МАТЕМАТИКА

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О ПЕРЕСЕЧЕНИИ ВСЕХ МАКСИМАЛЬНЫХ $\mathfrak{F}$-ПОДГРУПП КОНЕЧНОЙ ГРУППЫ

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ON THE INTERSECTION OF ALL MAXIMAL $\mathfrak{F}$-SUBGROUPS OF A FINITE GROUP

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Пусть $\mathfrak{F}$ класс групп. Подгруппа H группы G называется максимальной $\mathfrak{F}$-подгруппой группы G если $H \in \mathfrak{F}$ и G не имеет такой подгруппы $E \in \mathfrak{F}$ что $H < E$. Символ $\Sigma_{\mathfrak{F}}(G)$ обозначает пересечение всех максимальных $\mathfrak{F}$-подгрупп группы G. Мы изучаем влияние подгруппы $\Sigma_{\mathfrak{F}}(G)$ на строение группы G.

Ключевые слова: *насыщенная формация, наследственная формация, минимальная подгруппа, максимальная $\mathfrak{F}$-подгруппа, $\mathfrak{F}$-гиперцентр, разрешимая группа, сверхразрешимая группа, S-квазинормальная подгруппа.*

Let $\mathfrak{F}$ be a class of groups. A subgroup H of a group G is said to be a maximal $\mathfrak{F}$-subgroup of G if $H \in \mathfrak{F}$ and G has no a subgroup $E \in \mathfrak{F}$ such that $H < E$. The symbol $\Sigma_{\mathfrak{F}}(G)$ denotes the intersection of all maximal $\mathfrak{F}$-subgroups of G. We study the influence of the subgroup $\Sigma_{\mathfrak{F}}(G)$ on the structure of G.

Keywords: *saturated formation, hereditary formation, minimal subgroup, maximal $\mathfrak{F}$-subgroup, $\mathfrak{F}$-hypercentre, soluble group, supersoluble group, S-quasinormal subgroup.*

1 General properties of the intersection of all maximal $\mathfrak{F}$-subgroups

Throughout this paper, all groups are finite. We use $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups of a group G with $G/N \in \mathfrak{F}$. A class $\mathfrak{F}$ of groups is said to be a formation if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $\mathfrak{F}$ contains every group G with $G^{\mathfrak{F}} \le \Phi(G)$. A formation $\mathfrak{F}$ is said to be hereditary if $\mathfrak{F}$ contains every subgroup of every its group. In this paper $\mathfrak{F}$ denotes some hareditary saturated formation containing all nilpotent groups. We use $\mathfrak{N}$ and $\mathfrak{U}$ to denote the formation of all nilpotent groups and the formation of all supersoluble groups, respectively.

A subgroup H of a group G is said to be a maximal $\mathfrak{F}$-subgroup of G if $H \in \mathfrak{F}$ and G has no a subgroup $E \in \mathfrak{F}$ such that $H < E$. We use $\Sigma_{\mathfrak{F}}(G)$ to denote the intersection of all maximal $\mathfrak{F}$-subgroups of G. Thus $\Sigma_{\mathfrak{N}}(G)$ is the intersection of all maximal nilpotent subgroups of G and $\Sigma_{\mathfrak{U}}(G)$ is the intersection of all maximal supersoluble subgroups of G.

Applications of the subgroup $\Sigma_{\mathfrak{F}}(G)$ is based on the following our theorem.

Theorem A. *Let H, E be subgroups of a group G, N a normal subgroup of G and $\Sigma = \Sigma_{\mathfrak{F}}(G)$.*

(a) $\Sigma_{\mathfrak{F}}(H)N/N \le \Sigma_{\mathfrak{F}}(HN/N)$.

(b) $\Sigma_{\mathfrak{F}}(H) \cap E \le \Sigma_{\mathfrak{F}}(H \cap E)$.

(c) *If $H/H \cap \Sigma \in \mathfrak{F}$, then $H \in \mathfrak{F}$.*

(d) *If $H \in \mathfrak{F}$, then $H\Sigma \in \mathfrak{F}$.*

(e) *If $N \le \Sigma$, then $\Sigma/N = \Sigma_{\mathfrak{F}}(G/N)$.*

(f) $\Sigma_{\mathfrak{F}}(G/\Sigma) = 1$.

(g) *If every minimal non-$\mathfrak{F}$-subgroup of G is soluble and $\psi_e(N) \le \Sigma$, then $N \le \Sigma$.*

In this theorem $\psi_e(G)$ denotes the subgroup of G generated by all its cyclic subgroups of prime order and of order 4 [1]. A group G is said to be a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but $H \in \mathfrak{F}$ for every proper subgroup H of G.

Corollary 1.1. *Suppose that every minimal non-$\mathfrak{F}$-subgroup of a group G is soluble. If every p-subgroup P of G is contained in the intersection of all maximal $\mathfrak{F}$-subgroups of $N_G(P)$, then $G \in \mathfrak{F}$.*

It is well known that every minimal non-supersoluble group and every minimal non-p-nilpotent group are soluble. Hence from Corollary 1.1 we obtain the following.

***Corollary* 1.2.** *Suppose that every p-subgroup P of a group G is contained in $\Sigma_{\mathfrak{U}}(N_G(P))$. Then G is supersoluble.*

***Corollary* 1.3.** *Suppose that every p-subgroup P of a group G is contained in the intersection of all maximal p-nilpotent subgroups of $N_G(P)$. Then G is p-nilpotent.*

***Corollary* 1.4** (Frobenius). *If $N_G(P)/C_G(P)$ is a p-group for every p-subgroup P of a group G, then G is p-nilpotent.*

Proof. Since by [2, Appendix C, Corollary 6.4], $O_p(N_G(P)/C_{N_G(P)}(H/K))=1$ for all chief factors H/K of $N_G(P)$ below P, we have $P\le Z_\infty(N_G(P))$. Hence for every p-nilpotent subgroup H of $N_G(P)$, the subgroup PH is p-nilpotent as well. Therefore G is p-nilpotent by Corollary 1.3.

Next applications of Theorem A are connected with minimal subgroups.

***Theorem* 1.5.** *Let G be a group and $\mathfrak{F}$ the class of all $2'$-supersoluble groups.*

(1) *If every minimal subgroup L of G of odd order has a supplement T in G such that $L\cap T\le \Sigma_{\mathfrak{F}}(T)$, then G is $2'$-supersoluble.*

(2) *If G is soluble and every subgroup of G of order 2 is complemented in G, then G is 2-nilpotent.*

***Corollary* 1.6** (Gaschütz [3, IV, Theorem 5.7]). *If every minimal subgroup of a group G is normal in G, then the commutator subgroup G' of G is 2-closed.*

***Corollary* 1.7** (Buckley [4]). *Let G be a group of odd order. If every minimal subgroup of G is normal in G, then G is supersoluble.*

***Corollary* 1.8** (Ballester-Bolinches, Guo [5]). *Let G be a group. If every minimal subgroup of G is complemented in G, then G is supersoluble.*

Recall that a subgroup H of a group G is said to be quasinormal (S-quasinormal) in G if $HE=EH$ for all subgroups E of G ($HP=PH$ for all Sylow subgroups P of G, respectively).

***Proposition* 1.9.** *Let R be the subgroup of a group G generated by the set of all cyclic quasinormal subgroups of G and R_s be the subgroup of G generated by the set of all cyclic S-quasinormal subgroups of G. Then $R_s\le\Sigma_{\mathfrak{U}}(G)$ and R is contained in the intersection of all maximal p-supersoluble subgroups of G for all primes p.*

***Corollary* 1.10** (Shaalan [6]). *Let G be a group and E a normal subgroup of G with supersoluble quotient G/E. Suppose that all minimal subgroups of E and all its cyclic subgroups with order 4 are S-quasinormal in G. Then G is supersoluble.*

Proof. By Proposition 1.9 we have $\psi_e(E)\le\Sigma_{\mathfrak{U}}(G)$. Hence $E\le\Sigma_{\mathfrak{U}}(G)$ by Theorem A (g) and so G is supersoluble by Theorem A (c).

Recall that the hyper-generalized-center $genz^*(G)$ of G coincides with the largest term of the chain of subgroups

$$1=Q_0\le Q_1\le Q_2\le\cdots,$$

where $Q_i(G)/Q_{i-1}(G)$ is the subgroup of $G/Q_{i-1}(G)$ generated by the set of all cyclic S-quasinormal subgroups of $G/Q_{i-1}(G)$ (see [2, page 22]).

***Corollary* 1.11** (Agrawal [7]). *The hyper-generalized-center $genz^*(G)$ of G is contained in $\Sigma_{\mathfrak{U}}(G)$.*

Proof. Let R be the subgroup of G generated by the set of all its cyclic S-quasinormal subgroups. By Proposition 1.9 we have $R\le\Sigma_{\mathfrak{U}}(G)$ and hence $\Sigma_{\mathfrak{U}}(G)/R=\Sigma_{\mathfrak{U}}(G/R)$ by Theorem A (e). Thus $genz^*(G/R)=genz^*(G)/R\le\Sigma_{\mathfrak{U}}(G/R)$ and so $genz^*(G)\le\Sigma_{\mathfrak{U}}(G)$.

Based on Theorem C we also proved the following result.

***Theorem* 1.12.** *Suppose that $\mathfrak{F}$ is either the class of all soluble groups or the class of all p-decomposable groups for some prime p. Suppose that G has three subgroups A_1, A_2 and A_3 whose indices $|G:A_1|$, $|G:A_2|$, $|G:A_3|$ are pairwise coprime. If $A_i\cap A_j\le\Sigma_{\mathfrak{F}}(A_i)\cap\Sigma_{\mathfrak{F}}(A_j)$ for all $i\ne j$, then $G\in\mathfrak{F}$.*

***Corollary* 1.13** (Wielandt [8]). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G:A_1|$, $|G:A_2|$, $|G:A_3|$ are pairwise coprime, then G is itself soluble.*

***Corollary* 1.14** (Kegel [9]). *If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G:A_1|$, $|G:A_2|$, $|G:A_3|$ are pairwise coprime, then G is itself nilpotent.*

In view of Proposition 1.9 and Corollary 1.10 the following natural questions arise:

(I) *Is there a group G such that $genz^*(G)\ne\Sigma_{\mathfrak{U}}(G)$?* (see [2, page 22] or [7, page 19])

(II) *Is there a group G such that $genz^*(G)$ is not contained in the intersection of all maximal p-supersoluble subgroups of G?*

The following examples give positive answers to these questions.

Example. Let p, q and r be primes such that $p\ne q\ne r$. Let C_r be a group of order r and Q be a simple $\mathbb{F}_q[C_r]$-module which is faithful for C_r. Let $H=Q\rtimes C_r$ and P be a simple $\mathbb{F}_p[H]$-module which is faithful for QC_r. Finally, let $G=P\rtimes H$. Then $P=C_G(P)$, so $P=F(G)$.

1. Suppose that q divides $p-1$, r divides $p-1$ and r does not divide $q-1$ ($p=31$, $q=5$ and r=3, for instance). Then the maximal subgroups PQ and PC_r of G are supersoluble but the subgroup H is not supersoluble. Hence $\Sigma_{\mathfrak{U}}(G)=P$. Suppose that $genz^*(G)\neq 1$. Then G has a non-identity cyclic S-quasinormal subgroup V. Hence V is subnormal in G by [10], so $V\leq P=F(G)$ by [11]. Hence $QC_r\leq N_G(V)$ and so V is normal in G. But then $V=P$ and hence $G/C_G(P)\simeq QC_r$ is cyclic. This contradiction shows that $genz^*(G)=1\neq\Sigma_{\mathfrak{U}}(G)$.

2. Now suppose that $p=r$ and q divides $p-1$. In this case $PQ=O^p(G)$ and QC_r is not supersoluble. We shall show that $P=genz^*(G)$. Indeed, since q divides $p-1$, then PQ is supersoluble by [2, Chapter 1, Theorem 1.4]. Hence by Maschke's theorem, $P=P_1\times P_2\times\ldots\times P_t$, where P_i is normal in $PQ=O^p(G)$ and $|P_i|=p$ for all $i=1,2,\ldots,t$. Hence P_i is S-quasinormal in G by Lemma 4.3 below and so $P\leq genz^*(G)$. On the other hand, if E is a cyclic S-qusinormal subgroup of G, then E is subnormal in G and hence $E\leq P=F(G)$. Thus P is the subroup of G generated by the set of all its cyclic S-quasinormal subrgroups. It is clear also that any cyclic S-quasinormal subgroup of $QC_r\simeq G/P$ is identity. Hence $P=genz^*(G)$. Since all maximal subgroups of G are p-supersoluble, then the intersection Σ of all such subgroups is identity. Hence $genz^*(G)\nleq\Sigma$.

2 On the $\mathfrak{F}$-hypercentre and the intersection of all maximal $\mathfrak{F}$-subgroups of a finite group

A chief factor H/K of a group G is called $\mathfrak{F}$-central provided $(H/K)\rtimes(G/C_G(H/K))\in\mathfrak{F}$ (see [12, p. 127–128] or [13, Def. 2.4.3]). The product of all normal subgroups of G whose G-chief factors are $\mathfrak{F}$-central in G is called the $\mathfrak{F}$-hypercentre of G and denoted by $Z_{\mathfrak{F}}(G)$ [14, p. 389].

Note that for any subgroup $E\in\mathfrak{F}$ of G, the subgroup $\Sigma_{\mathfrak{F}}(G)E$ belongs to $\mathfrak{F}$ as well (see Theorem A 3.1 (d)). Moreover, we shall also show that $Z_{\mathfrak{F}}(G)\leq\Sigma_{\mathfrak{F}}(G)$ and the condition $G/\Sigma_{\mathfrak{F}}(G)\in\mathfrak{F}$ always implies $G\in\mathfrak{F}$ (Theorem A (c), (h)). Therefore the subgroup $\Sigma_{\mathfrak{F}}(G)$ is similar in properties to the subgroup $Z_{\mathfrak{F}}(G)$. Nevertheless, the following simple example shows that in general we have $Z_{\mathfrak{F}}(G)\neq\Sigma_{\mathfrak{F}}(G)$.

Example. Let p, q and r be primes such that q divides $p-1$, r divides $p-1$ and r does not divide $q-1$ ($p=31$, $q=5$ and $r=3$, for instance). Let C_r be a group of order r and Q be a simple $\mathbb{F}_q[C_r]$-module which is faithful for C_r. Let $H=Q\rtimes C_r$ and P be a simple $\mathbb{F}_p[H]$-module which is faithful for H. Finally, let $G=P\rtimes H$. Then the maximal subgroups PQ and PC_r of G are supersoluble but the subgroup H is not supersoluble. Hence $\Sigma_{\mathfrak{U}}(G)=P\neq Z_{\mathfrak{U}}(G)=1$.

In connection with these observations it is natural to ask:

Question. *What one can say about a hereditary saturated formation $\mathfrak{F}$ if the equality $Z_{\mathfrak{F}}(G)=\Sigma_{\mathfrak{F}}(G)$ is true in each group G?*

In order to give the answer to this question we write $\mathfrak{F}(p)$ to denote the intersection of all formations containing the set $\{G/O_{p',p}(G)\mid G\in\mathfrak{F}\}$ and we write $F(p)$ to denote the class of groups G such that $G^{\mathfrak{F}(p)}$ is a p-group. It is not difficult to show that $F(p)$ is a formation for all primes p, and any extension of any p-group P by a group $G\in F(p)$ belongs to $F(p)$ as well.

We say that the formation $\mathfrak{F}$ is a *formation with Property* (*) if $\mathfrak{F}$ contains each group whose maximal subgroups belongs to $F(p)$, at least for one prime p. We say that $\mathfrak{F}$ is a *formation with Property (*) in the class of all soluble groups* if $\mathfrak{F}$ contains each soluble group whose maximal subgroups belongs to $F(p)$, at least for one prime p.

The following our theorems give an answer to above question.

Theorem B. *The equality $Z_{\mathfrak{F}}(G)=\Sigma_{\mathfrak{F}}(G)$ is true in each group G if and only if $\mathfrak{F}$ is a formation with Property (*).*

Theorem C. *The equality $Z_{\mathfrak{F}}(G)=\Sigma_{\mathfrak{F}}(G)$ is true in each soluble group G if and only if $\mathfrak{F}$ is a formation with Property (*) in the class of all soluble groups.*

The proofs of these two theorems consist of many steps and are based on the following lemmas.

Lemma 2.1 [15, Theorem 1]. *Let $\mathfrak{F}$ be a formation containing all nilpotent groups. Then $\mathfrak{F}$ is saturated if and only if $F(p)\subseteq\mathfrak{F}$ for all primes p.*

From Thereom 17.14 in [12] we get

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation containing all nilpotent groups. A chief factor H/K of a group G is $\mathfrak{F}$-central if and only if*

$G/C_G(H/K) \in F(p)$ for all prime divisors p of $|H/K|$.

In view of [15, Remark 1] and Proposition 3.16 in [14, IV] we get also

***Lemma* 2.3.** *For any prime p, the formation $F(p)$ is hereditary.*

We shall need in our proofs the following properties of the $\mathfrak{F}$-hypercentre.

***Lemma* 2.4.** *Let G be a group and $H \le G$.*

(1) *If H is normal in G, then $Z_{\mathfrak{F}}(G)H/H \le Z_{\mathfrak{F}}(G/H)$.*

(2) $Z_{\mathfrak{F}}(G) \cap H \le Z_{\mathfrak{F}}(H)$.

(3) *If $G/Z_{\mathfrak{F}}(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. (1) This follows from the G-isomorphism $Z_{\mathfrak{F}}(G)H/H \simeq Z_{\mathfrak{F}}(G)/Z_{\mathfrak{F}}(G) \cap H$.

(2) Let $1 = Z_0 < Z_1 < ... < Z_{t-1} < Z_t = Z_{\mathfrak{F}}(G)$ be a chief series of G below $Z_{\mathfrak{F}}(G)$ and $C_i = C_G(Z_i/Z_{i-1})$. Let p be a prime divisor of $|Z_i \cap H/Z_{i-1} \cap H| = |Z_{i-1}(Z_i \cap H)/Z_{i-1}|$. Then p divides $|Z_i/Z_{i-1}|$, so $G/C_i \le F(p)$ by Lemma 2.2. Hence by Lemma 2.3, $H/H \cap C_i \simeq C_iH/C_i \in F(p)$. But $H \cap C_i \le C_H(Z_i \cap H/Z_{i-1} \cap H)$. Hence $H/C_H(Z_i \cap H/Z_{i-1} \cap H) \in F(p)$ for all primes p dividing $|Z_i \cap H/Z_{i-1} \cap H|$. Thus $Z_{\mathfrak{F}}(G) \cap H \le Z_{\mathfrak{F}}(H)$ by Lemma 2.2.

(3) This is evident.

The following lemma is a corollary of general results on f-hypercentral action (see [16, Chapter 2] or [14, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

***Lemma* 2.5.** *Let E be a normal p-subgroup of a group G. If $E \le Z_{\mathfrak{F}}(G)$, then $G/C_G(E) \in F(p)$.*

Proof. Let $1 = E_0 < E_1 < ... < E_t = E$ be a chief series of G below E. Let $C_i = C_G(E_i/E_{i-1})$ and $C = C_1 \cap C_2 \cap ... \cap C_t$. Then $C_G(E) \le C$ and by Corollary 3.3 in [17, Chapter 5], $C/C_G(E)$ is a p-group. On the other hand, $G/C_i \in F(p)$ by Lemma 2.2, so $G/C \in F(p)$. Hence $G/C_G(E) \in F(p)$.

***Lemma* 2.6.** *Let G be a group and p a prime such that $O_p(G) = 1$. If G has the only minimal normal subgroup, then there is a simple $\mathbb{F}_p[G]$-module which is faithful for G.*

Proof. Let $A = C_p \wr G = [K]G$, where C_p is a group of order p and K is the base group of the regular product A. Let

$$1 = K_1 < K_2 < ... < K_t = K, \qquad (**)$$

where K_i/K_{i-1} is a chief factor of A for all $i = 1, 2, ..., t$. Let $C_i = C_A(K_i/K_{i-1})$, N be a minimal normal subgroup of G and $C = C_1 \cap C_2 \cap ... \cap C_t$. Suppose that $C_i \cap G \ne 1$ for all $i = 1, 2, ..., t$. Then $N \le C \cap G$. Hence N stabilizes Series (**), so N is a p-group by Corollary 3.3 in [17, Chapter 5], which implies $N \le O_p(G)$. This contradiction shows that for some i we have $C_A(K_i/K_{i-1}) = A$. The lemma is proved.

***Lemma* 2.7.**

(1) *If $\mathfrak{F} = \mathfrak{G}_p\mathfrak{F}$, then $F(p) = \mathfrak{F}$.*

(2) *If $\mathfrak{F} = \mathfrak{N}\mathfrak{H}$ for some non-empty formation $\mathfrak{H}$, then $F(p) = \mathfrak{G}_p\mathfrak{H}$ for all primes p.*

Proof. (1) In view of Lemma 2.1 we need only to prove that $\mathfrak{F} \subseteq F(p)$. Suppose that this is false and let A be a group of minimal order in $\mathfrak{F} \setminus F(p)$. Then $R = A^{F(p)}$ is the only minimal normal subgroup of A and $O_p(A) = 1$. By Lemma 2.6 there is a simple $\mathbb{F}_p[A]$-module P which is faithful for A. Then $G = P \rtimes A \in \mathfrak{G}_p\mathfrak{F} = \mathfrak{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$, a contradiction.

(2) The inclusion $F(p) \subseteq \mathfrak{N}_p\mathfrak{H}$ is evident. Suppose that $\mathfrak{N}_p\mathfrak{H} \not\subseteq F(p)$ and let A be a group of minimal order in $\mathfrak{N}_p\mathfrak{H} \setminus F(p)$. Let L be a minimal normal subgroup of A. Then L is a unique minimal normal subgroup of A and $O_p(A) = 1$. Hence $A \in \mathfrak{H}$ and there is a simple $\mathbb{F}_p[A]$-module P which is faithful for A. Then $G = P \rtimes A \in \mathfrak{G}_p\mathfrak{H} \subseteq \mathfrak{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$, a contradiction. The lemma is proved.

A group G is said to be a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but $H \in \mathfrak{F}$ for every proper subgroup H of G.

In what follows we shall need the following result about minimal non-$\mathfrak{F}$-groups.

***Lemma* 2.8** [16, Chapter VI, Theorem 25.4]. *Let G be a minimal non-$\mathfrak{F}$-group such that $G^{\mathfrak{F}}$ is soluble.*

(a) *$P = G^{\mathfrak{F}}$ is a p-group for some prime p and P is of exponent p or of exponent 4 (if P is a non-abelian 2-group).*

(b) *$P/\Phi(P)$ is a chief factor of G and $(P/\Phi(P)) \rtimes (G/C_G(P/\Phi(P))) \notin \mathfrak{F}$.*

(c) *If P is abelian, then $\Phi(P) = 1$.*

Let H amd K be subgroups of a group G. If $HK = G$, then K is called a supplement of H in G. If, in addition, $HT \ne G$ for all proper subgroups T of K, then K is called a minimal supplement of H in G.

3 Some classes of formations with Property (*)

Classes of soluble groups with limited nilpotent length. Following [14, Chapter VII, Definitions 6.9] we write $l(G)$ to denote the nilpotent length of the group G. Recall that $\mathfrak{N}^r$ is the product of r copies of $\mathfrak{N}$; $\mathfrak{N}^0$ is the class of groups of order 1 by definition. It is well known that $\mathfrak{N}^r$ is the class of all soluble groups G with $l(G) \le r$.

***Proposition* 3.1.** *For any $r \in \mathbb{N}$, the class $\mathfrak{N}^r$ is a hereditary saturated formation with Property (*) in the class of the soluble groups. The formation $\mathfrak{N}$ is a formation with Property (*).*

Proof. Let $\mathfrak{F} = \mathfrak{N}^r$. It is clear that $\mathfrak{F}$ is a hereditary formation. Besides, by [14, Chapter A, Theorem 9.3 (c)], $\mathfrak{F}$ is saturated. Moreover, $F(p) = \mathfrak{N}_p\mathfrak{N}^{r-1}$ by Lemma 2.7 (2). Therefore in the case $r = 1$ the formation $\mathfrak{F} = \mathfrak{N}$ is a hereditary saturated formation with Property (*).

Now suppose that $r > 1$. We shall show that $\mathfrak{F}$ has Property (*) in the class of all soluble groups. Suppose that this is false and let $(\mathfrak{N}^r, G)$ be a counterexample with minimal $r|G|$. Then $G \notin \mathfrak{F}$ and there is a prime p such that every maximal subgroup of G belongs to $F(p)$. Then $\Phi(G) = 1$. Indeed, suppose that $\Phi(G) \ne 1$. Then $|G/\Phi(G)| < |G|$ and every maximal subgroup of $G/\Phi(G)$ belongs to $F(p)$. Hence $G/\Phi(G) \in \mathfrak{F}$ by the choice of G, so $G \in \mathfrak{F}$ since the formation $\mathfrak{F}$ is saturated. This contradiction shows that $\Phi(G) = 1$. Let R and N be any minimal normal subgroup of G. Suppose that $R \ne N$. Then $G = R \rtimes M$ and $N \rtimes L$ for some maximal subgroups M and L of G, so G/R, $G/N \in F(p) = \mathfrak{G}_pF(p)$. Hence $G \simeq G/R \cap N \in F(p) \subseteq \mathfrak{F}$, a contradiction. Therefore $R = N = C_G(R)$ is a unique minimal normal subgroup of G and R is a q-group for some prime $q \ne p$.

Let M_1 be any maximal subgroup of M. Then $RM_1 \in F(p) = \mathfrak{N}_p\mathfrak{N}^{r-1}$. Since $R = C_G(R)$, $O_{q'}(RM_1) = 1$. Hence $O_{q',q}(RM_1) = O_q(RM_1)$ and $O_p(RM_1) = 1$. Hence $RM_1 \in \mathfrak{N}^{r-1}$. Thus

$$M_1/M_1 \cap RO_q(M_1) \simeq RM_1/RO_q(M_1) =$$
$$= RM_1/O_q(RM_1) = RM_1/O_{q',q}(RM_1) \in \mathfrak{N}_q\mathfrak{N}^{r-2}.$$

Hence $M_1 \in \mathfrak{N}_q\mathfrak{N}^{r-2}$. Therefore every maximal subgroup of M belongs to $\mathfrak{N}_q\mathfrak{N}^{r-2}$. Hence $M \in \mathfrak{N}^{r-1}$ by the choice of $(\mathfrak{N}^r, G)$. Thus

$$G = [R]M \in \mathfrak{F} = \mathfrak{N}^r.$$

This contradiction completes the proof of the proposition.

***Proposition* 3.2.** *Let $\{\pi_i \mid \imath \in I\}$ be any partition of $\mathbb{P}$ and $\mathfrak{F}$ the class of all groups G such that $G \in \mathfrak{F}$ if and only if G is the direct product of its Hall π_i-subgroups. Then $\mathfrak{F}$ is a hereditary saturated formation with Property (*).*

Proof. It is clear that the class $\mathfrak{F}$ is closed under taking subgroups, homomorphic images and direct products. Hence $\mathfrak{F}$ is a hereditary formation. Moreover, this formation $\mathfrak{F}$ is saturated. We show that for any prime p, $F(p) = \mathfrak{G}_{\pi_i}$, where $p \in \pi_i$. It is clear that $F(p) \subseteq \mathfrak{G}_{\pi_i}$. Suppose that the inverse conclusion is not true and let A be a group of minimal order in $\mathfrak{G}_{\pi_i} \setminus F(p)$. Let L be a minimal normal subgroup of A. Then L is a unique minimal normal subgroup of A and $O_p(A) = 1$. Hence there is a simple $\mathbb{F}_p[A]$-module P which is faithful for A. Then $G = P \rtimes A \in \mathfrak{G}_{\pi_i} \subseteq \mathfrak{F}$, so $A \simeq G/P = G/O_{p',p}(G) \in F(p)$. This contradiction shows that $F(p) = \mathfrak{G}_{\pi_i}$. Now let G be a group such that every maximal subgroup of G belongs to $F(p) = \mathfrak{G}_{\pi_i}$. Then either G belongs to $F(p) \subseteq \mathfrak{F}$ or $|G| = q \notin \pi_i$ is a prime, so again we have $G \in \mathfrak{F}$. Hence $\mathfrak{F}$ is a formation with Property (*).

***Proposition* 3.3.** *Let $\{\pi_i \mid \imath \in I\}$ be any partition of $\mathbb{P}$ and $\mathfrak{F}$ be a class of all soluble groups G such that $G \in \mathfrak{F}$ if and only if G is the direct product of its Hall π_i-subgroups. Then $\mathfrak{F}$ is a hereditary saturated formation with Property (*) in the class of all soluble groups.*

Proof. See the proof of Proposition 3.2.

Lattice formations. A subgroup H is said to be $\mathfrak{F}$-subnormal in a group G if either $H = G$ or there exists a chain of subgroups

$$H = H_0 < H_1 < \ldots < H_t = G$$

such that H_{i-1} is a maximal subgroup of H_i and $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 1,2,\ldots,t$. A formation $\mathfrak{F}$ is said to be a lattice formation (see [18, Section 6]) if the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the lattice of all subgroups in every group.

We use $\mathfrak{S}$ to denote the class of all soluble groups.

***Proposition* 3.4.** *Every lattice formation $\mathfrak{F}$ with $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$ is a hereditary saturated formation with Property (*) in the class of all soluble groups.*

Proof. By [18, Corollary 6.3.1], there exists a partition $\{\pi_i \mid \imath \in I\}$ of $\mathbb{P}$ such that $G \in \mathfrak{F}$ if and only if G is the direct product of its Hall π_i-subgroups. Hence by Proposition 3.3, $\mathfrak{F}$ is a hereditary saturated formation with Property (*) in the class of all soluble groups.

***Corollary* 3.5.** *If either $\mathfrak{F} = \mathfrak{N}^r$, for some $r \in \mathbb{N}$, or $\mathfrak{F}$ is a lattice formation with $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$, then $Z_{\mathfrak{F}}(G) = \Sigma_{\mathfrak{F}}(G)$ in every soluble group G.*

From Proposition 3.2 we also get

***Corollary* 3.6.** *If either $\mathfrak{F} = \mathfrak{N}$ is the class of all nilpotent groups or $\mathfrak{F}$ is the class of all p-decomposable groups, for some prime p, then $Z_{\mathfrak{F}}(G) = \Sigma_{\mathfrak{F}}(G)$ in every group G.*

***Proposition* 3.7.** *Let $\mathfrak{F}$ be the class of all groups with nilpotent the commutator subgroup G'. Then $\mathfrak{F}$ is a hereditary saturated formation with Property (*).*

Proof. Suppose that this proposition is false and let G be a counterexample with minimal $|G|$. Then G' is not nilpotent and there is a prime p such that every maximal subgroup of G belongs to $F(p)$. It is clear that $\mathfrak{F} = \mathfrak{N}\mathfrak{A}$, where $\mathfrak{A}$ is the formation of all abelian groups. Hence by Lemma 2.7 (2), $F(p) = \mathfrak{G}_p\mathfrak{A}$ for all primes p. First we show that G is soluble. Suppose that this is false. Then for every Schmidt subgroup H of G we have $H \neq G$. Let $q \neq p$ be any prime divisor of $|G|$. Suppose that G is not q-nilpotent. Then G has a q-closed Schmidt subgroup $H = Q \rtimes R$ [3, Chapter IV, Satz 5.4], where Q is a Sylow q-subgroup of H, R is a cyclic Sylow r-subgroup of H. Since $H \neq G$, $H \leq M$, where $M \in F(p)$ is a maximal subgroup of G. Then $M' \leq O_p(M)$ and hence $H' \leq Q \cap O_p(H) = 1$. Hence H is abelian. This contradiction shows that G is q-nilpotent for all primes $q \neq p$, so $G^{\mathfrak{N}}$ is a Sylow p-subgroup of G. Hence G is soluble. Let R be any minimal normal subgroup of G. Then every maximal subgroup of G/R belongs to $F(p)$, so $(G/R)' \leq F(G/R)$ by the choice of G. Therefore R is the only minimal normal subgroup of G and $R \not\leq \Phi(G)$. Hence $G = R \rtimes M$ for some maximal subgroups M of G, $R = C_G(R) = O_q(G)$ for some prime $q \neq p$ (see the proof of Proposition 3.1). Let M_1 be any maximal subgroup of M. Then $RM_1 \in F(p)$, so RM_1 is abelian since $R = C_G(R)$. Hence $M_1 = 1$, so $G' = R$ is nilpotent. This contradiction completes the proof of the result.

***Corollary* 3.8.** *If $\mathfrak{F}$ is the class of all groups with nilpotent the commutator subgroup G'. Then $Z_{\mathfrak{F}}(G) = \Sigma_{\mathfrak{F}}(G)$ in every group G.*

4 Some classes of formations not having Property (*)

***Corollary* 4.1** *Suppose that for some prime p we have $F(p) = \mathfrak{F}$. Then $\mathfrak{F}$ does not have Property (*).*

Proof. Let G be a minimal non-$\mathfrak{F}$-group. Then $G \notin \mathfrak{F}$ but every maximal subgroup of G is in $\mathfrak{F} = F(p)$. Hence $\mathfrak{F}$ does not have Property (*).

Similarly one can prove the following

***Corollary* 4.2.** *Suppose that $\mathfrak{F} \subseteq \mathfrak{S}$ and for some prime p we have $F(p) = \mathfrak{F}$. Then $\mathfrak{F}$ does not have Property (*) in the class of all soluble groups.*

***Corollary* 4.3** *Suppose that $\mathfrak{F}$ is one of the following formations:*

(1) *The formation of all p-soluble groups.*

(2) *The formation of all p-supersoluble groups.*

(3) *The formation of all p-nilpotent groups.*

(4) *The formation of all soluble groups.*

Then $\mathfrak{F}$ does not have Property (*).

Proof. It is clear that for any prime $q \neq p$ we have $\mathfrak{F} = \mathfrak{G}_q\mathfrak{F}$. Hence $F(q) = \mathfrak{F}$ by Lemma 2.7 (1). Now we use Corollary 4.1 (4).

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