G) \times O_{\pi'}(G)) \le F(O_\pi(G)) \times O_{\pi'}(G)$.

(2) *If* $O_{\pi'}(G) = 1$, *then* $C_G(F(G)) \le F(G)$.

In the case, when σ is the smallest partition of $\mathbb{P}$, we get from Theorem 4.1 and Corollaries 4.2 and 4.3 the following known results.

Corollary **4.14** (See [4, X, 13.13]). $F^*(G)/F(G)$ *is the group generated by all minimal normal subgroup of* $C_G(F(G))F(G)/F(G)$.

Corollary **4.15** (See [4, X, 13.10]). $F^*(G)$ *is quasinilpotent and every subnormal quasinilpotent subgroup of G is contained in* $F^*(G)$.

Corollary **4.17** (See [5, A, 8.8]). $F(G)$ *is generated by all subnormal nilpotent subgroup of G.*

Corollary 4.18 (See [4, X, 13.15]).

$$F(G) = C_{F^*(G)}(E(G)).$$

5 Further applications

Let $\mathcal{L}$ be some non-empty set of subgroups of *G* and *E* a subgroup of *G*. Then a subgroup *A* of *G* is called $\mathcal{L}$*-permutable* if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$*-permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and $x \in E$.

If $\mathcal{L}$ is a *complete Sylow* π*-set* of *G* (that is, $\mathcal{L}$ contains exact one Sylow *p*-subgroup for every $p \in \pi$ such that *p* divides $|G|$), then an $\mathcal{L}^G$-permutable subgroup is called π*-permutable* or π*-quasinormal* [19] in *G*. Recall also that $\pi(G)$-permutable subgroups are also called *S-permutable* or *S-quasinormal* in *G*.

In this section we deal with the following generalization of these concepts.

Definition **5.1.** We say that a subgroup *H* of *G* is Π*-permutable* in *G* if *G* possess a complete Hall Π-set $\mathcal{H}$ such that *H* is $\mathcal{H}^G$-permutable.

Example **5.2.** (i) If *G* is nilpotent, then Sylow subgroups of *G* are normal in *G*, so every subgroup of *G* is σ-permutable in *G* for every partition σ of $\mathbb{P}$.

In more general case, when *G* is σ-nilpotent, every subgroup of *G* is Π-permutable in *G* for every $\Pi \subseteq \sigma$.

(ii) Now let p, q, r be different primes, where q divides $p-1$. Let $H = Q \rtimes R$ be a non-abelian group of order qr, P a simple $\mathbb{F}_pH$-module which is faithful for H, and $G = P \rtimes H$.

Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{p, r\}$ and $\sigma_2 = \{p, r\}'$. Then *G* is not σ-nilpotent and $|P| > p$. Since q divides $p-1$, PQ is supersoluble and hence for some normal subgroup L of PQ we have $1 < L < P$. Then for every Hall σ_1-subgroup V of G we have $L \le P \le V$, so $QV = V = VQ$. On the other hand, for every Hall σ_2-subgroup W of *G* we have $W \le PQ$, so $QW = WQ$. Hence Q is σ-permutable in *G*. It is also clear that *L* is not normal in G, so $LR \ne RL$, which implies that *L* is not *S*-permutable in *G*.

Theorem **5.3** (Skiba [20]). *Let H be a* Π*-subgroup of G.*

(i) *If G is a* Π*-full group and H is* Π*-permutable in* G, *then H is* σ*-subnormal in G and* H^G *is a* Π*-group.*

(iii) *If G is a* Π'*-full group and H is* Π'*-permutable in G*, *then* H^G *has a* σ*-nilpotent Hall* Π'*-subgroup.*

Corollary **5.4** (Kegel [19]). *If a* π*-subgroup H of G is* π*-permutable in G*, *then H is subnormal in G.*

A subgroup *H* of *G* is called a *S-semipermutable* in *G* if *H* permutes with all Sylow subgroups *P* of *G* such that $(|H|, |P|) = 1$.

Corollary **5.5** (Isaacs [21]). *If a* π*-subgroup H of G is S-semipermutable in G*, *then the normal* *closure H^G of H in G possess a nilpotent π-complement.*

Theorem 5.3 was applied in the proofs of many results about Π-permutable subgroups. In particular, on the basis of this result the following fact can be proved.

***Theorem* 5.6** (Skiba [2]). *Let G be a σ-full group and $D = G^{\mathfrak{N}_\sigma}$. If a subgroup H of G is σ-permutable in G, then H^G / H_G is σ-nilpotent and $D \le N_G(H)$.*

***Corollary* 5.7** (Deskins [22]). *If a subgroup H of G is S-permutable in G, then H / H_G is nilpotent.*

As a direct consequence of Theorem 5.6, we also have

***Corollary* 5.8.** *Suppose that G is a σ-full group of Sylow type. If $G^{\mathfrak{N}_\sigma} = G$, then every σ-permutable subgroup of G is normal.*

It is not difficult to show that if H / N is Π-permutable in G / N and G is a Π-full group of Sylow type, then *H* is Π-permutable in *G* as well. On the other hand, in view of Example 5.2 (i), every subgroup of every σ-nilpotent group is σ-permutable. Hence we also get from Theorem 5.6 the following facts.

***Corollary* 5.9.** *Suppose that G is a σ-full group of Sylow type and let H be a subgroup of G. If H is σ-permutable in G, then $N_G(H)$ is also σ-permutable in G.*

***Corollary* 5.10** (Schmid [23]). *If a subgroup H of G is S-permutable in G, then $N_G(H)$ is also S-permutable.*

A group *G* is said to be a *π-decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$, that is, *G* is the direct product of its Hall π-subgroup and Hall π'-subgroup.

Taking in Theorem 5.6 $\sigma = \{\pi, \pi'\}$, we get

***Corollary* 5.11.** *Assume that $G = A_1 A_2$, where A_1 are A_2 are Hall π-subgroup and Hall π'-subgroup of G, respectively. If a subgroup H of G permutes with A_i^x for all $x \in G$ and $i = 1, 2$, then H^G / H_G is π-decomposable.*

***Corollary* 5.12.** *Assume that G has a p-complement. If a subgroup H of G permutes with every Sylow p-subgroup of G and every p-complement of G, then H^G / H_G is p-decomposable.*

It is well-known that in general the set of all quasinormal subgroups of *G* is not a sublattice of the lattice of all subgroups of *G* (Ito). Nevertheless, as another application of Theorem 5.3, the following result is proved.

***Theorem* 5.13** (Skiba [20]). *Let G be a Π-full group of Sylow type. Then the set of all σ-subnormal Π-permutable subgroups of G forms a sublattice of the lattice of all σ-subnormal subgroups of G.*

***Corollary* 5.14** (Kegel [19]). *The set of all subnormal π-permutable subgroups of G forms a sublattice of the lattice of all subnormal subgroups of G.*

In view of Theorem 5.6, we get from Theorem 5.13 the following result.

***Corollary* 5.15** (Skiba [2]). *Let G be a σ-full group of Sylow type. Then the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all subgroups of G.*

***Corollary* 5.16** (Kegel [19]). *The set of all $\pi(G)$-permutable subgroups of G forms a sublattice of the lattice of all subgroups of G.*

Note that Corollary 5.15 not only generalizes Corollary 5.16 but also gives a shorter proof of it.

Groups in which σ-permutability is a transitive relation. A group *G* is called a *PST*-*group* if *S*-permutability is a transitive relation on *G*, that is, every *S*-permutable subgroup of an *S*-permutable subgroup of *G* is *S*-permutable in *G*. In view of the Corollary 5.14 the class of all *PST*-groups coincides with the class of all groups, in which every subnormal subgroup is *S*-permutable.

The description of *PST*-groups was first obtained by Agrawal [24], for the soluble case, and by Robinson in [25], for the general case. In the further publications, authors (see, for example, the recent papers [26]–[35]) have found out and described many other interesting characterizations of soluble *PST*-groups.

The results of such kind are the motivations for the following

***Question* 5.17.** *Let G be a σ-full group. What is the structure of G provided that every σ-subnormal subgroup of G is σ-permutable?*

The answer to this question for the case of an arbitrary σ-full group *G* is not known now. But a complete classification of such groups in the universe of all σ-soluble groups is known.

***Theorem* 5.18** (Skiba [2]). *Let G be a σ-soluble group. Then every σ-subnormal subgroup of G is σ-permutable if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is an abelian σ-Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.*

***Corollary* 5.19** (Agrawal [24]). *Let G be a soluble group. Then G is a PST-group if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is an abelian Hall subgroup of odd order of G such that every element of M induces a power automorphism of D.*

Two characterizations of σ-permutability. Now we give two characterizations of the σ-permutable subgroups. The first of them uses the idea of description of the quasinormal subgroups which dates back to Theorem 5.1.1 in [36].

***Theorem* 5.20** (Skiba [20]). *Let G be a σ-full group of Sylow type. Then a subgroup A of G is σ-permutable in G if and only if A is σ-subnormal and, for each $i \in I$, the equality*

$$E \cap \langle A, H \rangle = \langle A, E \cap H \rangle$$

holds for every Hall σ_i *-subgroup H of G and every subgroup E of G containing A.*

Theorem 5.20 remains to be new also in the case when $\sigma = \{\{2\},\{3\},\{5\},...\}$.

***Corollary* 5.21.** *A subgroup A of G is S-permutable in G if and only if A is subnormal in G and the equality* $E \cap \langle A, P \rangle = \langle A, E \cap P \rangle$ *holds for every Sylow subgroup P of G and every subgroup E of G containing A.*

By making some small changes in the proof of Theorem 4.1 in [2], one can prove the following result.

***Theorem* 5.22.** *Let G be a* Π *-full group of Sylow type. Then a subgroup* A *of G is* Π *-permutable in G if and only if* A *is σ-subnormal in G and* A *is* Π *-permutable in* $\langle A, x \rangle$ *for all* $x \in G$.

In the case when σ is the smallest partition of $\mathbb{P}$ we get from Theorem 5.22 the following fact.

***Corollary* 5.23.** *A* π *-subgroup A of G is* π *-permutable in G if and only if A is subnormal in G and* A *is* π *-permutable in* $\langle A, x \rangle$ *for all* $x \in G$.

Since a subgroup *A* of *G* is subnormal in *G* if and only if A is subnormal in $\langle A, x \rangle$ for all $x \in G$ (Wielandt), from Theorem 5.22 we get also the following known result.

***Corollary* 5.24** (Ballester-Bolinches and Esteban-Romero [37]). *A subgroup A of G is S-permutable in G if and only if A is S-permutable in* $\langle A, x \rangle$ *for all* $x \in G$.

The σ-permutable closure and the σ-core of subgroups. Let *H* be a subgroup of a Π -full group G. Then we use $H_{\Pi G}$ to denote the Π *-core* of H, that is, the subgroup of *H* generated by all those subgroups of *H* which are Π -permutable in G. We use $H^{\Pi G}$ to denote the Π *-permutable closure* of *H* in G, that is, the intersection of all Π -permutable subgroups of *G* containing *H*.

In the case, when $\Pi = \sigma$ and σ is the smallest partition of $\mathbb{P}$, these two constructions proved useful in the analysis of many aspects of the theory of groups (see, for example, [38]–[41]).

A subgroup *H* of *G* is called respectively *Hall normally embedded*, *Hall subnormally embedded* [42], *Hall S -qusinormally embedded* [43] in *G* if *H* is a Hall subgroup of respectively the normal closure H^G, the subnormal closure $H^{\cdot G}$ [5, A], the *S*-permutable closure H^{sG} [40] of *H* in G.

By analogy with it we say that a subgroup *H* of a σ-full group *G* is called *Hall σ-permutable embedded* in *G* if *H* is a σ-Hall subgroup of the σ-permutable closure $H^{\sigma G}$ of *H* in *G*. We say also that a subgroup *H* of a group *G* is called *Hall σ-subnormally embedded* in *G* if *H* is a σ-Hall subgroup of the σ-subnormal closure $H^{sub\sigma G}$ of *H* in G.

***Theorem* 5.25** (Skiba [44]). *Let G be a σ-full group. Then every subgroup of G is Hall σ-subnormally embedded in G if and only if every σ-subnormal subgroup E of G is a σ-soluble group of the form* $E = D \rtimes M$, *where* $D = E^{\mathfrak{N}_\sigma}$ *is a σ-Hall subgroup of E with* $|\sigma(D)| = |\pi(D)|$, M *is a σ-Carter subgroup of E and for every chief factor* H/K *of E below D there is a Sylow subgroup P of D such that* $H = K \rtimes P$, *so M acts irreducibly on every M-invariant Sylow subgroup of D.*

On the basis of Theorem 5.25 can be proved the following useful result.

***Theorem* 5.26.** *Let G be a σ-full group of Sylow type. Then every subgroup of G is Hall σ-quasinormally embedded in G if and only if* $G = D \rtimes M$, *where* $D = G^{\mathfrak{N}_\sigma}$ *is a σ-Hall cyclic subgroup of G of square-free order.*

***Corollary* 5.27** (Li and Liu [42]). *Every subgroup of G is Hall normally embedded in G if and only if* $G = D \rtimes M$ *is a split extension of a cyclic subgroup D of square-free order by a Dedekind group M, where D and M are both Hall subgroups of G.*

Proof. First assume that every subgroup of *G* is Hall normally embedded in *G*. Then by Theorem 5.26, $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is a Hall cyclic subgroup of *G* of square-free order. On the other hand, *G* is clearly a *T*-group, so $M \simeq G/D$ is a Dedekind group [45, Ch. 2, 2.1.11].

Conversely, if $H \le G$, then $H^G \le DH$ since G/D is a Dedekind group. Hence *H* is a Hall subgroup of DH, so *H* is a Hall subgroup of $H^{\sigma G}$. The corollary is proved.

Groups with given σ-cofactors of subgroups. Recall that the *cofactor* of the subgroup $H \le G$ is the factor group H/H_G. By analogy with it, we say that $H/H_{\Pi G}$ is a Π *-cofactor* of H.

The structure of groups with given restrictions on the cofactors of subgroups were studied by many authors (see, for example, [46]–[51]).

Recall that *G* is said to be an *A-group* provided all Sylow subgroups of *G* are abelian. The class of all *A*-groups is a formation. We denote this formation by the symbol $\mathfrak{A}^*$.

***Theorem* 5.28** (Skiba [44]). *If the σ-cofactor of every subgroup of G is a cyclic σ-primary group, then G is σ-soluble and* $G^{\mathfrak{A}^*} \le Z_\sigma(G)$.

From Theorem 5.28 we get

***Corollary* 5.29** (Poland [48]). *If the cofactor of every subgroup of G is a cyclic primary group, then G is soluble and* $G^{\mathfrak{A}^*} \le Z_\infty(G)$.

Groups with maximal subgroups of Hall subgroups σ-permutably embedded. We say that a subgroup *H* of *G* is said to be *σ-permutably embedded* in *G* if, for every $\sigma_i \in \sigma(H)$, every Hall σ_i -subgroup of H is also a Hall σ_i-subgroup of some σ-permutable subgroup of G. In particular, H of G is said to be *S-permutably embedded* in G [52] if, for every $p \in \pi(H)$, every Sylow p-subgroup of H is also a Sylow p-subgroup of some S-permutable subgroup of G.

Srinivasan proved [53] that G is supersoluble if every maximal subgroup of every Sylow subgroup of G is S-permutable in G. In the paper [54], Walls obtained a description of groups in which every maximal subgroup of every Sylow subgroup is normal. In the other direction, this result was amplified in the paper [52] where the authors have proved that G is supersoluble provided that every maximal subgroup of every Sylow subgroup is S-permutably embedded. These results are motivations for our two next results.

***Theorem* 5.30** (Skiba [20]). *Let G be a σ-full group of Sylow type and* $\mathcal{H} = \{H_1, \ldots, H_t\}$ *be a complete Hall σ-set of G such that* $H_i^{\mathfrak{N}}$ *is a Hall subgroup of* H_i. *Every maximal subgroup of every member of* $\mathcal{H}$ *is σ-permutably embedded in G if and only if* $G = D \rtimes M$, *where D and M are σ-Hall subgroups of G*, $D = G^{\mathfrak{N}_\sigma}$ *is nilpotent of odd order and every element of M induces a power automorphism on* $D/\Phi(D)$.

***Corollary* 5.31.** *Every maximal subgroup of every Sylow subgroup of G is S-permutably embedded in G if and only if* $G = D \rtimes M$, *where D and M are Hall nilpotent subgroups of G, D is of odd order and every element of M induces a power automorphism on* $D/\Phi(D)$.

On the basis of Theorem 3.30 the following generalization of the Walls result was obtained in [20].

***Theorem* 5.32.** *Let G be a σ-full group and* $\mathcal{H} = \{H_1, \ldots, H_t\}$ *a complete Hall σ-set of G such that* $H_i^{\mathfrak{N}}$ *is a Hall subgroup of* H_i. *Every maximal subgroup of every member of* $\mathcal{H}$ *is σ-per-mutable in G if and only if* $G = (A \times B) \rtimes C$, *where* (*i*) A, B *and* C *are σ-Hall subgroups of G*, (*ii*) A *is a normal nilpotent subgroup of G of odd order*, B *is a normal σ-nilpotent subgroup of G and* C *is a cyclic subgroup of* G *such that* $\pi(C) = \sigma(C)$ *and* $[B, C] = 1$, (*iii*) *the generators of Sylow subgroups of* C *induce power automorphisms on* $A/\Phi(A)$ *and automorphisms of order dividing a prime on A.*

***Corollary* 5.33** (Srinivasan [53]). *If every maximal subgroup of every Sylow subgroup of G is S-permutable in G, then G is supersoluble.*

***Corollary* 5.34** (Walls [54]). *Every maximal subgroup of every Sylow subgroup of G is normal in G if and only if* $G = H \rtimes \langle x \rangle$, *where* (*i*) *H is a normal nilpotent Hall subgroup of G*, (*ii*) *the generators of Sylow subgroups of* $\langle x \rangle$ *induce power automorphisms on* $H/\Phi(H)$ *and automorphisms of order dividing a prime on H.*

***Corollary* 5.35** (Ballester-Bolinches and Pedraza-Aguilera [52]). *If every maximal subgroup of every Sylow subgroup of G is S-permutable in G, then G is supersoluble.*

6 Final remarks and some open questions

1. In the case, when G is σ-soluble, Theorem 5.6 can be improved [20].

***Theorem* 6.1** (See [23, Theorem C]). *Let G be a σ-soluble group and H is a σ-permutable subgroup of G. If H permutes also with some σ-system normalizer of G, then* $H^G / H_G \leq Z_\sigma(G/H_G)$.

2. One of the key properties of σ-subnormal subgroups we get from the following (see Theorem 3.1 (7)).

***Lemma* 6.2.** *If* A *is σ-subnormal in* G, *then* $A \cap H$ *is a Hall* Π*-subgroup of* A *for every Hall* Π*-subgroup* H *of* G.

Moreover, the following fact is true.

***Proposition* 6.3** (Skiba [3]). *If G is a σ-soluble, then a subgroup* A *of G is σ-subnormal in G if and only if* $A \cap H$ *is a Hall* σ_i*-subgroup of* A *for every Hall* σ_i*-subgroup H of G and every* $i \in I$.

In view of these observations, it seems natural to ask:

***Question* 6.4.** *Is it true that a subgroup A of the σ-full group G is a σ-subnormal in G if and only if* $H \cap A$ *is a Hall* σ_i*-subgroup of* A *for every Hall* σ_i*-subgroup H of G and every* $i \in I$*?*

The answer to this question in the case when σ is the smallest partition of $\mathbb{P}$ is positive [55].

The remarks before Corollary 5.24 make natural the following question.

***Question* 6.5.** *Suppose that for every* $x \in G$, *the subgroup H of G is σ-subnormal in* $\langle H, x \rangle$. *Is it true then that H is σ-subnormal in G?*

Recall that the well-known Wielandt theorem states that

***Theorem* 6.6** (See [56, Ch. 4, 4.1.2]). *If H and* K *are subnormal subgroups of* G *such that* $\pi(H/H^{\mathfrak{N}}) \cap \pi(K/K^{\mathfrak{N}})$ *is empty, then* $HK = KH$.

In this theorem $H^{\mathfrak{N}}$ denotes the nilpotent residual of H.

Theorem 6.6 allows us to hope that the answer to the following question is positive.

***Question* 6.7.** *Let H and K be σ-subnormal subgroups of G such that* $\pi(H/H^{\mathfrak{N}_\sigma}) \cap \pi(K/K^{\mathfrak{N}_\sigma})$ *is empty. Is it true then that* $HK = KH$*?*

3. It is known [57] that if a subgroup H of G is subnormal and H permutes with all members of some complete set of Sylow subgroups of G, then H/H_G is nilpotent. Nevertheless, we do not know the answer to the following question.

***Question* 6.8.** *Let G be a σ-full group and H a subgroup of G. Suppose that H is σ-subnormal in G and it permutes with all members of some complete Hall σ-set of G. Is it true then that* H/H_G *is σ-nilpotent?*

4. Theorem 2.8 is a motivation for the following

***Question* 6.9.** *Let G be a σ-full group and* $\mathcal{H} = \{H_1, ..., H_t\}$ *a complete Hall σ-set of G. What is the structure of G provided that for every chief factor* H/K *of G and every* $A \in \mathcal{H}$ *the number* $|G : N_G((A \cap H)K)|$ *is σ-primary?*

Note that the answer to this question in the case when σ is the smallest partition of $\mathbb{P}$ is known [10].

5. The final stage in the proof of Theorem 5.3 (ii) is based on two useful observations.

The first of them is a σ-generalization of Wielandt's theorem on groups with a nilpotent Hall subgroup.

***Proposition* 6.10** (Skiba [7]). *If G possess a σ-nilpotent Hall* Π *-subgroup H, then every* Π *-subgroup of G is contained in a conjugate of H.*

In its turn, Proposition 6.10 has required the use of the following interesting result.

***Proposition* 6.11** (Skiba [7]). *Let G be σ-soluble and* $\pi = \sigma_i$. *If G is not* π' *-closed but every proper subgroup of G is* π' *-closed, then G is a Schmidt group.*

6. In the paper [58], V.A. Vedernikov proved the following important result.

***Theorem* 6.12** (Vedernikov [58]). *Let G be a* D_π*-group. If G is not* π*-decomposable but every proper subgroup of G is* π*-decomposable, then G is a Schmidt group.*

***Corollary* 6.13.** *Let G be a* D_σ *-group. If G is not σ-nilpotent but every proper subgroup of G is σ-nilpotent, then G is a Schmidt group.*

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