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On some classes of finite quasi-$\mathscr{F}$-groups

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Abstract. Let G be a finite group and $\mathscr{F}$ a class of finite groups. We say that G is a *quasi-$\mathscr{F}$-group* if for every $\mathscr{F}$-eccentric chief factor H/K of G and every $x \in G$, x induces an inner automorphism on H/K. In this paper, the general theory of quasi-$\mathscr{F}$-groups is given and some of its applications are considered. In particular, some characterizations of quasisoluble groups and quasisupersoluble groups are obtained.

1 Introduction

Throughout this paper, all groups are finite. We use $\mathscr{S}$, $\mathscr{U}$ and $\mathscr{N}$ to denote the classes of all soluble groups, supersoluble groups and nilpotent groups, respectively. The symbol $A \rtimes B$ denotes the semidirect product of A and B, where B is an operator group on the group A.

Let $\mathscr{F}$ be a class of groups. A chief factor H/K of a group G is called *$\mathscr{F}$-central* if $H/K \rtimes (G/C_G(H/K)) \in \mathscr{F}$ (see [13, pp. 127–128] or [3, Definition 2.4.3]). Otherwise, it is called *$\mathscr{F}$-eccentric*.

Recall that a group G is said to be quasinilpotent if for every chief factor H/K of G and every $x \in G$, x induces an inner automorphism on H/K (see [5, p. 124]). Note that since for every central chief factor H/K an element of G induces the trivial automorphism on H/K, one can say that a group G is quasinilpotent if for every *non-central* chief factor H/K of G and every $x \in G$, x induces an inner automorphism on H/K. This elementary observation allows us to consider the following analogue of quasinilpotent groups.

Definition 1.1. Let $\mathscr{F}$ be a class of groups and G a group. We say that *G is a quasi-$\mathscr{F}$-group* if for every $\mathscr{F}$-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner.

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By analogy with p-quasinilpotent groups (defined in [6]), we introduce also the following local version of Definition 1.1.

Definition 1.2. Let $\mathscr{F}$ be a class of groups, G a group and p a prime. We say that G *is a p-quasi-$\mathscr{F}$-group* if for every $\mathscr{F}$-eccentric chief factor H/K of G of order divisible by p, the automorphisms of H/K induced by all elements of G are inner.

We use $\mathscr{F}^*$ and $\mathscr{F}_p^*$ to denote the class of all quasi-$\mathscr{F}$-groups and the class of all p-quasi-$\mathscr{F}$-groups, respectively.

Recall that a formation $\mathscr{F}$ is a homomorph of groups such that each group G has a smallest normal subgroup (denoted by $G^{\mathscr{F}}$) whose quotient is still in $\mathscr{F}$. A formation $\mathscr{F}$ is said to be (solubly) saturated (see [2, IV, Definition 4.9]) if it contains each group G with $G/\Phi(N) \in \mathscr{F}$ for some normal (soluble) subgroup N of G. It is known that the class of all quasinilpotent groups is a solubly saturated formation (see [12]) and every normal subgroup of any quasinilpotent group is a quasinilpotent group (see [5, X, (13.3)]).

The following theorem shows that the classes $\mathscr{F}^*$ and $\mathscr{F}_p^*$ have the same properties.

Theorem A. *Suppose that $\mathscr{F}$ is a saturated formation containing $\mathscr{N}$. Then both $\mathscr{F}^*$ and $\mathscr{F}_p^*$ are solubly saturated formations. Moreover, if $\mathscr{F}$ contains every normal subgroup of every group in $\mathscr{F}$, then every normal subgroup of any quasi-$\mathscr{F}$-group* (*resp. any p-quasi-$\mathscr{F}$-group*) *is also a quasi-$\mathscr{F}$-group* (*resp. a p-quasi-$\mathscr{F}$-group*).

Corollary 1.3. *For every saturated formation $\mathscr{F}$ containing $\mathscr{N}$, both $\mathscr{F}^*$ and $\mathscr{F}_p^*$ are Baer-local formations.*

For the definition of Baer-local formations, see [2, IV, Definition 4.9].

Theorem A is the basis for our other results. In particular, using this theorem we prove the following result which shows that in the definition of quasi-$\mathscr{F}$-groups we need only consider chief factors between $\Phi(F(G))$ and $F^*(G)$.

Theorem B. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{N}$ and G a group. Then G is a quasi-$\mathscr{F}$-group if and only if for every $\mathscr{F}$-eccentric G-chief factor H/K between $\Phi(F(G))$ and $F^*(G)$, the automorphisms of H/K induced by all elements of G are inner.*

Corollary 1.4. *If for every non-central G-chief factor H/K between $\Phi(F(G))$ and $F^*(G)$, all automorphisms of H/K induced by elements of G are inner, then G is quasinilpotent.*

The normal structure of quasinilpotent groups is well known: a group G is quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple (see, for example, [5, X, Theorem 13.6]); recall that a group G is called semisimple if either $G = 1$ or G is a direct product of non-abelian simple groups.

For the p-quasi-$\mathscr{F}$-groups and for the quasi-$\mathscr{F}$-groups, we give the following result.

Theorem C. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{N}$ and let p be a prime. Suppose that $\mathscr{F}$ contains every normal subgroup of every group in $\mathscr{F}$. Then*

(1) *G is a p-quasi-$\mathscr{F}$-group if and only if $G/\mathrm{Z}_{\mathscr{F}_p}(G)$ is semisimple and the order of each composition factor of $G/\mathrm{Z}_{\mathscr{F}_p}(G)$ is divisible by p, and*

(2) *G is a quasi-$\mathscr{F}$-group if and only if $G/\mathrm{Z}_{\mathscr{F}}(G)$ is semisimple.*

In this theorem, $\mathrm{Z}_{\mathscr{F}}(G)$ denotes the $\mathscr{F}$-hypercenter of G (see [2, p. 389]); $\mathrm{Z}_{\mathscr{F}_p}(G)$ denotes the product of all normal subgroups H of G such that every G-chief factor of H of order divisible by p is $\mathscr{F}$-central in G.

Following Robinson [9], we call a group G an SC-group if every chief factor of G is a simple group. By Theorem C, we see that every quasisupersoluble group is an SC-group.

On the basis of Theorem C, one can easily obtain examples of quasisupersoluble and p-quasisupersoluble groups. For example, let $A = C \rtimes \langle\alpha\rangle$, where $|C| = 7$ and α is an automorphism of C with $|\alpha| = 3$. Let $B = A \times A_7$ and $G = A_5 \wr B$. Then by Theorem C, G is 7-quasisupersoluble but not 7-quasinilpotent. The group B is quasisupersoluble and not quasinilpotent. The group $C = B \rtimes \langle\beta\rangle$, where β is an inner automorphism of A_7 with $|\beta| = 2$ and β acts trivially on A, is an SC-group but not a quasisupersoluble group.

Theorem A, B and C are proved in Section 2. In Section 3, we give some characterizations of quasisupersoluble groups and of quasisoluble groups.

All unexplained notation and terminology is standard, as used for example in [1]–[3] and [5].

2 Proofs of Theorems A, B and C

Lemma 2.1. *For any class $\mathscr{F}$ of groups, the classes $\mathscr{F}^*$ and $\mathscr{F}_p^*$ are non-empty formations.*

Proof. This follows in an obvious way by using the proof of [5, X, Lemma 13.3]. □

A function f of the form $f : \mathbb{P} \to \{\text{group formations}\}$ (where $\mathbb{P}$ is the set of all primes) is called a formation function. The symbol $\mathrm{LF}(f)$ denotes the set of all groups G such that either $G = 1$ or $G \neq 1$ and $G/\mathrm{C}_G(H/K) \in f(p)$ for every chief factor H/K of G and every prime p dividing $|H/K|$. For a formation $\mathscr{F}$, if there exists a formation function f such that $\mathscr{F} = \mathrm{LF}(f)$, then $\mathscr{F}$ is called a local formation. It is well known that a non-empty formation is local if and only if it is saturated. A formation is called normally hereditary if it contains every normal subgroup of each of its groups.

Lemma 2.2. *Let $L \leqslant K \leqslant H \leqslant D \leqslant N \leqslant G$, where L, D, N are normal subgroups of G and K, H are normal subgroups of N. Suppose that D/L is a chief factor of G and*

H/K is a chief factor of N. If $\mathscr{F}$ is a normally hereditary saturated formation and $D/L \rtimes (G/C_G(D/L)) \in \mathscr{F}$, then $H/K \rtimes (N/C_N(H/K)) \in \mathscr{F}$.

Proof. Since $\mathscr{F}$ is normally hereditary saturated formation, by [2, IV, Theorem 3.16] there exists a formation function f such that $\mathscr{F} = LF(f)$ and any value $f(p)$ of f is a normally hereditary formation contained in $\mathscr{F}$. Since $D/L \rtimes (G/C_G(D/L)) \in \mathscr{F}$ we have $G/C_G(D/L) \in f(p)$ for all primes p dividing $|D/L|$ by [3, Theorem 3.1.6] or [13, Theorem 17.14]. Since the formation $\mathscr{F}$ is normally hereditary by hypothesis, $N/C_N(H/K) \in f(p)$ for all primes p dividing $|H/K|$. Then by [3, Theorem 3.1.6] again, we obtain that $H/K \rtimes (N/C_N(H/K)) \in \mathscr{F}$. □

Lemma 2.3. *Let $L \leqslant K \leqslant H \leqslant D \leqslant N \leqslant G$ where L, D, N are normal subgroups of G and K, H are normal subgroups of N. Suppose that D/L is a chief factor of G and H/K is a chief factor of N. If $x \in N$ and x induces an inner automorphism on D/L, then x induces an inner automorphism on H/K.*

Proof. See the proof of [5, X, Lemma 13.1]. □

Following Doerk and Hawkes [2, IV, (4.10)], we write $C^p(G)$ for the intersection of the centralizers of all abelian p-chief factors of the group G, with $C^p(G) = G$ if G has no such chief factors.

For every function f of the form

$$f : \mathbb{P} \cup \{0\} \to \{\text{group formations}\}, \qquad (*)$$

following [14] we put

$$\mathrm{CLF}(f) = \{G \mid G/G_{\mathscr{S}} \in f(0) \text{ and } G/C^p(G) \in f(p) \text{ for any } p \in \pi(\mathrm{Com}(G))\}.$$

Here, $G_{\mathscr{S}}$ denotes the $\mathscr{S}$-radical of G (i.e., the largest normal soluble subgroup of G); $\mathrm{Com}(G)$ denotes the class of all abelian simple groups A such that $A \cong H/K$ for some composition factor H/K of G.

Lemma 2.4. *Let H/K be a chief factor of a group G. Suppose that the automorphism of H/K induced by an element g of G is inner; then $gK \in (H/K)C_{G/K}(H/K)$.*

Proof. Since the automorphism of H/K induced by g is also induced by some element xK of H/K we have $gx^{-1}K \in C_G(H/K)$. Hence the result follows. □

Let p be a prime and $\mathscr{F}$ a non-empty class of groups. We write $\mathscr{F}(p)$ for the intersection of all formations containing the set $\{G/O_{p',p}(G) \mid G \in \mathscr{F}\}$. The symbol $\mathscr{N}_p$ denotes the class of all p-groups.

Proposition 2.5. *Let $\mathscr{F}$ be a saturated formation containing all nilpotent groups and let p be a prime. Then*

(1) $\mathscr{F}_p^* = \mathrm{CLF}(f_p^*)$, *where* $f_p^*(p) = \mathscr{N}_p\mathscr{F}(p) \subseteq \mathscr{F}$ *and* $f_p^*(0) = \mathscr{F}_p^* = f_p^*(q)$ *for all primes* $q \neq p$, *and*

(2) $\mathscr{F}^* = \mathrm{CLF}(f^*)$, *where* $f^*(0) = \mathscr{F}^*$ *and* $f^*(q) = \mathscr{N}_q\mathscr{F}(q) \subseteq \mathscr{F}$ *for all primes* q.

Proof. Since $\mathscr{F}$ is a saturated formation containing all nilpotent groups, by [2, IV, Theorem 4.6] and by [3, Theorem 3.1.15] we have $\mathscr{F} = \mathrm{LF}(f)$, where $f(q) = \mathscr{F}(q)$ for all primes q. Then by [3, Corollary 3.1.17] we have $\mathscr{F} = \mathrm{LF}(F)$, where $F(q) = \mathscr{N}_q f(q) \subseteq \mathscr{F}$ for all primes q.

(1) Let f_p^* be a function of the form $(*)$ such that $f_p^*(p) = \mathscr{N}_p\mathscr{F}(p) = F(p)$ and $f_p^*(0) = \mathscr{F}_p^* = f_p^*(q)$ for all primes $q \neq p$. Put $\mathscr{M}_p = \mathrm{CLF}(f_p^*)$. Then we only need to prove that $\mathscr{F}_p^* = \mathscr{M}_p$. Suppose that $\mathscr{F}_p^* \nsubseteq \mathscr{M}_p$ and let G be a group of minimal order in $\mathscr{F}_p^* \backslash \mathscr{M}_p$. Then $R = G^{\mathscr{M}_p}$ is the only minimal normal subgroup of G.

Suppose that R is an abelian p-group and let $C = C_G(R)$. If $R/1$ is $\mathscr{F}$-eccentric, then $G = RC = C$ by Lemma 2.4 since $G \in \mathscr{F}_p^*$. This means that $R/1$ is $\mathscr{F}$-central since $\mathscr{N} \subseteq \mathscr{F}$. This contradiction shows that R is $\mathscr{F}$-central. If $C = R$, then $R = C^p(G)$ and so

$$G/C = G/C^p(G) \in F(p) = f_p^*(p).$$

It follows that $G \in \mathscr{M}_p$, a contradiction. Hence $R \neq C$. Since R is $\mathscr{F}$-central, we have $T = R \rtimes (G/C) \in \mathscr{F} \subseteq \mathscr{F}_p^*$. But since $|T| < |G|$ we have $T \in \mathscr{M}_p$ by the choice of G. Hence $G/C \in F(p) = f_p^*(p)$. Since $R = G^{\mathscr{M}_p}$ we have $(G/R)/C^p(G/R) \in f^*(p)$. But obviously $C \cap C_0 = C^p(G)$, where $C_0/R = C^p(G/R)$. Hence $G/C^p(G) \in f_p^*(p)$. This implies that $G \in \mathscr{M}_p$, a contradiction. Hence R is non-abelian and so $R \leqslant C^q(G)$ for all primes q. Then

$$G/C^q(G) \cong (G/R)/(C^q(G)/R) = (G/R)/C^q(G/R) \in f_p^*(q)$$

for all primes q. On the other hand, since $G \in \mathscr{F}_p^*$ we have $G/G_{\mathscr{S}} \in \mathscr{F}_p^* = f_p^*(0)$. Thus $G \in \mathscr{M}_p$. This contradiction shows that $\mathscr{F}_p^* \subseteq \mathscr{M}_p$.

Next suppose that $\mathscr{M}_p \nsubseteq \mathscr{F}_p^*$ and let G be a group of minimal order in $\mathscr{M}_p \backslash \mathscr{F}_p^*$. Then $R = G^{\mathscr{F}_p^*}$ is the only minimal normal subgroup of G. If $R/1$ is $\mathscr{F}$-central or is a p'-group, then every $\mathscr{F}$-eccentric chief factor of G of order divisible by p is above R. Since $G/R \in \mathscr{F}_p^*$, every element of G/R induces an inner automorphism on each $\mathscr{F}$-eccentric chief factor of G/R of order divisible by p. Hence $G \in \mathscr{F}_p^*$ by the Jordan–Hölder theorem. This contradiction shows that the factor $R/1$ is $\mathscr{F}$-eccentric of order divisible by p. Suppose that R is non-abelian. Then $G_{\mathscr{S}} = 1$. Since $G \in \mathscr{M}_p$ we have $G \cong G/G_{\mathscr{S}} \in f^*(0) = \mathscr{F}_p^*$. This contradiction shows that R is an abelian p-group. Let $C = C_G(R)$. By [2, IV, (1.5)], we have $T = R \rtimes (G/C) \in \mathscr{M}_p$. Suppose that $R \neq C$. Then $|T| = |R \rtimes (G/C)| < |G|$. The minimal choice of G implies that $T \in \mathscr{F}_p^*$. Obviously $C_T(R) = R$ and so $R \rtimes (T/C_T(R)) \cong T = R \rtimes (G/C)$. Then since $R/1$ is $\mathscr{F}$-eccentric in G, it is $\mathscr{F}$-eccentric in T. Thus $T = CR = C$ by Lemma 2.4. It follows that $R/1$ is $\mathscr{F}$-central in T since $\mathscr{N} \subseteq \mathscr{F}$ by hypothesis. This contradiction shows that $R = C$ and hence $R = C^p(G)$. Since $G \in \mathscr{M}_p$ we conclude that $G/R \in F(p) = f_p^*(p)$ and consequently $G \in \mathcal{N}_p F(p) \subseteq \mathcal{F} \subseteq \mathcal{F}_p^*$. This contradiction shows that $\mathcal{F}_p^* = \mathcal{M}_p$.

(2) Let f^* be a function of the form $(*)$ such that $f^*(q) = F(q) = \mathcal{N}_q \mathcal{F}(q)$ for all primes q and $f^*(0) = \mathcal{F}^*$. Then as above it may be proved that $\mathcal{F}^* = \mathrm{CLF}(f^*)$ and so (2) holds. □

Proof of Theorem A. By [2, IV, (4.11)], for any normal soluble subgroup N of a group G, we have $C^p(G/\Phi(N)) = C^p(G)/\Phi(N)$. Hence by Proposition 2.5, the formations $\mathcal{F}_p^*$ and $\mathcal{F}^*$ are solubly saturated.

Now suppose that $\mathcal{F}$ is a normally hereditary formation. We prove that $\mathcal{F}^*$ and $\mathcal{F}_p^*$ are also normally hereditary. Let N be a normal subgroup of the quasi-$\mathcal{F}$-group (resp. of the p-quasi-$\mathcal{F}$-group) G. If $L \leqslant K \leqslant H \leqslant D \leqslant N$, where D/L is a chief factor of G and H/K is an $\mathcal{F}$-eccentric chief factor of N (resp. H/K is an $\mathcal{F}$-eccentric chief factor of N of order divisible by p), then by Lemma 2.2, D/L is an $\mathcal{F}$-eccentric chief factor of G (resp. D/L is an $\mathcal{F}$-eccentric chief factor of G of order divisible by p). By hypothesis, every element $n \in N$ induces an inner automorphism in D/L. Then by Lemma 2.3, n induces an inner automorphism in H/K. Therefore N is a quasi-$\mathcal{F}$-group (resp. is a p-quasi-$\mathcal{F}$-group). This completes the proof. □

Proof of Corollary 1.3. This follows from Theorem A and [2, IV, (4.17)]. □

Proof of Theorem B. Let $F = F(G)$. We only need to show that if for every $\mathcal{F}$-eccentric G-chief factor H/K between $\Phi(F)$ and $F^*(G)$ every automorphism of H/K induced by an element of G is inner, then $G \in \mathcal{F}^*$. Suppose that this is false and let G be a counter-example of minimal order. By Theorem A, $F^*(G/\Phi(F)) = F^*(G)/\Phi(F)$. Hence the hypothesis holds for $G/\Phi(F)$. If $\Phi(F) \neq 1$, then the minimal choice of G implies that $G/\Phi(F) \in \mathcal{F}^*$. Then by Theorem A again, $G \in \mathcal{F}^*$. This contradiction shows that $\Phi(F) = 1$. Therefore for every $\mathcal{F}$-eccentric G-chief factor H/K of $F^*(G)$, every automorphism of H/K induced by an element of G is inner. It follows that every G-chief factor of F is central.

Now let f^* be a function of the form $(*)$ such that $f^*(0) = \mathcal{F}^*$ and $f^*(q) = \mathcal{N}_q \mathcal{F}(q) \subseteq \mathcal{F}$ for all primes q. Then $\mathcal{F}^* = \mathrm{CLF}(f^*)$ by Proposition 2.5. Hence $G/\mathrm{C}_G(H/K) \in f^*(|H/K|)$ for every G-chief factor H/K of F. On the other hand, if H/K is a chief factor of G between F and $F^*(G)$, then

$$\mathrm{C}_{G/K}(H/K)(H/K) = (\mathrm{C}_G(H/K)/K)(H/K) = G/K$$

by hypothesis and Lemma 2.4. Hence $G/\mathrm{C}_G(H/K) \cong H/K$ is semisimple. Consequently $G/\mathrm{C}_G(H/K) \in \mathcal{F}^* = f^*(0)$. Since, by Corollary 1.3, $\mathcal{F}^*$ is a Baer-local formation, by using the analogue of [3, Theorem 3.1.6] for Baer-local formations and the well-known Schmid–Shemetkov theorem on $\mathcal{F}^*$-stable automorphism groups (see [11, II, Theorem 9.3] or [3, Theorem 3.2.6]), we obtain that $G/\mathrm{C}_G(F^*(G)) \in \mathcal{F}^*$. But by [5, X, (13.12)], $\mathrm{C}_G(F^*(G)) \leqslant F$, hence $G \in \mathcal{F}^*$ by Theorem A. This contradiction completes the proof. □

Proof of Theorem C. (1) The proof is similar to the proof of [5, X, Theorem 13.6]. We only need to prove that if G is a p-quasi-$\mathscr{F}$-group, then $G/\mathrm{Z}_{\mathscr{F}_p}(G)$ is semisimple and the order of each of its composition factors is divisible by p. Let $Z = \mathrm{Z}_{\mathscr{F}_p}(G)$. If $Z \neq 1$, then the inductive hypothesis may be applied to G/Z by Lemma 2.1, and the assertion holds. Now assume that $Z = 1$. Let R be a minimal normal subgroup of G and $C = \mathrm{C}_G(R)$. Then p divides $|R|$, since otherwise $R \leqslant Z$, which is impossible. Since $\mathscr{F}$ contains all nilpotent groups, $\mathrm{Z}(G) = 1$ and hence $C \neq G$. By Theorem A, C is a p-quasi-$\mathscr{F}$-group. Hence by the inductive hypothesis, $C/\mathrm{Z}_{\mathscr{F}_p}(C)$ is semisimple and the order of each composition factor is divisible by p. Since $Z = 1$, R is $\mathscr{F}$-eccentric. Therefore $G = RC$ by Lemma 2.4 and so $R \cap C \leqslant \mathrm{Z}(G) = 1$. It follows that $G = R \times C$. Hence R is non-abelian and $\mathrm{Z}_{\mathscr{F}_p}(C) = 1$. This shows that G is semisimple and the order of each of its composition factors is divisible by p.

(2) This follows from (1). □

3 Some characterizations of quasisoluble groups and quasisupersoluble groups

The characterizations of quasisupersolubility and quasisolubility of groups in this section are based on the following concept.

Definition 3.1. Let H be a subgroup of a group G. We say that H is *nearly normal* in G if G has a normal subgroup T such that $T \cap H \leqslant H_G$ and $HT = H^G$.

The following lemma can be proved by direct calculations.

Lemma 3.1. *Let G be a group and $H \leqslant K \leqslant G$.*

(1) *Suppose that H is normal in G. Then K/H is nearly normal in G/H if and only if K is nearly normal in G.*

(2) *If H is nearly normal in G, then H is nearly normal in K.*

(3) *Suppose that H is normal in G. Then HE/H is nearly normal in G/H for every nearly normal subgroup E of G satisfying $(|H|, |E|) = 1$.*

Lemma 3.2. *Let X be a normal subgroup of a group G. Suppose that every maximal subgroup of X is nearly normal in G. Then X is soluble.*

Proof. We prove the result by induction on $|G|$. Let N be a minimal normal subgroup of G contained in X. Then X/N is soluble. Indeed, if $N = X$, this is clear. Otherwise, by Lemma 3.1 (1) the hypothesis holds for G/N and so X/N is soluble by induction. If G has a minimal normal subgroup $R \neq N$ with $R \leqslant X$, then $X \cong X/1 = X/(N \cap R)$ is soluble. Therefore we may assume that N is the only minimal normal subgroup of G contained in X. Now let p be a prime dividing $|N|$ and N_p be a Sylow p-subgroup of N. Then $N_p = N \cap P$ for some Sylow p-subgroup P of X. Obviously $P \leqslant \mathrm{N}_X(N_p)$. Also, by the Frattini argument, $X = N\mathrm{N}_X(N_p)$. Suppose that $N \neq N_p$. Then for some maximal subgroup M of X we have $\mathrm{N}_X(N_p) \leqslant M$. Hence $N \nleqslant M$ and p does not divide $|X : M|$. It follows that $M_G = 1$. By hypothesis M is nearly normal in G. Let T be a normal subgroup of G such that $X = MT = M^G$ and $T \cap M \leqslant M_G = 1$. Then $X = T \rtimes M$ and $N \leqslant T$. It is also clear that T is a minimal normal subgroup of G. Hence $N = T$ and so p divides $|X : M| = |N|$. This contradiction shows that N is a p-group. Consequently X is soluble. $\square$

Theorem 3.3. *A group G is quasisoluble if and only if G has a normal subgroup X such that G/X is semisimple, every maximal subgroup of X is nearly normal in G and for every $x \in G$ and every G-chief factor H/K of X, the automorphism of H/K induced by x is also induced by some element of X.*

Proof. We first prove the 'if' part by induction on $|G|$. Let N be a minimal normal subgroup of G contained in X. We claim that G/N is quasisoluble. Indeed, if $N = X$, this is clear. Otherwise, by Lemma 3.1 (2) the hypothesis holds for G/N and so by induction G/N is quasisoluble. By Lemma 3.2, X is soluble and hence N is a p-group for some prime p. Now let $C = C_G(N)$ and $g \in G$. Then by the hypothesis the automorphism of N induced by g is induced by some element x of X. Hence $gx^{-1} \in C$ and so $G = CX$. Then $G/C \cong X/(X \cap C)$ is soluble. This means that the factor $N/1$ is $\mathscr{S}$-central. But since G/N is quasisoluble, by Theorem C, we obtain that G is quasisoluble.

Now we prove the necessity part. Let $X = Z_{\mathscr{S}}(G)$ be the $\mathscr{S}$-hypercenter of the quasisoluble group G. Then by Theorem C, G/X is a semisimple group. Moreover, for any G-chief factor H/K of X the group $H/K \rtimes (G/C_G(H/K))$ is soluble. Hence $G/C_G(H/K)$ is soluble and so $XC_G(H/K) = G$. It follows that for every $x \in G$ the automorphism of H/K induced by x is also induced by some element of X. Finally, we prove that every maximal subgroup M of X is nearly normal in G. Suppose that $M_G \neq 1$. Then by induction M/M_G is nearly normal in G/M_G. Hence by Lemma 3.1 (1), M is nearly normal in G. Now let $M_G = 1$ and let N be a minimal normal subgroup of G such that $NM = X$. Let $D = N \cap M$. Since $N \leqslant Z_{\mathscr{S}}(G)$, N is abelian. Hence D is normal in X. On the other hand, $C_G(N) \leqslant N_G(D)$. This implies that D is normal in $G = XC_G(N)$ and so $D \leqslant M_G = 1$. Thus M is nearly normal in G and the theorem is proved. $\square$

Recall that a subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leqslant H_G = \mathrm{core}_G(H)$ (see [15]).

Corollary 3.4 (Wang [15]). *A group G is soluble if and only if every maximal subgroup M of G is c-normal in G.*

Proof. Suppose that G is soluble and M is a maximal subgroup of G. If M is normal in G, then $G = GM$ and $G \cap M = M = M_G$. Now assume that M is not normal in G and let T/M_G be a chief factor of G. Then T/M_G is abelian. Hence $T \cap M \leqslant M_G$ and $MT = G$. The converse is obvious by Lemma 3.2 since by the hypothesis we have $M^G = M(M^G \cap T)$. $\square$

Corollary 3.5. *A group G is soluble if and only if every maximal subgroup of G is nearly normal in G.*

Lemma 3.6. *Let P be a normal p-subgroup of a group G. If P is elementary abelian and every maximal subgroup of P is nearly normal in G, then every minimal normal subgroup of G contained in P has prime order.*

Proof. Let N be any minimal normal subgroup of G contained in P. Suppose that $|N| > p$ and let M be a maximal subgroup of P such that $NM = P$. Then M is not normal in G and so $M^G = P$. Let T be a normal subgroup of G such that $P = MT$ and $T \cap M \leqslant M_G$. Suppose that $M_G \neq 1$. By Lemma 3.1 the hypothesis holds for G/M_G and so $|NM_G/M_G| = |N| = p$ by induction, a contradiction. Hence $M_G = 1$ and consequently $T \cap M = 1$. This implies that $|T| = p$ and $T \neq N$. But by induction, we have also that $|TN/T| = |N| = p$. This contradiction completes the proof. □

Lemma 3.7. *Suppose that every maximal subgroup M of every non-cyclic Sylow subgroup of a group G is nearly normal in G. Then G is soluble.*

Proof. Suppose that this lemma is false and let G be a counter-example of minimal order. Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then $p = 2$ by the Feit–Thompson theorem on groups of odd order. Clearly P is not cyclic (see [10, (10.1.9)]). Suppose that for some maximal subgroup V of P we have $V_G \neq 1$. Then by Lemma 3.1 the hypothesis holds for G/V_G and so G/V_G is soluble, which implies the solubility of G. Therefore $V_G = 1$ for all maximal subgroups V of P. Let $P = V_1V_2$ for some maximal subgroups V_1 and V_2 of P. By hypothesis G has a normal subgroup T_i such that $D_i = V_i^G = V_iT_i$ and $T_i \cap V_i \leqslant (V_i)_G = 1$. Clearly, $P \cap T_i$ is a Sylow 2-subgroup of T_i. But since $T_i \cap V_i = 1$ we have $|T_i \cap P| \leqslant 2$. Hence T_i is soluble and so V_i^G is soluble. It follows that $D = V_1^G V_2^G$ is soluble and therefore G is soluble since G/D is a $2'$-group. This contradiction completes the proof. □

Theorem 3.8. *The following properties are equivalent*:

(1) *G is quasisupersoluble*;

(2) *G has a normal subgroup E such that G/E is semisimple and every maximal subgroup M of every Sylow subgroup of $F^*(E)$ is nearly normal in G*;

(3) *G has a normal subgroup E such that G/E is quasisupersoluble and every maximal subgroup M of every Sylow subgroup of $F^*(E)$ is nearly normal in G.*

Proof. (1) ⇒ (2) Let $E = Z_{\mathscr{U}}(G)$. By Theorem C, G/E is semisimple. Since E is supersoluble, $F^*(E) = F(E)$. Let M be a maximal subgroup of some Sylow subgroup P of $F(E)$. Since $F(E)$ is characteristic in E and P is characteristic in $F(E)$, P is normal in G. We now prove that M is nearly normal in G. If M is normal in G, this is clear. Hence let $M \neq M^G$. Suppose that $\Phi = \Phi(G) \cap P \neq 1$. Let L be a minimal normal subgroup of G contained in Φ. Then $F(E)/L = F^*(E)/L = F^*(E/L)$ by [4, III, (3.5)] and M/L is a maximal subgroup of the Sylow subgroup P/L of $F^*(E/L)$. Hence by induction M/L is nearly normal in G/L. It follows from Lemma 3.1 that M is nearly normal in G. Now suppose that $\Phi = 1$. Then by [12, II, Lemma 7.9], P is a product of minimal normal subgroups $N_1, N_2, \ldots, N_t$ of G. Obviously, for some i we have $N_i \nleqslant M$. Since $N_i \leqslant Z_{\mathfrak{U}}(G)$, $|N_i|$ is a prime. Hence $N_i \cap M = 1$ and $MN_i = M^G = P$. This shows that M is nearly normal in G.

(2) $\Rightarrow$ (3) This is obvious since a semisimple group is clearly quasisupersoluble.

(3) $\Rightarrow$ (1) Suppose that G has a normal subgroup E such that G/E is quasisupersoluble and every maximal subgroup M of every Sylow subgroup of $F^*(E)$ is nearly normal in G. We shall prove that G is quasisupersoluble. Suppose that this is false and let G be a counter-example with minimal $|G|\,|E|$. Let p be prime dividing $|F^*(E)|$ and let P be a Sylow p-subgroup of $F^*(E)$. In view of Lemma 3.7 we have $F^*(E) = F(E)$.

We first show that no minimal subgroup of P is normal in G. Suppose that some minimal subgroup L of P is normal in G and let $C = C_E(L)$. We claim that the hypothesis is true for (G, C). Indeed, by Lemma 2.1, $G/C = G/(E \cap C_G(L))$ is quasisupersoluble. In addition, since $F^*(E) = F(E) \leqslant C$, we have $F^*(C) = F^*(E)$. Hence by Lemma 3.1, the hypothesis holds for (G, C). The choice of (G, E) implies that $C = E$. It follows that $L \leqslant Z(E)$. Thus by Theorem C we have $F^*(C/L) = F^*(E)/L$. Now, by Lemma 3.1, the hypothesis holds for $(G/L, C/L)$. Hence G/L is quasisupersoluble and so G is quasisupersoluble, a contradiction. Therefore no minimal subgroup of P is normal in G.

If $\Phi(P) = 1$, then P is elementary abelian p-group. Hence by Lemma 3.6 for every minimal normal subgroup L of G contained in P we have $|L| = p$, which is a contradiction. Thus $\Phi(P) \neq 1$. By Theorem A, we have $F^*(E/\Phi(P)) = F^*(E)/\Phi(P)$. Then by Lemma 3.1, the hypothesis holds for $(G/\Phi(P), E/\Phi(P))$. But $|G/\Phi(P)| < |G|$ and hence $G/\Phi(P)$ is quasisupersoluble by the choice of G. Now by Theorem A again, G is quasisupersoluble. This contradiction completes the proof. □

In view of Lemma 3.7, we obtain from Theorem 3.8 the following new characterization of the supersoluble groups.

Corollary 3.9. *A group G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of $F^*(G)$ is nearly normal in G.*

Corollary 3.10 (Ramadan [8]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(G)$ are normal in G, then G is supersoluble.*

Corollary 3.11 (Li and Guo [7]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

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