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Form-factors of relativistic bound state systems of two scalar particles with one-boson exchange potential

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Получены численные решения релятивистских интегральных уравнений для связанных s-состояний систем двух скалярных частиц с потенциалом однобозонного обмена. На основании полученных решений рассчитаны форм-факторы упругого рассеяния и форм-факторы аннигиляции. Установлено, что для всех рассмотренных случаев число нулей упругого форм-фактора равно числу нулей соответствующей волновой функции.
Ключевые слова: интегральное уравнение, двухчастичная система, связанное состояние, скалярная частица, волновая функция, функция Грина, потенциал однобозонного обмена, собственное значение, форм-фактор.

Numerical solutions of relativistic integral equations are obtained for bound s-state systems of two scalar particles with one-boson exchange potential. The form-factors of elastic scattering and annihilation form-factors are calculated on the basis of solutions obtained. It is ascertained that in all cases under consideration the zero number of elastic form-factor is equal to the zero number of respective wave function.
Keywords: integral equation, two-particle system, bound state, scalar particle, wave function, Green function, one-boson exchange potential, eigenvalue, form-factor.

In this paper we discuss numerical solutions of relativistic integral equations of quantum field theory (QFT), describing the bound s-states of two scalar particles [1; 2] with one-boson exchange potential [2]. Afterwards on this foundation the elastic form-factors [3] and the annihilation form-factors of two-particle system [4] are found.

The two-particle equations of QFT for bound s-state wave functions in the momentum representation (MR) $\psi_{(j)}(w,\chi)$ have the form [5]:

$$\psi_{(j)}(w,\chi)=\frac{2\lambda}{\pi m}G_{(j)}(w,\chi)\int_0^\infty d\chi' V(\chi,\chi')\psi_{(j)}(w,\chi'), \tag{1}$$

where index $j=1,2,3,4$ corresponds to the four variants of quasipotential type equations: $j=1$ ($j=3$) – the Logunov-Tavkhelidze equation (modified), $j=2$ ($j=4$) – the Kadyshevsky equation (modified). The value χ in equation (1) is the rapidity associated with the momentum p by the relation $p=m\sinh\chi$ (m is the mass of each particle), w is associated with the two-particle system energy $2E$ by the relation $2E=2m\cos w$, $\lambda>0$ is the coupling constant, $V(\chi,\chi')$ is the relativistic potential, $G_{(j)}(w,\chi)$ are the Green functions (GF), which have the following form [1, 2]:

$$G_{(1)}(w,\chi)=\left[\cosh^2\chi-\cos^2 w\right]^{-1};\;\; G_{(2)}(w,\chi)=\left[2\cosh\chi\left(\cosh\chi-\cos w\right)\right]^{-1};$$
$$G_{(3)}(w,\chi)=\cosh\chi\left[\cosh^2\chi-\cos^2 w\right]^{-1};\;\; G_{(4)}(w,\chi)=\left[2\left(\cosh\chi-\cos w\right)\right]^{-1}.$$

In the spherically symmetric case after integration over angles the scalar one-boson exchange potential turns to be

$$V(\chi,\chi')=\frac{1}{4}\ln\left(\frac{\cosh(\chi+\chi')-\cos\alpha}{\cosh(\chi-\chi')-\cos\alpha}\right),$$

where the value α is associated with the mass μ of the exchange boson by the relation [2]

$$\cos\alpha = 1 - \mu^2/2m^2 \ .$$

For determining the elastic form-factors it is necessary to know the wave functions in the relativistic configurational representation (RCR). The equations corresponding to (1) for the wave functions in the RCR can be written in the form [6]:

$$\psi_{(j)}(w,r) = -\lambda\int_0^\infty dr' G_{(j)}(w,r,r')V(r')\psi_{(j)}(w,r') \ , \qquad (2)$$

where r is the radius-vector modulus, the functions $\psi_{(j)}(w,r)$, $G_{(j)}(w,r,r')$, $V(r)$ are related to respective functions in the MR by the transformations

$$\psi_{(j)}(w,r) = \int_0^\infty d\chi \sin(\chi m r)\psi_{(j)}(w,\chi) \ ,$$

$$G_{(j)}(w,r,r') = \frac{-2}{\pi m}\int_0^\infty d\chi \sin(\chi m r)\, G_{(j)}(w,\chi)\sin(\chi m r') \ , \qquad (3)$$

$$V(\chi,\chi') = \int_0^\infty dr \sin(\chi m r)V(r)\sin(\chi' m r) \ . \qquad (4)$$

The computation of integrals for GF (3) gives the following expressions in the RCR [6]:

$$G_{(j)}(w,r,r') = G_{(j)}(w,r-r') - G_{(j)}(w,r+r') \ ;$$

$$G_{(1)}(w,r) = \frac{-1}{m\sin 2w}\frac{\sinh(\pi/2-w)mr}{\sinh \pi m r/2} \ ; \qquad G_{(3)}(w,r) = \frac{-1}{2m\sin w}\frac{\cosh(\pi/2-w)mr}{\cosh \pi m r/2} \ ;$$

$$G_{(2)}(w,r) = \frac{(4m\cos w)^{-1}}{\cosh \pi m r/2} - \frac{1}{m\sin 2w}\frac{\sinh(\pi-w)mr}{\sinh \pi m r} \ ; \quad G_{(4)}(w,r) = \frac{-1}{2m\sin w}\frac{\sinh(\pi-w)mr}{\sinh \pi m r} \ .$$

The inverse transformation to (4) gives the potential $V(r)$ in the RCR [2]

$$V(r) = \frac{\cosh(\pi-\alpha)mr}{r\sinh \pi m r} \ ,$$

which turns to be the Yukawa potential in the non-relativistic limit.

To find numerical solutions of the integral equations in the RCR we use the composite Gaussian quadrature rule for computing the integrals [7] after replacing the infinity limit of integration to a large value R. Alternatively the solutions of integral equations in the MR are obtained by the Chebyshev quadrature rule after reduction of the half-infinity interval of integration to the interval $[-1;1]$ by the variable substitution $\chi = -\ln\left[(1-x)/2\right]$. Using the quadrature rules for integral equations (1) and (2) gives homogeneous systems of linear algebraic equations, which we represent in the general form for the MR and for the RCR as $M\psi = \lambda^{-1}\psi$, where ψ is the vector of values of the wave function in the quadrature node, M is a matrix obtained from the integral equation kernel. The finding of linear algebraic equations eigenvalues λ [7, 8] (for each concrete energy $2E = 2m\cos w$) gives the dependence λ on $2E$ (or $2E$ on λ). The parallel solution of equations in the MR and in the RCR allows to control accuracy of the eigenvalues obtained. We show the $2E-\lambda$ dependence at $\mu = m = 1$ and at $\mu = 0.1m = 0.1$ in figures 1 and 2. The results of numerical calculations for eigenvalues in the MR and in the RCR coincide with the accuracy 10^{-8} for the first (minimum) eigenvalues λ and with the accuracy 10^{-6} for the second and for the third eigenvalues λ. As an example we represent the results of numerical calculations of the wave functions at $\mu = m = 1$, $2E = 1$ in figure 3. As one can see the number of wave function zeros at $r \neq 0$ is equal to the number of state minus one (no zeros for the first state).

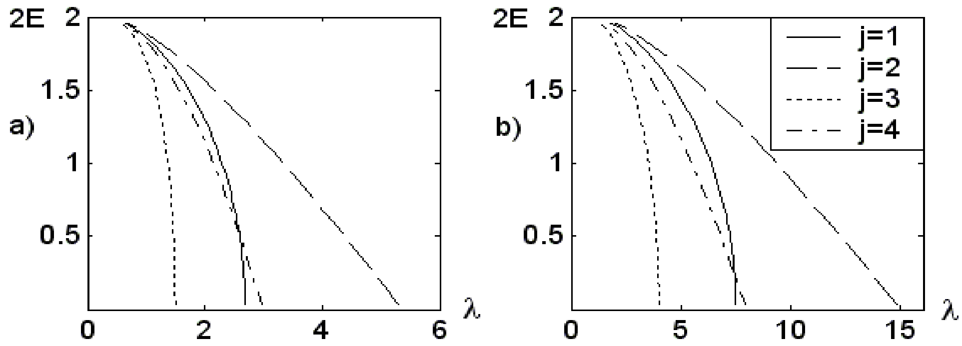

Figure 1 – The bound states energy at $\mu = 0.1m = 0.1$:
a) the first states, b) the second states

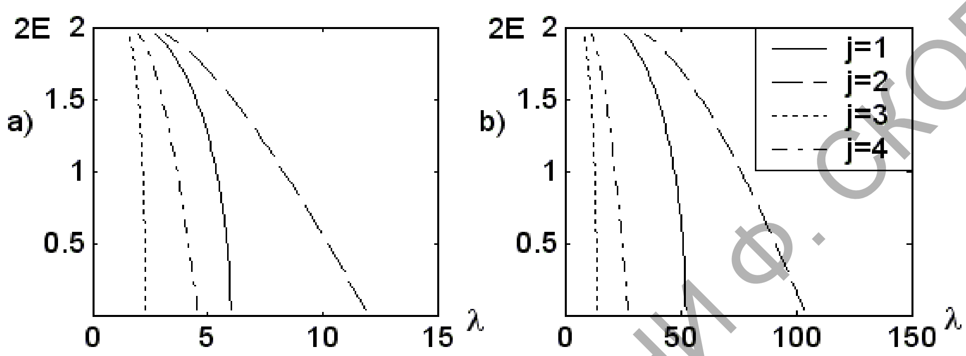

Figure 2 – The bound states energy at $\mu = m = 1$:
a) the first states, b) the second states

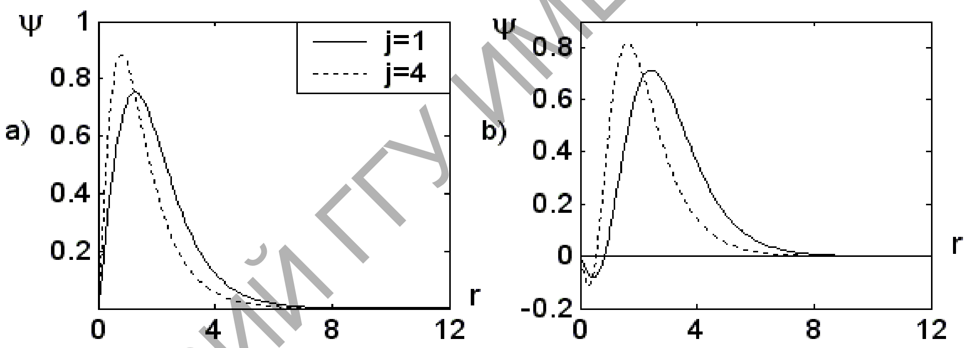

Figure 3 – The wave functions at $\mu = m = 1$, $2E = 1$:
a) the first states, b) the second states

Availability of wave functions in the RCR and in the MR and energy of bound states makes it possible to determine the form-factors of elastic scattering and annihilation. The elastic scattering form-factor of two spinless particle system was obtained based on the following interaction Hamiltonian [3]

$$H(x) = -z_1\varphi_1^+(x)\varphi_1(x)A(x) - z_2\varphi_2^+(x)\varphi_2(x)A(x),$$

where $\varphi_{1,2}(x)$, $A(x)$ are scalar fields, $z_{1,2}$ are coupling constants. In the s-wave case the expression for the elastic form-factor $F_{(j)}(\chi_q)$ has the form [3]:

$$F_{(j)}(\chi_q) = \frac{4\pi(z_1+z_2)}{m\sinh\chi_q}\int_0^\infty dr\,\frac{\sin\chi_q mr}{r}\,|\psi_{(j)}(w,r)|^2, \tag{5}$$

where χ_q is the rapidity associated with the square of the four-momentum transfer $t = (p'-p)^2$ by the relation $t = 4p^2\sinh^2(\chi_q/2)$, where p and p' are the four-momenta of the two-particle system before and after collision respectively (and we assume that $z_1 + z_2 = 1$). The results of numerical calculations for elastic form-factors (5) at $\mu = m = 1$, $2E = 1$ are shown in figures 4 and 5. It is seen in the figures that the form-factors vanish once for the second states and vanish twice for the third states. Thus numerical calculations show that the same is true also for the next states and the number of the form-factor zeros is equal to the index number of state minus one for all under study j (the form-factors of the first states have no zeros).

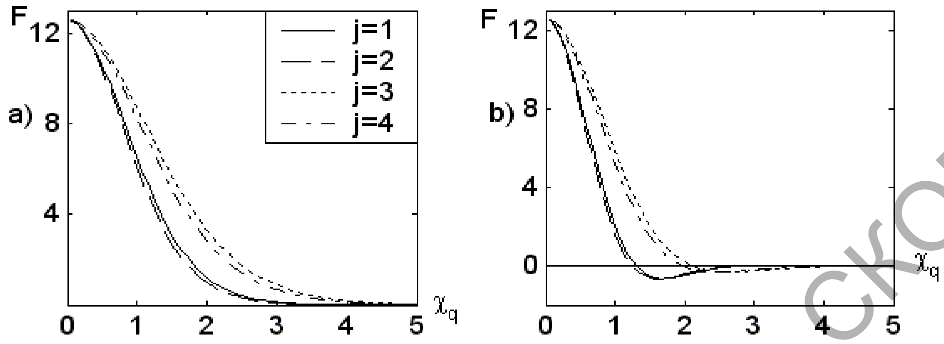

Figure 4 – The elastic form-factors at $\mu = m = 1$:
a) the first states, b) the second states

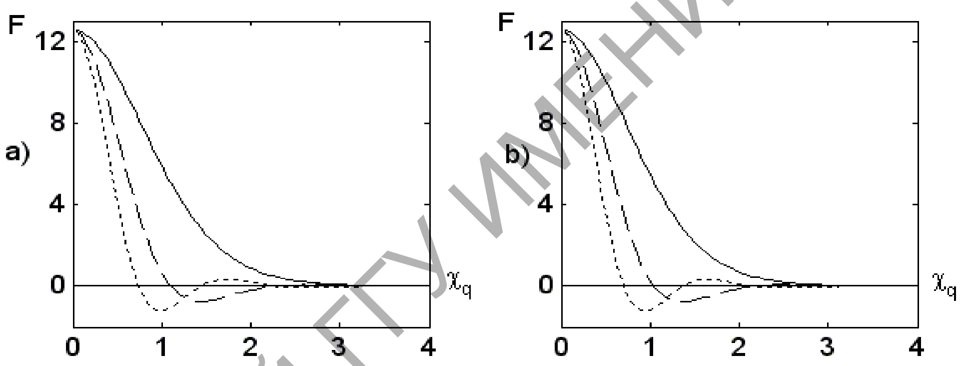

Figure 5 – The elastic form-factors for $j = 1$ (a), $j = 2$ (b) at $\mu = 0.5m = 0.5$:
solid line – the first states, dashed line – the second states,
dotted line – the third states

The expression for the form-factor of two-particle system annihilation $f_{(j)}(2E)$ has the following form [4]:

$$f_{(j)}(2E) = \frac{-4\sqrt{2\pi}\lambda}{2E} \int_0^\infty d\chi \chi \psi_{(j)}(\mathrm{Arc}\cos(E/m), \chi)\,. \qquad (6)$$

In figures 6 and 7 we represent the results of numerical calculations for expressions (6) at $\mu = m = 1$ and at $\mu = 0.1m = 0.1$. The same quadrature formulae have been used to calculate the integrals in expressions (5) and (6).

Thus, in this paper the numerical solutions of relativistic integral equations in the momentum representation and in the relativistic configurational representation describing the bound s-state of two scalar particle systems with one-boson exchange potential have been obtained (eigenvalues and wave functions). The form-factors of elastic scattering and two-particle systems annihilation have been determined on the basis of solutions obtained. It was found out that in all cases under consideration the zero number of the elastic form-factors $F_{(j)}(\chi_q)$ coincides with the zero number of the wave functions $\psi_{(j)}(w,r)$.

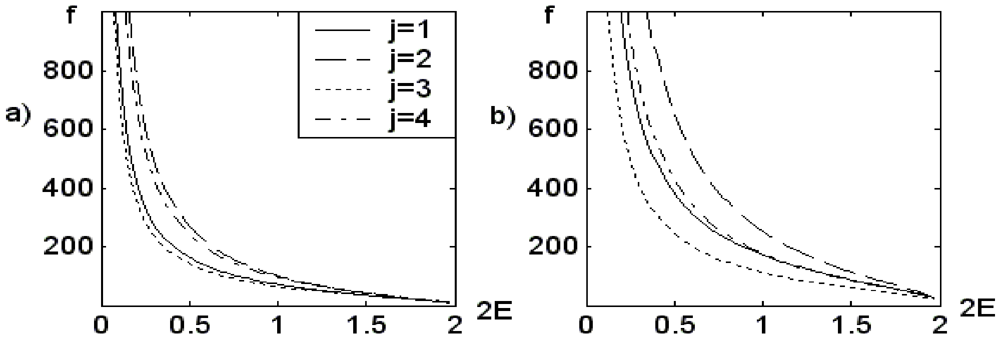

Figure 6 – The annihilation form-factors at $\mu = m = 1$:
a) the first states, b) the second states

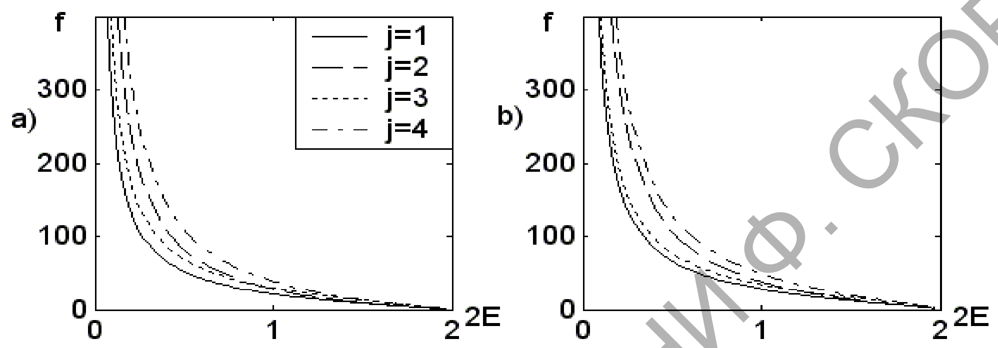

Figure 7 – The annihilation form-factors at $\mu = 0.1m = 0.1$:
a) the first states, b) the second states

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