Список использованных источников

1 Демидович, Б. П. Лекции по математической теории устойчивости / Б. П. Демидович. – Москва : Наука, 1967. – 472 с.

2 Хартман, Ф. Обыкновенные дифференциальные уравнения / Ф. Хартман. – Москва : Мир, –1990. – 720 с. (Перевод с английского: Hartman Philip. Ordinary Differential Equations. – John Wiley & Sons, New York–London–Sidney, 1964.)

3 Арнольд, В. И. Обыкновенные дифференциальные уравнения: учеб. пособие для вузов / В. И. Арнольд. – 3-е изд. – Москва : Наука, 1984. – 272 с.

4 Хенри, Д. Геометрическая теория полулинейных параболических уравнений / Д. Хенри. – М.: Мир, 1985. – 370 с. (Перевод с английского: Henry Dan. Geometric Theory of semilinear Parabolic Equations. – Springer–Verlag, Berlin–Heidelberg–New York, 1981.

5 Мироненко, В. И. Отражающая функция и классификация периодических дифференциальных систем / В. И. Мироненко // Дифференциальные уравнения. – 1984. –Т. XX, № 9. – С. 1635–1638.

6 Мироненко, В. И. Отражающая функция и периодические решения дифференциальных уравнений / В. И. Мироненко. – Минск : изд-во «Университетское», 1986. – 76 с.

7 Мироненко, В. И. Отражающая функция и исследование многомерных дифференциальных систем / В. И. Мироненко. – Гомель : ГГУ им. Ф. Скорины, 2004. – 196 с.

8 Zhou, Zhengxin. The Theory of Reflecting Function of Differential Equations and Applications / Zhengxin Zhou. – China Machine Press, Beijing. – 2014. – 218 p.

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SOME RESENT TRENDS IN THE THEORY OF HAUSDORFF OPERATORS

Hausdorff operators were introduced by Hardy on the segment, and by Liflyand and Moricz [1] on the whole real line. Later Liflyand and Lerner [2] considered their multidimensional generalizations. Now it is an active research area. It is enough to note that the Google search by request "Hausdorff operator" gives more then 1 200 000 results. See also survey articles [3, 4] for historical remarks and the state-of-the-art up to 2014.

This note is a survey of some recent results in this theory. We correct also the statements of Theorem 2 in [5] and Lemma 2 and Theorem in [6].

The n-dimensional Hausdorff operator looks as follows:

$$(\mathcal{H}f)(x) = \int_{\mathbb{R}^m} \Psi(u) f(A(u)x) du,$$

where $\Psi : \mathbb{R}^m \to \mathbb{C}$ is a locally integrable function, $A(u)$ stands for a family of non-singular $n \times n$-matrices, $x \in \mathbb{R}^n$, a column vector.

In [7] the next generalization of this notion to topological groups was proposed.

Definition 1 [7]. Let (Ω, μ) be a measure space, G a topological group, $A : \Omega \to \mathrm{Aut}(G)$ a measurable map, and $\Psi \in L^1_{loc}(\Omega)$. We define the *Hausdorff operator* with the kernel Ψ over the group G by the formula

$$(\mathcal{H}_{\Psi,A} f)(x) = \int_{\Omega} \Psi(u) f(A(u)(x)) d\mu(u).$$

The classical definition of a Hausdorff operator on $\mathbb{R}^n$ mentioned above, the Harish-Chandra transform, and the Delsarte generalized shift are special cases of this notion.

Hereafter G stands for a locally compact metrizable group.

The following lemma is a corrected version of Lemma 2 from [6].

Lemma 1. *There is a left invariant metric ρ' which is compatible with the topology of G such that every automorphism $A \in \mathrm{Aut}(G)$ is Lipschitz with respect to every left invariant metric ρ that is strongly equivalent to ρ'. Moreover, one can choose the Lipschitz constant to be*

$$L_A = \kappa_\rho \mathrm{mod} A,$$

where the constant κ_ρ depends on the metric ρ only.

The next theorem is a corrected version of the main result in [6].

Theorem 1 *Let a left invariant metric ρ be as in Lemma 1, the doubling condition holds for the corresponding metric measure space (G, ρ, ν), $d = \dim G$, and $k(u) = \mathrm{mod}(A(u))^{-1}$. For $\Psi \in L^1(\Omega, k^d d\mu)$ the Hausdorff operator $\mathcal{H}_{\Psi,A}$ is bounded on the real Hardy space $H^1(G)$ and*

$$\| \mathcal{H}_{\Psi,A} \| \le \mathrm{Const} \| \Psi \|_{L^1(\Omega, k^d d\mu)}.$$

Theorem 2 [7]. *Let F be a locally compact field equipped with the norm (e.g., $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{Q}_p$). For the Hausdorff operator $\mathcal{H}_{\Psi,A}$ on the real Hardy space $H^1(F^n)$ the next estimate holds*

$$\| \mathcal{H}_{\Psi,A} \| \le \mathrm{Const} \int_{\Omega} |\Psi(u)| \| A(u)^{-1} \|^n d\mu(u).$$

In [5] and [8] the following definition of a Hausdorff operator over homogeneous spaces of groups was proposed.

For a compact subgroup K of G from now on we put $\dot{x} := xK$, $\dot{A}(\dot{x}) := \pi_K(A((x))$ ($A \in \mathrm{Aut}(G)$, $x \in G$), and

$$\mathrm{Aut}_K(G) := \{\dot{A} : A \in \mathrm{Aut}(G), A(K) = K\}.$$

Definition 2 [5]. Let (Ω, μ) be a measure space, $(\dot{A}(u))_{u \in \Omega} \subset \mathrm{Aut}_K(G)$ a family of homeomorphisms of G/K, and $\Psi \in L^1_{loc}(\Omega, \mu)$. For a function f on G/K we define a *Hausdorff operator on* G/K by the formula

$$(\mathcal{H}_{\Psi,\dot{A}} f)(\dot{x}) := \int_{\Omega} \Psi(u) f(\dot{A}(u)(\dot{x})) d\mu(u).$$

Example 1. One can identify the Lobachevsky plane $\mathbb{H}^2$ with the homogeneous space $SL(2)/SO(2)$. If we in turn identify $\mathbb{H}^2$ with the upper half-plane in $\mathbb{C}$, a Hausdorff operator is as follows:

$$(\mathcal{H}_\Psi f)(z) = \int_0^{2\pi} \Psi(\theta) f\left(\frac{z\cos\theta + \sin\theta}{-z\sin\theta + \cos\theta}\right) d\mu(\theta).$$

Several results on the boundedness of such operators were obtained by the author.

Theorem 3 [5]. Suppose that the conditions of Definition 2 are fulfilled, $p \in [1, \infty]$, and

$$\|\Psi\|_{p,A} := \int_\Omega |\Psi(u)|\,(\mathrm{mod}A(u))^{-1/p}\, d\mu(u) < \infty.$$

Then $\mathcal{H}_{\Psi,\dot{A}}$ is bounded on $L^p(G/K)$ and

$$\|\mathcal{H}_{\Psi,\dot{A}}\|_{\mathcal{L}(L^p(G/K))} \le \|\Psi\|_{p,A}.$$

Definition 3 [5]. We define the *Hardy space* $H^1(G/K)$ as a space of such functions $f = \dot{g}$ on G/K that g admits an atomic decomposition of the form $g = \sum_{j=1}^{\infty} \alpha_j a_j$, where a_j are right-K-invariant $(1, r)$ atoms and $\sum_{j=1}^{\infty} |\alpha_j| < \infty$. In this case,

$$\|f\|_{H^1(G/K)} := \inf \sum_{j=1}^{\infty} |\alpha_j|,$$

and infimum is taken over all decompositions above of g. Thus a function $f = \dot{g}$ from $H^1(G/K)$ admits an atomic decomposition $f = \sum_{j=1}^{\infty} \alpha_j \dot{a}_j$ such that $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, and $\|f\|_{H^1(G/K)} = \|g\|_{H^1(G)}$.

Theorem 4 [8] *If $G \neq K$ the space $H^1(G/K)$ is nontrivial and Banach.*

Thereafter we put

$$k(u) := \mathrm{mod}A(u)^{-1}.$$

The following theorem is a corrected version of the Theorem 2 in [5].

Theorem 5 *Let a left invariant metric ρ be as in Lemma 1 and the doubling condition holds for the corresponding metric measure space (G, ρ, ν), $d = \dim G$. Under assumptions of Definition 2 let (Ω, q, μ) be a σ-compact quasi-metric space with positive Radon measure μ. Then for the operator $\mathcal{H}_{\Psi,\dot{A}}$ the next estimate holds*

$$\|\mathcal{H}_{\Psi,\dot{A}}\|_{\mathcal{L}(H^1(G/K))} \le \mathrm{Const}\|\Psi\|_{L^1(k^d\mu)}.$$

Theorem 6 [8]. *Let Ω be σ-compact quasi-metric space with positive Radon measure μ. Let G be a locally compact group with left Haar measure ν and a left invariant metric ρ as in Lemma 1 and the local doubling and approximate midpoint conditions hold. Let the family $(\dot{A}(u))_{u\in\Omega} \subset \mathrm{Aut}_K(G)$ satisfies $k(u) \le C$. Then for $\Psi \in L^1(\Omega, k^{1/r}\mu)$ ($r \in (1, \infty]$) the Hausdorff operator $H_{\Psi,\dot{A}}$ is bounded on the real Hardy space $H^1(G/K)$ and for some constant $\gamma_{\rho,C} > 0$ the next estimate holds*

$$\| \mathcal{H}_{\Psi,\dot{A}}\| \le \gamma_{\rho,C}\| \Psi\|_{L^1(\Omega,k^{1/r}\mu)} .$$

Until 2019 all known results on general Hausdorff operators refer to the boundedness of such operators in various settings only (exceptions are several papers in which some spectra were calculated for special cases in the one-dimensional setting). In particular, multidimensional normal Hausdorff operators have not been studied. Below we describe spectral representation for such operators. In [9] and [10] the notion of a matrix symbol of a normal Hausdorff operator was introduced and the following theorem proved.

Theorem 7 [9], [10]. *Let* $A(u)$ *be a commuting family of non-singular self-adjoint* $n\times n$ *-matrices, and* $|\det A(u)|^{-1/2}\ \Psi(u)\in L^1(\Omega)$. *Then*

(i) *the Hausdorff operator* $\mathcal{H}_{\Psi,A}$ *in* $L^2(\mathbb{R}^n)$ *with matrix symbol* Φ *is normal and unitary equivalent to the operator* M_Φ *of multiplication by the normal matrix* Φ *in the space* $L^2(\mathbb{R}^n;\mathbb{C}^{2^n})$ *of* $\mathbb{C}^{2^n}$ *-valued functions;*

(ii) *the spectrum of* $\mathcal{H}_{\Psi,A}$ *equals to the spectrum* $\sigma(\Phi)$ *of* Φ *in the matrix algebra* $\mathrm{Mat}_{2^n}(C_b(\mathbb{R}^n))$,

$$\sigma(\mathcal{H}_{\Psi,A})=\{\lambda\in\mathbb{C}:\inf_{s\in\mathbb{R}^n}|\det(\lambda-\Phi(s))|=0\};$$

(iii) *the point spectrum* $\sigma_p(\mathcal{H}_{\Psi,A})$ *of* $\mathcal{H}_{\Psi,A}$ *consists of such complex numbers* λ *for which the closed set*

$$E(\lambda):=\{s\in\mathbb{R}^n:\det(\lambda-\Phi(s))=0\}$$

has positive Lebesgue measure. The residual spectrum of $\mathcal{H}_{\Psi,A}$ *is empty.*

Corollary [10]. *Let the matrices* $A(u)$ *be positive definite. Then the operator* $\mathcal{H}_{K,A}$ *is unitary equivalent to the operator of coordinate-wise multiplication by a function* $\varphi\in C_b(\mathbb{R}^n)$ (the scalar symbol of $\mathcal{H}_{K,A}$) *in the space* $L^2(\mathbb{R}^n,\mathbb{C}^{2^n})$. *In particular,*

(i) *the spectrum, the point spectrum, and the continuous spectrum of* $\mathcal{H}_{K,A}$ *equal to the spectrum (i. e., to the closure of the range of* φ*), to the point spectrum, and to the continuous spectrum of the operator* M'_φ *of multiplication by* φ *in* $L^2(\mathbb{R}^n)$ *respectively, the residual spectrum of* $\mathcal{H}_{K,A}$ *is empty;*

(ii) $\| \mathcal{H}_{K,A}\| = \sup|\varphi|$.

The problem of compactness of Hausdorff operators was posted in [3]. There is a conjecture that a nontrivial Hausdorff operator on Lebesgue space is non-compact. In [11] we formulated the more strong conjecture that a nontrivial Hausdorff operator on Lebesgue space is non-Riesz and proved the following.

Theorem 8 [11] *Let* $A(u)$ *be a commuting family of real self-adjoint* $n\times n$*-matrices* ($u\in\mathrm{supp}(\Psi)$), *and* $(\det A(u))^{-1/p}\Psi(u)\in L^1(\Omega)$. *Then every nontrivial Hausdorff operator* $\mathcal{H}_{\Psi,A}$ *in* $L^p(\mathbb{R}^n)$ *(*$1\le p\le\infty$*) is a non-Riesz operator* (*and in particular it is not a sum of a quasinilpotent and compact operator*).

References

1 Liflyand, E. The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$ / E. Liflyand, F. Moricz // Proc. Amer. Math. Soc. – 2000. – Vol. 128. – P. 1391–1396.

2 Lerner, A. Multidimensional Hausdorff operators on the real Hardy space / A. Lerner and E. Liflyand // J. Austr. Math. Soc. – 2007. – Vol. 83. – P. 79–86.

3 Liflyand, E. Hausdorff operators on Hardy spaces / E. Liflyand // Eurasian Math. J. – 2013. – Vol. 4. – № 4. – P. 101–141.

4 Chen, J. Hausdorff operators on Euclidean space (a survey article) / J. Chen, D. Fan, S. Wang, // Appl. Math. J. Chinese Univ. Ser. B. – 2014. – Vol. 28. – №4. – P. 548–564.

5 Mirotin, A. R. Hausdorff operators on homogeneous spaces of locally compact groups / A. R. Mirotin // Journal of the Belarusian State University. Math. Mech. – 2020. – № 2. – P. 28–35.

6 Mirotin, A. R. Addendum to «Boundedness of Hausdorff operators on Hardy spaces H^1 over locally compact groups» / A. R. Mirotin // J. Math. Anal. Appl. – 2019. – Vol. 479. – № 1. – P. 872–874.

7 Mirotin, A. R. Boundedness of Hausdorff operators on Hardy spaces H^1 over locally compact groups / A. R. Mirotin // J. Math. Anal. Appl. – 2019. – Vol. 473. – P. 519–533. – (Preprint arXiv:1808.08257v2 [math. FA] 1 Sep 2018).

8 Mirotin, A. R. Hausdorff operators on real Hardy spaces H^1 over homogeneous spaces with local doubling property / A. R. Mirotin // Analysis. Math. (submitted).

9 Mirotin, A. R. The structure of normal Hausdorff operators on Lebesgue spaces / A. R. Mirotin // Forum Math. – 2020. – Vol. 32. – № 1. – P. 111–119.

10 Mirotin, A. R. On the Structure of Normal Hausdorff Operators on Lebesgue Spaces / A.R. Mirotin // Functional Anal. and Appl. – 2019. – Vol. 53. – № 4. – P. 261–269.

11 Mirotin, A. R. Is every nontrivial Hausdorff operator on Lebesgue space a non-Riesz operator? To appear in: Operator Theory and Harmonic Analysis / A. R. Mirotin // Vol. 1: New General Trends and Advances of the Theory. Proceedings OTHA 2020, Springer Proceedings in Mathematics and Statistics (2020).