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Finite groups with permutable complete Wielandt sets of subgroups

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Abstract. Let $\mathcal{W}$ be a set of nilpotent Hall subgroups of a finite group G. We say that $\mathcal{W}$ is a *Wielandt set* if for each prime p dividing $|G|$ there is exactly one subgroup $W \in \mathcal{W}$ such that p divides $|W|$. In this paper, we study the influence of Wielandt sets on the structure of G. In particular, we give conditions under which a normal subgroup of a finite group is hypercyclically embedded.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. A subgroup A of G is said to be S*-permutable* if A permutes with every Sylow subgroup P of G, that is, $AP = PA$. The symbol $G^{\mathfrak{N}}$ denotes the nilpotent residual of G, that is, the smallest normal subgroup of G with nilpotent quotient.

In what follows, $\sigma = \{\pi_i : i \in I\}$ is some partition of the set $\mathbb{P}$ of all primes, that is, $\mathbb{P} = \bigcup_{i \in I} \pi_i$ and $\pi_i \cap \pi_j = \varnothing$ for all $i \neq j$.

If G has a set $\mathcal{W} = \{W_1, \dots, W_t\}$ of nilpotent Hall subgroups W_k such that for any k, W_k is a π_k-subgroup, then we say $\mathcal{W}$ is a *complete Wielandt set of subgroups of type* σ or simply $\mathcal{W}$ is a *complete Wielandt set of subgroups* of G. If $\mathcal{W}$ is such that $W_i W_j = W_j W_i$ for all i, j, then we say that $\mathcal{W}$ is a *Wielandt system of type* σ of G or simply $\mathcal{W}$ is a *Wielandt system* of G.

In view of [5], the class of all soluble groups having a complete Wielandt set of subgroups of type σ is a saturated formation, for any partition σ on $\mathbb{P}$.

Many results are connected with the study of the groups in which some Sylow subgroups permute with given subgroups of the group. For example, the classic result in this vein, due to P. Hall, states: *G is soluble if and only if it has a Sylow basis, that is, a complete set of pairwise permutable Sylow subgroups*. From the

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famous Wielandt theorems on the products of nilpotent groups (see [12, Chapter VI, Theorems 4.3, 4.9]) it follows that the analogue of this result remains true if we use the Wielandt systems of any fixed type σ in place of the Sylow bases.

In [10], Huppert proved that a soluble group is supersoluble if it has a Sylow basis $\mathcal{S}$ such that every subgroup in $\mathcal{S}$ permutes with all maximal subgroups of each other member in $\mathcal{S}$. In [11] (see also [12, Chapter VI, Section 3]), Huppert described the group in which every complete set of Sylow subgroups forms a Sylow basis of the group. Note, in passing, that if $G = P \rtimes (QR)$, where $|P| = 7$, $|Q| = 3$, $|R| = 2$ and $QR \leq \mathrm{Aut}(P)$, then some complete set of Sylow subgroups is not a Sylow basis of G, but every complete Wielandt set of subgroups of type σ, where $\sigma = \{7\} \cup \{3, 2\}$, is clearly a Wielandt system of G.

The above-mentioned results and many other known results of this kind make it natural to ask:

Question (I). What can we say about the group G provided G has a complete Wielandt set of subgroups $\mathcal{W}$ such that every member of $\mathcal{W}$ permutes with all maximal subgroups of any other subgroup in $\mathcal{W}$?

Question (II). What can we say about the soluble group G provided every complete Wielandt set of subgroups of G forms a Wielandt system?

Our main goal here is to give an answer to Question (I).

Theorem A. *A group G has a complete Wielandt set of subgroups $\mathcal{W}$ such that every member in $\mathcal{W}$ permutes with all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$ if and only if $G = D \rtimes M$ is a supersoluble group where $D = G^{\mathfrak{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G.*

Theorem A shows that the class of the groups satisfying the hypothesis in Question (I) is very close to the class of so-called PST-groups. Recall that a group G is called a PST-*group* if S-permutability is a transitive relation on G, that is, every S-permutable subgroup of an S-permutable subgroup of G is S-permutable in G. In view of [1] (see also [4, Chapter 2]), a soluble group G is a PST-group if and only if G is a supersoluble group such that $D = G^{\mathfrak{N}}$ is an abelian Hall subgroup of G of odd order and every subgroup of D is normal in G.

The following result was obtained by Asaad and Heliel (see [3, Theorem 3.1]) by applying the classification of the finite simple groups. Their theorem can be deduced from Theorem A. We note that the proof of Theorem A only requires the classification of the finite simple groups with abelian Sylow 2-subgroups and not the full classification of the finite simple groups.

Corollary 1.1. *If a group G has a complete set of Sylow subgroups $\mathcal{S}$ such that every Sylow subgroup P in $\mathcal{S}$ permutes with all maximal subgroups of any member of $\mathcal{S}$, then G is supersoluble.*

Recall that a normal subgroup E of G is called *hypercyclically embedded* in G (see [16, p. 217]) if every chief factor of G below E is cyclic. Hypercyclically embedded subgroups play a important role in the theory of soluble groups (see [4, 16, 22]) and the conditions under which a normal subgroup is hypercyclically embedded in G were found by many authors (see the books [4, 16, 22] and, for example, the recent papers [2, 9, 15, 17–20]).

On the base of Theorem A, we prove the following result in this vein.

Theorem B. *Let E be a normal subgroup of a group G. Suppose that G has a complete Wielandt set of subgroups $\mathcal{W} = \{W_1, \dots, W_t\}$ such that the maximal subgroups of $W_i \cap E$ permute with all members of $\mathcal{W}$, for $i = 1, \dots, t$. Then E is hypercyclically embedded in G.*

By Theorem B, we may directly obtain the following results.

Corollary 1.2. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G has a complete Wielandt set of subgroups $\mathcal{W} = \{W_1, \dots, W_t\}$ such that the maximal subgroups of $W_i \cap E$ permute with all members of $\mathcal{W}$, for $i = 1, \dots, t$. Then $G \in \mathfrak{F}$.*

Corollary 1.3 (see [3, Theorem]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G has a complete set of Sylow subgroups $\mathcal{S} = \{P_1, \dots, P_t\}$ such that the maximal subgroups of $P_i \cap E$ permute with all members of $\mathcal{S}$, for all $i = 1, \dots, t$. Then $G \in \mathfrak{F}$.*

2 Proofs of theorems

We shall need the following known fact.

Lemma 2.1 (see [14]). *Let H, K and N be pairwise permutable subgroups of G, and suppose that H is a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Proof of Theorem A. We say that G is a *generalized* PST-*group* if $G = D \rtimes M$ is a supersoluble group, where $D = G^{\mathfrak{N}}$ is a nilpotent Hall subgroup of G of odd order whose maximal subgroups are normal in G.

First assume that there is a complete Wielandt set $\mathcal{W}_\sigma = \{W_1, \dots, W_t\}$ of type σ of G such that every member of $\mathcal{W}$ permutes with each maximal subgroup of every non-cyclic subgroup S in $\mathcal{W}_\sigma$. We show that in this case G is a generalized PST-group. Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G and $D = G^{\mathfrak{N}}$ the nilpotent residual of G. We proceed via the following steps.

(1) *The hypothesis holds for G/R. Hence G/R is a generalized* PST-*group.*

As $W_i R/R \simeq W_i/W_i \cap R$ is nilpotent and $W_i R/R$ is a Hall subgroup of G/R, it follows that

$$\overline{\mathcal{W}} = \{W_1 R/R, \dots, W_t R/R\}$$

is a Wielandt set of type σ of G/R.

Now assume that $W_k R/R$ is non-cyclic and V_1/R and V_2/R are distinct maximal subgroups of $W_k R/R$. Then $V_i = R(V_i \cap W_k)$, and so $V_1 \cap W_k \neq V_2 \cap W_k$. Moreover, as W_k is nilpotent and

$$\begin{aligned} |(W_k R/R) : (V_i/R)| &= |W_k R : (V_i \cap W_k)R| \\ &= |W_k : W_k \cap (V_i \cap W_k)R| \\ &= |W_k : (V_i \cap W_k)(W_k \cap R)|, \end{aligned}$$

it follows that $(V_i \cap W_k)(W_k \cap R)$ is a maximal subgroup of W_k. Therefore

$$(V_i \cap W_k)(W_k \cap R)W_j = W_j(V_i \cap W_k)(W_k \cap R),$$

so

$$(V_i/R)(W_j R/R) = (W_j R/R)(V_i/R)$$

for all j. This shows that the hypothesis holds for G/R. Hence G/R is a generalized PST-group by the choice of G.

(2) *D is not a Hall nilpotent subgroup of G.*

Assume that this is false. We first show that in this case every maximal subgroup of D is normal in G. Since D is nilpotent, it is enough to show that every maximal subgroup V of any Sylow subgroup P of D is normal in G. If P is cyclic, it is evident. Now suppose that P is not cyclic. Let $P \leq W_i$ and E be the complement of P to W_i. Then V is normal in W_i and VE is maximal in W_i, so $VEW_j = W_j VE$ for all $j \neq i$ by hypothesis. Hence

$$PW_j \cap W_j VE = W_j(P \cap W_j VE) = W_j V = VW_j$$

is a subgroup of G. Then since $V = P \cap VW_j$, we have $W_j \leq N_G(V)$. This shows that V is normal in G. Therefore every maximal subgroup of D is normal in G. Assume that 2 divides $|D|$. Then D has a maximal subgroup E with $|D : E| = 2$.

Moreover, E is normal in G and $C_G(D/E) = G$. Hence $G/E \in \mathfrak{N}$ and so $D \leq E$, a contradiction. Hence $|D|$ is odd and so G is a generalized PST-group by the Schur–Zassenhaus Theorem. This contradiction shows that we have (2).

(3) *G is not supersoluble.*

Assume that G is supersoluble. Then D is nilpotent by [12, Chapter VI, Theorem 9.1]. We now show that D is a Hall subgroup of G. Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$ for some Sylow p-subgroup G_p of G. Then $|G_p| > p$. Let N be a minimal normal subgroup of G contained in D. Then N is a group of order r for some prime r. By (1), G/N is a generalized PST-group, so $D/N = (G/N)^{\mathfrak{N}}$ is a Hall subgroup of G/N by the choice of G. It follows that $r = p$ and $N = P = D$. Since $|G_p| > p$ and $G/N = G/N^{\mathfrak{N}} = G/P$ is nilpotent, G_p is normal in G. If $O_{p'}(G) \neq 1$ and R is a minimal normal subgroup of G contained in $O_{p'}(G)$, then $R \cap D = 1$. Moreover, $(G/R)^{\mathfrak{N}} = DR/R \simeq D$ is a Hall subgroup of G/R by (1). But then $P = G_p$, a contradiction. Hence $O_{p'}(G) = 1$ and thereby $F(G) = G_p$. Then since $C_G(F(G)) \leq F(G)$ (see, for example, [8, Theorem 1.8.18]), $G_p \in \mathcal{W}$.

Suppose that $\Phi(G_p) \neq 1$ and let R be a minimal normal subgroup of G contained in $\Phi(G_p)$. Then $R \leq \Phi(G)$. If $R = D = G^{\mathfrak{N}}$, then G is nilpotent and so $D = 1$, a contradiction. Hence $R \neq D$. It is also clear that $RD \neq G_p$. But as the hypothesis holds for G/R by (1), we have that $DR/R = G_p/R$ by the choice of G. Thus $RD = G_p$. This contradiction shows that $\Phi(G_p) = 1$. Hence G_p is elementary abelian and so G_p is not cyclic since $|G_p| > p$.

Let V be a maximal subgroup of G_p and let $W_i \neq G_p$. By hypothesis, we have that $W_i V = V W_i$ is a subgroup of G and $G_p \cap V W_i = V$. Hence $V W_i \leq N_G(V)$. On the other hand, V is normal in G_p. Therefore V is normal in G. This shows that every subgroup of G_p is normal in G since G_p is elementary abelian. We may, therefore, assume that $G_p = \langle a \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$, where $\langle a_i \rangle$ is a minimal normal subgroup of G and $\langle a \rangle = D$. Write $a_1 = a a_2 \dots a_t$. Then since $\langle a_1 \rangle \cap \langle a_2 \rangle \dots \langle a_t \rangle = 1$, we have $G_p = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$. Note that $\langle a_1 \rangle$ is normal in G. Therefore from the G-isomorphism $D\langle a_1 \rangle / D \simeq \langle a_1 \rangle$ we have $\langle a_1 \rangle \leq Z(G)$. Similarly, $\langle a_2 \rangle \times \cdots \times \langle a_t \rangle \leq Z(G)$. Hence $G_p \leq Z_\infty(G)$, which implies that G is nilpotent, a contradiction. The above contradiction shows that D is a nilpotent Hall subgroup of G, contrary to Claim (2). Hence we have (3).

(4) *G is not soluble.*

Assume that this is false. Then in view of (1) and (3), R is the unique minimal normal subgroup of G, R is non-cyclic, $R \nleq \Phi(G)$ and $C_G(R) = R$. Let $p \in \pi_k$. Then W_k is a non-cyclic Sylow p-subgroup of G since $R \leq W_k$. Hence every member of $\mathcal{W}$ permutes with each maximal subgroup of W_k. Since $R \nleq \Phi(G)$, for some maximal subgroup V of W_k we have $RV = W_k$ and $E = R \cap V \neq 1$ since $|R| > p$. We claim that $W_i \leq N_G(E)$ for all $i = 1, \dots, t$. Indeed, in the case when $i = k$, this follows from the fact that V is normal in W_k. Now assume that $i \neq k$. Then since VW_i is a subgroup of G, $R \cap VW_i = R \cap V = E$ and so $W_i \subseteq N_G(E)$. Therefore E is normal in G, which contradicts the minimality of R.

(5) *If at least one of the two subgroups W_i and W_k is non-cyclic, say W_i, then $W_i W_k = W_k W_i$*. (This follows from the fact that every maximal subgroup of W_i permutes with W_k.)

(6) Suppose that the smallest prime divisor of $|G|$ belongs to $\pi(W_1)$. Then *$W = W_1$ is not cyclic*. (In view of the Feit–Thompson Theorem this directly follows from Claim (4) and [12, Chapter IV, Theorem 2.8].)

In view of (6), the set $\mathcal{W}_\sigma$ contains non-cyclic subgroups. Without loss of generality, we may assume that $W_1, \dots, W_r$ are non-cyclic subgroups and all subgroups $W_{r+1}, \dots, W_t$ are cyclic.

(7) The subgroups $E = W_1 \dots W_r$ *and $W_1 W_k$ are generalized* PST-*groups for each $k > r$*.

By Claim (5) and Wielandt's theorems [12, Chapter VI, Theorems 4.3 and 4.9], both E and $W_1 W_k$ are soluble groups. Hence Claim (4) implies that both these subgroups are proper subgroups of G. The choice of G implies that E and $W_1 W_k$ are generalized PST-groups.

(8) *The subgroup R is a non-abelian simple group.*

By (1), G/R is a generalized PST-group. Hence in view of (4), R is non-abelian. It follows from (7) that $R \nleq E$. Therefore for some prime p dividing $|R|$, there is $k > r$ such that $p \in \pi(W_k)$. But W_k is cyclic, so a Sylow p-subgroup of R is cyclic. This implies that R is a non-abelian simple group.

(9) *The subgroup R has a Hall $\{2, p\}$-subgroup for each p dividing $|R|$.*

It is clear in the case when $p \in \pi(W_1)$. Now assume that $p \in \pi(W_i)$ for some $i > 1$. Claims (5) and (6) imply that $B = W_1 W_i$ is a soluble Hall subgroup of G. Hence B has a Hall $\{2, p\}$-subgroup V and $V \cap R$ is a Hall $\{2, p\}$-subgroup of R.

(10) *A Sylow 2-subgroup R_2 of R is non-abelian.*

Assume that this is false. Then by Claims (8) and (9) and [13, Chapter XI, Theorem 13.7], R is isomorphic to one of the following groups:

(a) $\mathrm{PSL}(2, 2^f)$,

(b) $\mathrm{PSL}(2, q)$, where 8 divides $q - 3$ or $q - 5$,

(c) the Janko group J_1,

(d) a Ree group.

But with respect to each of these groups it is well known that the group has no Hall $\{2, r\}$-subgroup for at least one odd prime r dividing its order (see, for example, [21, Theorem 1]). Hence we have (10).

(11) *If $q \in \pi(W_k)$ for some $k > r$, then q does not divide $|R : N_R((R_2)')|$.*

By (7), $B = W_1 W_k$ is supersoluble. Let P be the Sylow 2-subgroup of W_1. Then there is a Sylow q-subgroup of Q of B such that PQ is a Hall $\{2, q\}$-subgroup of B. Hence $W = PQ \cap R = (P \cap R)(Q \cap R) = R_2(Q \cap R)$ is a Hall supersoluble subgroup of R with cyclic Sylow q-subgroup $Q \cap R$. By [12, Chapter VI, Theorem 9.1], $Q \cap R$ is normal in W, and $W/C_W(Q \cap R)$ is an abelian group by [7, Chapter 5, Theorem 4.1]. Hence

$$R_2 C_W(Q \cap R)/C_W(Q \cap R) \simeq R_2/R_2 \cap C_W(Q \cap R)$$

is abelian and so $(R_2)' \leq C_W(Q \cap R)$. Hence $Q \cap R \leq N_R((R_2)')$.

Final contradiction.

In view of (11), $R = (E \cap R)N_R((R_2)')$. Hence

$$((R_2)')^R = ((R_2)')^{(E \cap R)N_R((R_2)')} = ((R_2)')^{E \cap R} \leq E \cap R.$$

But by (7), $E \cap R$ is soluble. On the other hand, Claim (10) implies that $(R_2)' \neq 1$ and so R is soluble, contrary to (8).

Conversely, assume now that $G = D \rtimes M$ is a generalized PST-group where $D = G^{\mathfrak{N}}$. We show that the group G has a complete Wielandt set of subgroups $\mathcal{W}_\sigma = \{W_1, \ldots, W_t\}$ of type σ such that every member of $\mathcal{W}$ permutes with all maximal subgroups of any non-cyclic subgroup S in $\mathcal{W}$. Let $\pi = \pi(M)$. Since G is soluble, for any i, there is an element $x_i \in G$ such that a Hall π-subgroup of $E_i = (W_i)^{x_i}$ is contained in M. Then $E_i = (E_i \cap D)(E_i \cap M)$. Now consider a complete Wielandt set of subgroups $\mathcal{E} = \{E_1, \ldots, E_t\}$ of type σ in G.

Let V be a maximal subgroup of a non-cyclic subgroup E_i. Clearly, $|E_i : V|$ is a prime. Assume that $|E_i : V| = p \in \pi$. Then $V = (V \cap D)(V \cap M)$ and $V \cap M$ is a maximal subgroup of $E_i \cap M$. Hence $E_i \cap M$ is normal in M since M is nilpotent. Therefore for any subgroup $E_j = (E_j \cap D)(E_j \cap M)$, we have

$$\begin{aligned} E_j V &= (E_j \cap D)(E_j \cap M)(V \cap D)(V \cap M) \\ &= (V \cap D)(V \cap M)(E_j \cap D)(E_j \cap M) \\ &= VE_j. \end{aligned}$$

This completes the proof. □

Proof of Theorem B. Assume that this theorem is false and let G be a counterexample with $|G| + |E|$ minimal. Then:

(1) *If S is a Hall normal subgroup of E, then the hypothesis holds for (G, S) and for (S, S). Hence E is supersoluble.*

Let V be a maximal subgroup of $W_i \cap S$ and let T be the complement of $W_i \cap S$ in $W_i \cap E$. Then VT is maximal in $W_i \cap E$, so $VTW_j = W_jVT$ for all $j = 1, \ldots, t$. Hence

$$\begin{aligned} SW_j \cap VTW_j &= W_j(S \cap VTW_j) \\ &= W_jV(S \cap TW_j) \\ &= W_jV(S \cap T)(S \cap W_j) \\ &= W_jV \\ &= VW_j. \end{aligned}$$

Therefore the hypothesis holds for (G, S). Similarly one can show that the hypothesis holds also for (S, S) and in particular for (E, E). Hence E is supersoluble by Theorem A.

(2) *The hypothesis holds on $(G/R, E/R)$ for any non-identity normal subgroup R of G contained in E. Therefore the choice of G implies that E/R is hypercyclically embedded in G/R.*

It is clear first of all that

$$\overline{\mathcal{W}} = \{W_1R/R, \ldots, W_tR/R\}$$

is a complete Wielandt set of subgroups of G/R. Let V/P be a maximal subgroup of

$$(W_iR/R) \cap (E/R) = R(W_i \cap E)/R.$$

Then for some maximal subgroup S of $W_i \cap E$ we have $V/R = SR/R$. By hypothesis, S permutes with W_j, so V/R permutes with W_jR/R for all j. Therefore the hypothesis holds for $(G/R, E/R)$. The choice of G implies that E/R is hypercyclically embedded in G/R.

(3) *The subgroup E is a p-group.*

Let p be the largest prime dividing $|E|$ and let P be a Sylow p-subgroup of E. Then Claim (1) and [12, Chapter VI, Theorem 9.2] imply that P is characteristic in E, so P is normal in G. Assume that $P \neq E$. Then (1) and the choice of (G, E) implies that P is hypercyclically embedded in G. It follows from Claim (2) and the Jordan–Hölder Theorem for chief series that E is hypercyclically embedded in G, a contradiction. Thus $E = P$.

(4) We have that $\Phi(E) = 1$. (This directly follows from [6, Chapter IV, Theorem 6.7], Claim (2) and the choice of G.)

Final contradiction.

Let R be a minimal normal subgroup of G contained in E. Then E/R is hypercyclically embedded in G/R by Claim (2). Hence R is not cyclic. Without loss of generality, we can assume that $E \leq W_1$. Let V be a maximal subgroup of R such that V is normal in W_1. Claim (4) implies that R has a complement S in E. Let $W = VS$. Then $V = R \cap W$ and W is maximal in E. Thus by the hypothesis, $WW_j = W_jW$ is a subgroup of G. Then by Lemma 2.1,

$$\begin{aligned} RW_j \cap W_jW &= W_j(R \cap W_jW) \\ &= W_j(R \cap W_j)(R \cap W) \\ &= W_j(R \cap W) \\ &= (R \cap W)W_j \\ &= VW_j \\ &= W_jV, \end{aligned}$$

and $V = R \cap VW_j$ is normal in VW_j. Thus $W_k \le N_G(V)$ for all $k = 1, \dots, t$. This implies that V is normal in G and so $V = 1$. This contradiction completes the proof. □

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