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FINITE GROUPS WITH ABNORMAL SCHMIDT SUBGROUPS

All groups are finite and G always denotes a finite group.

A subgroup H of G is said to be (1) *abnormal* in G if $x \in \langle H, H^x\rangle$ for all $x \in G$.

We say that G is an *SA-group* if G is a non-nilpotent group in which every Schmidt subgroup is abnormal.

Recall that G is said to be an *SC-group* if G is soluble and the set of all Carter subgroups of G coincides with the set of all systems normalizers of G.

We proved the following two results.

Theorem 1. *G is a soluble SA-group if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is the nilpotent residual of G, and the following conditions hold:*

(i) *D is nilpotent, $M = \langle x\rangle$ is a cyclic Sylow q-subgroup of G for some prime q dividing $|G|$, and $F(G) = D\langle x^q\rangle$.*

(ii) *For every prime p dividing $|D|$ and for the Sylow p-subgroup subgroup D_p of D we have*

(a) *$Q \leq N_G(D_p)$ and $[Q, D_p] \neq 1$.*

(b) *$D_p = (D_pQ)^{\mathfrak{N}}$.*

(c) *G has normal subgroups $Z = \Phi_t < \Phi_{t-1} < \dots < \Phi_2 < \Phi_1 < \Phi_0 = D_p$ ($t = t(p)$), where $\Phi_i = \Phi(\Phi_{i-1})$ for all $i = 1,...,t$, such that $[Q, Z] = 1$ and $[Q, \Phi_i] \neq 1$ for all $i < t$.*

(d) *For every i, the factor Φ_i/Φ_{i+1}, regarded as a G-module, is completely reducible.*

(e) *All chief factors H/K of G between Z and P are G-isomorphic with $C_G(H/K) = F(G)$ and $|H/K| = p^n$, where n is the smallest integer such that q divides $p^n - 1$.*

(f) *G is an SC-group and ZQ is a Carter subgroup of G and $Z = N_G(Q)\cap D \leq Z_\infty(G)$. Moreover, if $Z \neq 1$, then $D = D_p$ is a Sylow subgroup of G.*

Furthermore, Conditions (b), (c), (d) and (e) hold for every p-subgroup P of G such that $Q \leq N_G(P)$ and $[Q, P] \neq 1$.

(iii) *A subgroup C of G is a Carter subgroup if and only if C is a maximal abnormal subgroup of some Schmidt subgroup of G. Moreover, if H is a Schmidt subgroup of G, then a maximal abnormal subgroup of H is a Carter subgroup of G.*

Theorem 2. *Assume that G is a non-soluble SA-group. Then the following hold.*

(a) *G is quasi-simple and $|Z(G)| \in\in \{2,4,3,9\}$.*

(b) *Every local subgroup $N_G(P)$ of G, where $|P| \notin \{2,4,3\}$, is soluble.*

Example. (1) Let E be an extraspecial group of order 3^7 and exponent 3. Then $\mathrm{Aut}(E)$ contains an element α of order 7 which operates irreducibly on E/Z_E and centralizes $Z(E)$ by Lemma 20.13 in [1, Ch. A]. Let E_1 and E_2 be two copies of the group E and let $P = E_1 \curlyvee E_2 := (E_1 \times E_2)/D$, where $D = \{(a,a^{-1}) \mid a \in Z(E)\}$ be the direct product of the groups E_1 and E_2 with joint center (see [2, p. 49]). Then α induces an automorphism of order 7 on P and for the group $G_1 = P \rtimes \langle\alpha\rangle$ all Conditions (i), (ii), (iii) and (iv) are fulfilled with $t = 1$.

(2) Let P be a cyclic group of order 29^3. Then Aut(E) contains an element α of order 7. Then Conditions (i), (ii), (iii) and (iv) are fulfilled for the group $G_2 = P \rtimes \langle\alpha\rangle$ with $t = 3$ and $Z = 1$.

(3) Let $\varphi_i : G_i \longrightarrow \langle\alpha\rangle$ be an epimorphism of G_i onto $\langle\alpha\rangle$ and let $G = G_1 \curlywedge G_2 = \{(g_1,g_2) \mid g_i \in G_i,$ $\emptyset_1\ (g_1) = \emptyset_2\ (g_2)$‘$\}$ be the direct product of the groups G_1 and G_2 with joint factorgroup $\langle\alpha\rangle$ (see [2, p. 50]). Then Conditions (i), (ii), (iii) and (iv) are fulfilled for G by Parts (1) and (2).

References

1 Doerk, K. Finite soluble groups / K. Doerk, T. Hawkes // Walter de Gruyter, Berlin–New York. – 1992. – 891 p.

2 Huppert, B. Endliche Gruppen I / B. Huppert. – Berlin–Heidelberg–New York : Springer-Verlag. – 1967 – 793 p.