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Finite groups with generalized $\mathbb{P}$-subnormal second maximal subgroups

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A subgroup H of a group G is said to be K-$\mathbb{P}$-*subnormal in* G [A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov, On finite groups with almost all K-$\mathbb{P}$-subnormal Sylow subgroups, in *Algebra and Combinatorics: Abstracts of Reports of the International Conference on Algebra and Combinatorics on Occasion the 60th Year Anniversary of A. A. Makhnev* (Ekaterinburg, 2013), pp. 19–20] if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \dots, n$. In this paper, we describe finite groups in which every second maximal subgroup is K-$\mathbb{P}$-subnormal.

Keywords: 2-maximal (second maximal) subgroup; soluble group; supersoluble group; minimal nonsupersoluble group; K-$\mathbb{P}$-subnormal subgroup; $\mathfrak{U}$-subnormal subgroup; permutable subgroup.

AMS Subject Classification: 20D10, 20D15, 20D20

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We write $\mathfrak{U}$ to denote the class of all supersoluble groups, $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups N of G with $G/N \in \mathfrak{U}$.

Recall that a subgroup H of G is said to be: (i) $\mathfrak{U}$-*subnormal* in G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$, for $i = 1, \dots, n$; (ii) $\mathfrak{U}$-*subnormal* in the sense of Kegel [15] or K-$\mathfrak{U}$-*subnormal* (see [3, p. 236] in G if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \dots, t$.

A subgroup H of G is called a 2-*maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define 3-*maximal subgroups*, and so on. If H is n-maximal in G but not

n-maximal in any proper subgroup of G, then H is said to be a *strictly* n-maximal subgroup of G.

One of the interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its n-maximal subgroups. In particular, there are many papers in which the structure of groups with given second maximal subgroups are described. One of the earliest publications in this direction is the paper of Huppert [12] who established the supersolubility of G whose all second maximal subgroups are normal. This result was developed by many authors. In particular, it was proved that G is supersoluble if every 2-maximal subgroup of G is either permutable with every maximal subgroup of G (Poljakov [23]) or S-quasinormal in G (Agrawal [1, 27]). In [2], Asaad proved that G is supersoluble if every strictly 2-maximal subgroup of G is normal. Flavell [6] obtained an upper bound for the number of maximal subgroups containing a strictly 2-maximal subgroup and classify the extremal examples.

Among the recent interesting results on 2-maximal subgroups we can mention the paper of Guo and Shum [9], where the solubility of groups is established in which all 2-maximal subgroups enjoy the cover-avoidance property, and the papers of Guo, Shum, Skiba and Li [10, 11, 20], where new characterizations of supersoluble groups in terms of 2-maximal subgroups were obtained. Li [19] gave a classification of nonnilpotent groups whose all 2-maximal subgroups are TI-subgroups. In [4], Ballester-Bolinches, Ezquerro and Skiba obtained a full classification of the groups in which the second maximal subgroups of the Sylow subgroups cover or avoid the chief factors of some of its chief series. Guo, Lutsenko and Skiba [8] gave a description of the groups in which every two second maximal subgroups are permutable. In [21], Lutsenko and Skiba described the groups whose all 2-maximal subgroups are subnormal. In [16], Kniahina and Monakhov studied those groups in which every 2-maximal subgroup permutes with every Schmidt subgroup.

Recall that a subgroup H of G is said to be $\mathbb{P}$-*subnormal in* G [25] if either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for all $i = 1, \ldots, n$.

Another important results on 2-maximal subgroups were obtained by Kovaleva and Skiba in [17, 18] and Monakhov and Kniahina in [22]. In [17], the authors described the groups whose all 2-maximal subgroups are $\mathfrak{U}$-subnormal. In [18], it was obtained a description of the groups with all 2-maximal subgroups $\mathfrak{F}$-subnormal for some saturated formation $\mathfrak{F}$. In [22], the groups with all 2-maximal subgroups $\mathbb{P}$-subnormal were studied.

In this paper, we consider the following generalization of $\mathbb{P}$-subnormality and subnormality.

Definition. A subgroup H of G is said to be K-$\mathbb{P}$-*subnormal in* G (Vasilyev, Vasilyeva and Tyutyanov [26]) if there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \ldots, n$.

It is easy to see that every $\mathfrak{U}$-subnormal subgroup of G is K-$\mathbb{P}$-subnormal in G. Moreover, if G is soluble, then every K-$\mathbb{P}$-subnormal subgroup of G is $\mathfrak{U}$-subnormal in G.

We prove the following result.

Theorem. *The following statements are equivalent*:

(1) *Every second maximal subgroup of G is K-$\mathbb{P}$-subnormal in G.*
(2) *If T is a second maximal subgroup of G such that T is not permutable with some second maximal subgroup of G, then T is K-$\mathbb{P}$-subnormal in G.*
(3) *G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Corollary 1 (Monakhov and Kniahina [22]). *Every second maximal subgroup of G is $\mathbb{P}$-subnormal in G if and only if G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Corollary 1 gives the answer to [25, Problem 1].

Corollary 2 (Kovaleva and Skiba [17] or [18]). *Every second maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if G is either supersoluble or a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.*

Corollary 3 (Huppert [12]). *If every second maximal subgroup of G is normal in G, then G is supersoluble.*

Corollary 4 (Agrawal [1]). *If every second maximal subgroup of G is S-quasinormal in G, then G is supersoluble.*

All unexplained notation and terminology are standard. The reader is referred to [3, 5, 7] if necessary.

2. Proof of Theorem

We need the following lemmas.

Lemma 2.1. *Let H and K be subgroups of G such that H is $\mathfrak{U}$-subnormal in G.*

(1) *If $G^{\mathfrak{U}} \leq K$, then K is $\mathfrak{U}$-subnormal in G* [3, *Lemma* 6.1.7(1)].
(2) *$H \cap K$ is $\mathfrak{U}$-subnormal in K* [3, *Lemma* 6.1.7(2)].

Lemma 2.2. *Let H and K be subgroups of G such that H is K-$\mathbb{P}$-subnormal in G.*

(1) *If N is a normal subgroup of G, then $H \cap N$ is K-$\mathbb{P}$-subnormal in N and HN/N is K-$\mathbb{P}$-subnormal in G/N.*
(2) *If K is K-$\mathbb{P}$-subnormal in H, then K is K-$\mathbb{P}$-subnormal in G.*
(3) *If $G^{\mathfrak{U}} \leq K$, then K is K-$\mathbb{P}$-subnormal in G.*

Proof. Since H is K-$\mathbb{P}$-subnormal in G, there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either H_{i-1} is normal in H_i or $|H_i : H_{i-1}|$ is a prime, for $i = 1, \ldots, n$.

(1) Consider the chain $H \cap N = H_0 \cap N \leq H_1 \cap N \leq \cdots \leq H_n \cap N = N$. If H_{i-1} is normal in H_i, then it is evident that $H_{i-1} \cap N$ is normal in $H_i \cap N$. Assume that $|H_i : H_{i-1}|$ is a prime. If $H_i \cap N = H_{i-1} \cap N$, then $H_{i-1} \cap N$ is normal in $H_i \cap N$. Suppose that $H_i \cap N \neq H_{i-1} \cap N$. Then

$$\begin{aligned}|H_i \cap N : H_{i-1} \cap N| &= |H_i \cap N : H_i \cap N \cap H_{i-1}| \\ &= |H_i \cap N| : |H_i \cap N \cap H_{i-1}| \\ &= |H_i \cap N| : ((|H_i \cap N||H_{i-1}|) : |H_{i-1}(H_i \cap N)|) \\ &= (|H_i \cap N||H_{i-1}(H_i \cap N)|) : (|H_i \cap N||H_{i-1}|) \\ &= |H_{i-1}(H_i \cap N) : H_{i-1}| \neq 1.\end{aligned}$$

Hence $H_{i-1}(H_i \cap N) \neq H_{i-1}$. Therefore, since $H_{i-1} < H_{i-1}(H_i \cap N) \leq H_i$ and $|H_i : H_{i-1}|$ is a prime, we have $H_{i-1}(H_i \cap N) = H_i$. Hence $|H_i \cap N : H_{i-1} \cap N| = |H_i : H_{i-1}|$ is a prime. Thus $H \cap N$ is K-$\mathbb{P}$-subnormal in N.

Now consider the chain $HN/N = H_0N/N \leq H_1N/N \leq \cdots \leq H_nN/N = G/N$. If H_{i-1} is normal in H_i, then $H_{i-1}N/N$ is normal in H_iN/N. Suppose that $|H_i : H_{i-1}|$ is a prime. If $H_{i-1}N/N = H_iN/N$, then $H_{i-1}N/N$ is normal in H_iN/N. Let $H_{i-1}N/N \neq H_iN/N$. Then

$$\begin{aligned}|H_iN/N : H_{i-1}N/N| &= |H_i/H_i \cap N| : |H_{i-1}/H_{i-1} \cap N| \\ &= |H_i : H_{i-1}| : |H_i \cap N : H_{i-1} \cap N| \neq 1,\end{aligned}$$

so $|H_i \cap N : H_{i-1} \cap N| = 1$. Hence $|H_iN/N : H_{i-1}N/N| = |H_i : H_{i-1}|$ is a prime. Thus HN/N is K-$\mathbb{P}$-subnormal in G/N.

(2) Since K is K-$\mathbb{P}$-subnormal in H, there exists a chain of subgroups $K = K_0 \leq K_1 \leq \cdots \leq K_t = H$ such that either K_{i-1} is normal in K_i or $|K_i : K_{i-1}|$ is a prime, for $i = 1, \ldots, t$. By considering the chain $K = K_0 \leq K_1 \leq \cdots \leq K_t = H = H_0 \leq H_1 \leq \cdots \leq H_n = G$, we see that K is K-$\mathbb{P}$-subnormal in G.

(3) Since $G^{\mathfrak{U}} \leq K$, K is $\mathfrak{U}$-subnormal in G by Lemma 2.1(1). Hence K is K-$\mathbb{P}$-subnormal in G. The lemma is proved. □

The next lemma is evident.

Lemma 2.3. *If G is supersoluble, then every subgroup of G is K-$\mathbb{P}$-subnormal in G.*

The following lemma is well-known.

Lemma 2.4. *Let A and B be proper subgroups of G such that $G = AB$. Then $G = AB^x$ and $G \neq AA^x$ for all $x \in G$.*

Proof of Theorem. The implication (1) $\Rightarrow$ (2) is evident.

(2) $\Rightarrow$ (3) Assume that G is not supersoluble. We prove that G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.

First prove that G is soluble. Assume that this is false and let G be a counterexample of minimal order. Suppose that G is simple. Let p be the largest prime divisor of $|G|$ and P a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $P \leq M$. Since G is not soluble and p is the largest prime divisor of $|G|$, $P \neq M$ by well-known Deskins–Janko–Thompson's Theorem [13]. Hence there is a maximal subgroup T of M such that $P \leq T$. Suppose that T is permutable with every 2-maximal subgroup of G. Then $TT^x = T^xT$ for every $x \in G$. Since P is a Sylow p-subgroup of T and P^x is a Sylow p-subgroup of T^x there is $y \in TT^x$ such that $P^y = PP^x$ by [13]. Hence $P = P^x$ for all $x \in G$ and so P is normal in G, a contradiction. Therefore, we can assume that T is not permutable with some 2-maximal subgroup of G. Then T is K-$\mathbb{P}$-subnormal in G by hypothesis and so there exists a proper subgroup H of G such that $T \leq H$ and either H is normal in G or $|G : H| = q$ is a prime. But in the first case we have that G is not simple, a contradiction. Therefore $|G : H| = q$. In view of inclusions $P \leq T \leq H$, $q \neq p$. Since G is simple, $H_G = 1$ and by considering the permutation representation of G on the right cosets of H, we see that G is isomorphic to some subgroup of the symmetric group S_q of degree q. Hence $|G| < q!$ and so q is the largest prime divisor of $|G|$. It follows that $q = p$. The final contradiction shows that G is not a simple group.

Let N be a minimal normal subgroup of G. If every two second maximal subgroups of G/N are permutable, then all maximal subgroups of G/N are nilpotent. Therefore, G/N is either a nilpotent group or a Schmidt group. Hence G/N is soluble in view of [7] or [24]. Let T/N be a 2-maximal subgroup of G/N such that T/N is not permutable with some 2-maximal subgroup E/N of G/N. Then T and E are 2-maximal subgroups of G and $TE \neq ET$. By hypothesis, T is K-$\mathbb{P}$-subnormal in G. Therefore, by Lemma 2.2(1), T/N is K-$\mathbb{P}$-subnormal in G/N. Thus the hypothesis holds for G/N. Hence G/N is soluble by the choice of G and so N is the only minimal normal subgroup of G and N is not soluble. Since G/N is soluble, there exists a normal maximal subgroup M/N of G/N. Then M is a normal maximal subgroup of G. We show that M is supersoluble. Let K be an arbitrary maximal subgroup of M. If K is permutable with every 2-maximal subgroup of G, then $KK^x = K^xK$ for all $x \in M$. In view of maximality of K^x in M, it follows that either $KK^x = M$ or $K^x = KK^x$. But the first case is impossible in view of Lemma 2.4. Therefore, $K^x = KK^x$ for every $x \in M$. It follows that $K \leq K^x$ for every $x \in M$ and hence $K = K_M$ is normal in M. Consequently, $|M : K|$ is a prime in view of maximality of K in M. Suppose that K is not permutable with some 2-maximal subgroup of G. Then K is K-$\mathbb{P}$-subnormal in G, hence K is K-$\mathbb{P}$-subnormal in M by Lemma 2.2(1). Therefore, either K is normal in M or $|M : K|$ is a prime. But in the first case we also see that $|M : K|$ is a prime in view of maximality of K in M. Since K is an arbitrary maximal subgroup of M, it follows that all maximal subgroups of

M have prime indices. Therefore, M is supersoluble and so N. This contradiction completes the proof of solubility of G.

Since G is soluble, every K-$\mathbb{P}$-subnormal subgroup of G is $\mathfrak{U}$-subnormal in G. We show that every maximal subgroup of G is supersoluble. Let M be a maximal subgroup of G and T any maximal subgroup of M. If T is permutable with every 2-maximal subgroup of G, then $TT^x = T^xT$ for all $x \in M$ and so arguing as above we get that $|M : T|$ is a prime. Assume that T is not permutable with some 2-maximal subgroup of G. Then by hypothesis, T is K-$\mathbb{P}$-subnormal in G. Hence T is $\mathfrak{U}$-subnormal in G, which implies the $\mathfrak{U}$-subnormality of T in M by Lemma 2.1(2). Therefore, $M/T_M \in \mathfrak{U}$, hence $|M : T|$ is a prime. Since T is an arbitrary maximal subgroup of M, we have that M is supersoluble. Thus all maximal subgroups of G are supersoluble, so G is a minimal nonsupersoluble group.

By [7] or [24], for some prime p, $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G such that $G_p/\Phi(G_p)$ is a chief factor of G. Moreover, by [7] or [24], G_p is the unique normal Sylow subgroup of G. Suppose that $\Phi(G_p) \neq 1$. Since G is not supersoluble, there exists an $\mathfrak{U}$-abnormal maximal subgroup L of G. By Lemma 2.1(1), $G_p \not\leq L$. It follows that $G = G_pL$ and $L = (G_p \cap L)G_{p'} = \Phi(G_p)G_{p'}$, where $G_{p'}$ is a Hall p'-subgroup of G. Since $\Phi(G_p)$ is a Sylow p-subgroup of L, $\Phi(G_p) \not\leq \Phi(L)$. Hence there is a maximal subgroup T of L such that $\Phi(G_p) \not\leq T$. Then $L = \Phi(G_p)T$ and so $G = G_pL = G_p\Phi(G_p)T = G_pT$. Suppose that T is permutable with every 2-maximal subgroups of G. Then $TT^x = T^xT$ for all $x \in G$. Since $G = G_pT \neq G_p$, for some prime $q \neq p$, there is a Sylow q-subgroup G_q of G such that $G_q \leq T$. Since G_q is a Sylow q-subgroup of T and G_q^x is a Sylow q-subgroup of T^x, there is $y \in TT^x$ such that $G_q^y = G_qG_q^x$ by [13]. Hence $G_q = G_q^x$ for all $x \in G$ and so G_q is normal in G, a contradiction. Thus we can assume that T is not permutable with some 2-maximal subgroup of G. But then T is $\mathfrak{U}$-subnormal in G, hence there is a proper subgroup H of G such that $T \leq H$ and $G/H_G \in \mathfrak{U}$. Therefore, we have $G_p = G^{\mathfrak{U}} \leq H_G$ and so $G = G_pT \leq H$, a contradiction. Thus $\Phi(G_p) = 1$ and hence G_p is a minimal normal subgroup of G. So (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) If G is supersoluble, then every subgroup of G is K-$\mathbb{P}$-subnormal in G by Lemma 2.3. Suppose that G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G. Let T be a 2-maximal subgroup of G and M a maximal subgroup of G such that T is a maximal subgroup of M. Since M is supersoluble, T is K-$\mathbb{P}$-subnormal in M by Lemma 2.3. If M is K-$\mathbb{P}$-subnormal in G, then T is K-$\mathbb{P}$-subnormal in G by Lemma 2.2(2). Assume that M is not K-$\mathbb{P}$-subnormal in G. Then $G^{\mathfrak{U}} \not\leq M$ by Lemma 2.2(3). Therefore, $G = G^{\mathfrak{U}} \rtimes M$ and $G^{\mathfrak{U}}T$ is a maximal subgroup of G. By Lemma 2.2(3), $G^{\mathfrak{U}}T$ is K-$\mathbb{P}$-subnormal in G. Since G is a minimal nonsupersoluble group, $G^{\mathfrak{U}}T$ is supersoluble and so T is K-$\mathbb{P}$-subnormal in $G^{\mathfrak{U}}T$ by Lemma 2.3. Hence by Lemma 2.2(2), T is K-$\mathbb{P}$-subnormal in G. The theorem is proved. □

Proof of Corollary 3. Suppose that this corollary is false and let G be a counterexample of minimal order. Since every 2-maximal subgroup of G is normal in G,

G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G by Theorem. By [7] or [24], $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $G = G_p \rtimes M$. Let T be a maximal subgroup of M. Assume that $T = 1$, then $|M| = q$ is a prime. Let P be a maximal subgroup of G_p. Since by hypothesis P is normal in G and G_p is a minimal normal subgroup of G, $P = 1$. It follows that G is supersoluble, a contradiction. Therefore $T \neq 1$. By hypothesis, T is normal in G. It is evident that the hypothesis holds for G/T. Hence G/T is supersoluble by the choice of G, which implies that $G \simeq G/T \cap G_p$ is supersoluble. This contradiction completes the proof of Corollary 3. □

Proof of Corollary 4. Suppose that this corollary is false and let G be a counterexample of minimal order. Since every S-quasinormal subgroup is subnormal by [14], all 2-maximal subgroups of G are subnormal in G. Hence all 2-maximal subgroups of G are K-$\mathbb{P}$-subnormal in G. Therefore, by Theorem, G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G. By [7] or [24], $G_p = G^{\mathfrak{U}}$ is a Sylow p-subgroup of G. Let M be a maximal subgroup of G such that $G = G_p \rtimes M$. Let T be a maximal subgroup of M. Then T is subnormal in G, so $T \leq M_G$. If $M_G = 1$, then $T = 1$ and hence $|M| = q$ is a prime. Thus M is a Sylow subgroup of G. Let K be a maximal subgroup of G_p. Since G is not supersoluble, $K \neq 1$. Moreover, K is a 2-maximal subgroup of G and so $MK = KM$, which contradicts the maximality of M. Thus $M_G \neq 1$. Hence $T = M_G$ is normal in G. In view of [14, Lemma 1], the hypothesis holds for G/T. Therefore, G/T is supersoluble by the choice of G. But then $G \simeq G/T \cap G_p$ is supersoluble. This contradiction completes the proof of Corollary 4. □

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References

1. R. K. Agrawal, Generalized center and hypercenter of a finite group, *Proc. Amer. Math. Soc.* **54** (1976) 13–21.
2. M. Asaad, Finite groups some whose n-maximal subgroups are normal, *Acta Math. Hung.* **54**(1–2) (1989) 9–27.
3. A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite Groups* (Springer-Verlag, 2006).
4. A. Ballester-Bolinches, L. M. Ezquerro and A. N. Skiba, On second maximal subgroups of Sylow subgroups of finite groups, *J. Pure Appl. Algebra* **215**(4) (2011) 705–714.
5. K. Doerk and T. Hawkes, *Finite Soluble Groups* (Walter de Gruyter, 1992).
6. P. Flavell, Overgroups of second maximal subgroups, *Arch. Math.* **64** (1995) 277–282.
7. W. Guo, *The Theory of Classes of Groups* (Science Press, Kluwer Academic Publishers, 2000).
8. W. Guo, Yu. V. Lutsenko and A. N. Skiba, On nonnilpotent groups with every two 3-maximal subgroups permutable, *Siberian Math. J.* **50**(6) (2009) 988–997.

9. X. Y. Guo and K. P. Shum, Cover-avoidance properties and the structure of finite groups, *J. Pure Appl. Algebra* **181** (2003) 297–308.
10. W. Guo, K. P. Shum and A. N. Skiba, X-semipermutable subgroups of finite groups, *J. Algebra* **315** (2007) 31–41.
11. W. Guo and A. N. Skiba, Finite groups with given s-embedded and n-embedded subgroups, *J. Algebra* **321** (2009) 2843–2860.
12. B. Huppert, Normalteiler and maximal Untergruppen endlicher Gruppen, *Math. Z.* **60** (1954) 409–434.
13. B. Huppert, *Endliche Gruppen*, Vol. 1 (Springer-Verlag, 1967).
14. O. H. Kegel, Sylow-Gruppen and Subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962) 205–221.
15. O. H. Kegel, Zur Struktur mehrfach faktorisierbarer endlicher Gruppen, *Math. Z.* **87** (1965) 409–434.
16. V. N. Kniahina and V. S. Monakhov, On the permutability of n-maximal subgroups with Schmidt subgroups, *Tr. Inst. Mat. Mekh. UrO RAN* **18**(3) (2012) 125–130.
17. V. A. Kovaleva and A. N. Skiba, Finite solvable groups with all n-maximal subgroups $\mathfrak{U}$-subnormal, *Siberian. Math. J.* **54**(1) (2013) 65–73.
18. V. A. Kovaleva and A. N. Skiba, Finite soluble groups with all n-maximal subgroups $\mathfrak{F}$-subnormal, *J. Group Theory* **17** (2014) 273–290.
19. S. Li, Finite non-nilpotent groups all of whose second maximal subgroups are TI-groups, *Math. Proc. R. Ir. Acad.* **100A**(1) (2000) 65–71.
20. B. Li and A. N. Skiba, New characterizations of finite supersoluble groups, *Sci. China Ser. A*: *Math.* **50**(1) (2008) 827–841.
21. Yu. V. Lutsenko and A. N. Skiba, Finite groups with subnormal second and third maximal subgroups, *Math. Notes* **91**(5) (2012) 680–688.
22. V. S. Monakhov and V. N. Kniahina, Finite groups with $\mathbb{P}$-subnormal subgroups, *Ricerche Mat.* **62**(2) (2013) 307–322.
23. L. Ja. Poljakov, Finite groups with permutable subgroups, in *Finite Groups* (Nauka i Tekhnika, 1966), pp. 75–88.
24. L. A. Shemetkov, *Formations of Finite Groups* (Nauka, 1978).
25. A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov, On finite groups similar to supersoluble groups, *Probl. Phys. Math. Tech.* **2**(3) (2010) 21–27.
26. A. F. Vasilyev, T. I. Vasilyeva and V. N. Tyutyanov, On finite groups with almost all K-$\mathbb{P}$-subnormal Sylow subgroups, in *Algebra and Combinatorics*: *Abstracts of Reports of the International Conference on Algebra and Combinatorics on Occasion the* 60*th year Anniversary of A. A. Makhnev* (Ekaterinburg, 2013), pp. 19–20.
27. M. Weinstein, *Between Nilpotent and Solvable* (Polygonal Publishing House, 1982).