Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **26** (2018). Number 1, pp. 90–96

On finite groups with Hall normally embedded Schmidt subgroups

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Communicated by L. A. Kurdachenko

To the 70th anniversary of Academician of the National Academy of Sciences of Belarus V. I. Yanchevskii

Abstract. A subgroup H of a finite group G is said to be Hall normally embedded in G if there is a normal subgroup N of G such that H is a Hall subgroup of N. A Schmidt group is a non-nilpotent finite group whose all proper subgroups are nilpotent. In this paper, we prove that if each Schmidt subgroup of a finite group G is Hall normally embedded in G, then the derived subgroup of G is nilpotent.

1. Introduction

All groups in this paper are finite. We use the standard notation and terminology of [1, 2].

A Schmidt group is a non-nilpotent group in which every proper subgroup is nilpotent. O. Y. Schmidt [3] initiated the investigations of such groups. He proved that a Schmidt group is biprimary (i. e. its order is divided by only two different primes), one of its Sylow subgroups is normal and other one is cyclic. In [3], it was also specified the index system of the chief series of a Schmidt group. Reviews on the structure of the Schmidt

2010 MSC: 20E28, 20E32, 20E34.
Key words and phrases: finite group, Hall subgroup, normal subgroup, derived subgroup, nilpotent subgroup.

groups and their applications in the theory of finite groups are available in [4, 5].

Since every non-nilpotent group contains a Schmidt subgroup, Schmidt groups are universal subgroups of groups. So naturally the properties of Schmidt subgroups contained in a group have a significant influence on the group structure. Groups with some restrictions on Schmidt subgroups was investigated in many papers. For example, groups with subnormal Schmidt subgroups were studied in [6]–[8], and groups with Hall Schmidt subgroups were described in [9].

The normal closure of a subgroup H in a group G is the smallest normal subgroup of G containing H. It is clear that the normal closure

$$H^G = \langle H^x \mid x \in G \rangle = \bigcap_{H \leqslant N \lhd G} N$$

coincides with the intersection of all normal subgroups of G containing H.

A subgroup H of a group G is said to be Hall normally embedded in G if there is a normal subgroup N of G such that $H \leqslant N$ and H is a Hall subgroup of N, i.e., $(|H|, |N : H|) = 1$. In this situation the subgroup H is a Hall subgroup of H^G. It is clear that all normal and all Hall subgroups of G are Hall normally embedded in G.

Groups in which some subgroups are normally embedded were studied, for example, in [10]–[13].

In this paper, we study groups with Hall normally embedded Schmidt subgroups. The following theorem is proved.

Theorem. *If each Schmidt subgroup of a group G is Hall normally embedded in G, then the derived subgroup of G is nilpotent.*

2. Preliminaries

Throughout this paper, p and q are always different primes. Recall that a p-closed group is a group with a normal Sylow p-subgroup, and a p-nilpotent group is a group of order $p^a m$, where p does not divide m, with a normal subgroup of order m. A pd-group is a group of the order divided by p. A group of order $p^a q^b$, where a and b are non-negative integers, is called a $\{p, q\}$-group.

If q divides $p^n - 1$ and does not divide $p^{n_1} - 1$ for all $1 \leqslant n_1 < n$, then we say that the positive integer n is the order of p modulo q.

Let G be a group. We denote by $\pi(G)$ the set of all prime divisors of the order of G. We use $Z(G)$, $\Phi(G)$ and $F(G)$ to denote the center, the Frattini subgroup and the Fitting subgroup of G, respectively. As usual, $O_p(X)$ and $O_{p'}(X)$ are the largest normal p- and p'-subgroups of X, respectively. We denote by $[A]B$ a semidirect product of two subgroups A and B with a normal subgroup A. The symbol $\square$ indicates the end of the proof.

We need the following properties of Schmidt groups.

Lemma 1 ([3,5]). *Let S be a Schmidt group. Then the following statements hold:*

(1) $\pi(S) = \{p, q\}$, $S = [P]\langle y\rangle$, *where P is a normal Sylow p-subgroup, $\langle y\rangle$ is a non-normal Sylow q-subgroup, $y^q \in Z(S)$;*

(2) $P/\Phi(P)$ *is a minimal normal subgroup of* $G/\Phi(P)$, $\Phi(P) = P' \leqslant Z(G)$;

(3) $|P/\Phi(P)| = p^n$, *n is the order of p modulo q.*

Following [6], a Schmidt group with a normal Sylow p-subgroup and a non-normal cyclic Sylow q-subgroup is called an $S_{\langle p,q\rangle}$-group. So if G is an $S_{\langle p,q\rangle}$-group, then $G = [P]Q$, where P is a normal Sylow p-subgroup and Q is a non-normal cyclic Sylow q-subgroup.

Lemma 2 ([6, Lemma 6]). (1) *If a group G has no p-closed Schmidt subgroups, then G is p-nilpotent.*

(2) *If a group G has no 2-nilpotent Schmidt $2d$-subgroups, then G is 2-closed.*

(3) *If a p-soluble group G has no p-nilpotent Schmidt pd-subgroups, then G is p-closed.*

Lemma 3. *Let A be a subgroup of a group G such that A is a Hall subgroup of A^G.*

(1) *If H is a subgroup of G, $A \leqslant H$, then A is a Hall subgroup of A^H.*

(2) *If N is a normal subgroup of G, then AN/N is a Hall subgroup of $(AN/N)^{(G/N)}$.*

Proof. 1. By the hypothesis, A is a Hall subgroup of A^G and $A \leqslant H \cap A^G$. Since A^G is normal in G, it follows that $H \cap A^G$ is normal in H. So $A^H \leqslant H \cap A^G \leqslant A^G$ and A is a Hall subgroup of A^H.

2. Since $A^G N$ is normal in G and $AN \leqslant A^G N$, so $(AN/N)^{(G/N)} \leqslant A^G N/N$. By the hypothesis, A is a Hall subgroup of A^G, thus AN/N is a Hall subgroup of $A^G N/N$. Therefore, AN/N is a Hall subgroup of $(AN/N)^{(G/N)}$. $\square$

Lemma 4. *Let K and D be subgroups of a group G such that D is normal in K. If K/D is an $S_{\langle p,q\rangle}$-subgroup, then each minimal supplement L to D in K has the following properties:*

(1) *L is a p-closed $\{p,q\}$-subgroup;*
(2) *all proper normal subgroups of L are nilpotent;*
(3) *L includes an $S_{\langle p,q\rangle}$-subgroup $[P]Q$ such that D does not include Q and $L = ([P]Q)^L = Q^L$;*
(4) *if $[P]Q$ is a Hall subgroup of $([P]Q)^G$, then $L = [P]Q$.*

Proof. Assertions (1)–(3) were established in [6, Lemma 2]. Let us verify assertion (4). If $[P]Q$ is a Hall subgroup of $([P]Q)^G$, then $[P]Q$ is a Hall subgroup of $([P]Q)^L = L$ by Lemma 3 (1), and $L = [P]Q$. □

Lemma 5. *If H is a subgroup of a group G generated by all $S_{\langle p,q\rangle}$-subgroups of G, then G/H has no $S_{\langle p,q\rangle}$-subgroups.*

Proof. Assume the contrary. Suppose that A/H is a $S_{\langle p,q\rangle}$-subgroup of G/H. By Lemma 4, in A there is an $S_{\langle p,q\rangle}$-subgroup S such that $S^A H = A$. However, $S^A \leqslant H$ by the choice of H, i. e $A = H$, a contradiction. □

Lemma 6. *Let each $S_{\langle p,q\rangle}$-subgroup of a group G be Hall normally embedded in G.*

(1) *If H is a subgroup of G, then each $S_{\langle p,q\rangle}$-subgroup of H is Hall normally embedded in H.*
(2) *If N is a normal subgroup of G, then each $S_{\langle p,q\rangle}$-subgroup of G/N is Hall normally embedded in G/N.*

Proof. 1. Let A be an $S_{\langle p,q\rangle}$-subgroup of H. Therefore, A is an $S_{\langle p,q\rangle}$-subgroup of G. By the hypothesis, A is a Hall subgroup of A^G. By Lemma 3 (1), A is a Hall subgroup of A^H.

2. Let K/N be an $S_{\langle p,q\rangle}$-subgroup of G/N, and let L be a minimal supplement to N in K. By Lemma 4 (4), L is an $S_{\langle p,q\rangle}$-subgroup, therefore, L is Hall normally embedded in G. By Lemma 3 (2), $LN/N = K/N$ is Hall normally embedded in G/N. □

Lemma 7. *Let G be a p-soluble group and $l_p(G) > 1$. If $l_p(H) \leqslant 1$ and $l_p(G/K) \leqslant 1$ for each $H < G$, $1 \neq K \lhd G$, then the following hold:*

(1) *$\Phi(G) = O_{p'}(G) = 1$;*
(2) *G has a unique minimal normal subgroup $N = F(G) = O_p(G) = C_G(N)$;*
(3) *$l_p(G) = 2$;*
(4) *$G = [N]S$, where $S = [Q]P$ is a p-nilpotent Schmidt subgroup, $|P| = p$.*

Proof. Assertions (1)–(2) follow from [2, VI.6.9]. As $l_p(N) = 1$ and $l_p(G/N) \leqslant 1$ we have $l_p(G) = 2$. It remains to prove assertion (4). Since G is a p-soluble non-p-closed group, we conclude from Lemma 2 (3) that in G there is an $S_{\langle q,p\rangle}$-subgroup $S = [Q]P$ for some $q \in \pi(G)$. Suppose that NS is a proper subgroup of G. Then $O_{p'}(NS) \leqslant C_G(N) = N$. Thus, $O_{p'}(NS) = 1$. By the hypothesis, $l_p(NS) = 1$, so NS is p-closed. This contradicts the fact that S is not p-closed. Therefore, $NS = G$. Moreover $N \cap S \lhd G$, $N \cap S = 1$, and S is a maximal subgroup of G. Since $O_p(S) = 1$, it follows from Lemma 1 that $|P| = p$. □

Lemma 8. *If each p-nilpotent Schmidt pd-subgroup of a p-soluble group G is Hall normally embedded in G, then $l_p(G) \leqslant 1$.*

Proof. Let G be a counterexample of minimal order. By Lemma 6, each proper subgroup and each non-trivial quotient group of G have a p-length $\leqslant$ 1. By Lemma 7,

$$G = [N]S,\ \Phi(G) = O_{p'}(G) = 1,\ N = O_p(G) = F(G) = C_G(N),$$

where $S = [Q]P$ is a maximal subgroup of G and is an $S_{\langle p,q\rangle}$-subgroup for some $q \in \pi(G)$. By the hypothesis, S is a Hall subgroup of S^G. Since $S^G = G$, it follows that N is a p'-subgroup, a contradiction. □

Lemma 9. *Let $n \geqslant 2$ be a positive integer, let r be a prime, and let π be the set of primes t such that t divides $r^n - 1$ but t does not divide $r^{n_1} - 1$ for all $1 \leqslant n_1 < n$. Then the group $\mathrm{GL}(n, r)$ contains a cyclic π-Hall subgroup.*

Proof. The group $G = \mathrm{GL}(n, r)$ is of order

$$r^{n(n-1)/2}(r^n - 1)(r^{n-1} - 1)\dots(r^2 - 1)(r - 1).$$

By Theorem II.7.3 [2], G contains a cyclic subgroup T of order $r^n - 1$. Its π-Hall subgroup T_π is a π-Hall subgroup of G, because t does not divide $r^{n_1} - 1$ for all $t \in \pi$ and all $1 \leqslant n_1 < n$. □

3. Proof of the theorem

We proceed by induction on the order of G. First, we verify that G is soluble. Assume the contrary. It follows that G is not 2-closed, and by Lemma 2 (2), in G there exists a 2-nilpotent Schmidt subgroup $S = [P]Q$ of even order, where P is a Sylow p-subgroup of order $p > 2$, Q is a cyclic Sylow 2-subgroup. By the hypothesis, S is a Hall subgroup of S^G, therefore, Q is a Sylow 2-subgroup of S^G, and S^G is 2-nilpotent. Thus, $S \leqslant S^G \leqslant R(G)$. Here $R(G)$ is the largest normal soluble subgroup of G. Since S is arbitrary, we conclude that all 2-nilpotent Schmidt subgroups of even order are contained in $R(G)$. By Lemma 5, the quotient group $G/R(G)$ has no 2-nilpotent Schmidt subgroups of even order. By Lemma 2 (2), the quotient group $G/R(G)$ is 2-closed, therefore, G is soluble.

Note that the derived subgroup G' is nilpotent if and only if $G \in \mathfrak{N}\mathfrak{A}$. Here $\mathfrak{N}$, $\mathfrak{A}$ and $\mathfrak{E}$ are the formations of all nilpotent, abelian and finite groups, respectively, and

$$\mathfrak{N}\mathfrak{A} = \{\ G \in \mathfrak{E} \mid G' \in \mathfrak{N}\}$$

is the formation product of $\mathfrak{N}$ and $\mathfrak{A}$. According to [14, p. 337], $\mathfrak{N}\mathfrak{A}$ is an s-closed saturated formation. The quotient group $G/N \in \mathfrak{N}\mathfrak{A}$ for each non-trivial normal subgroup N of G by Lemma 6 (2). A simple check shows that

$$G = [N]M,\ N = O_p(G) = F(G) = C_G(N),\ |N| = p^n,\ M_G = 1,$$

where N is a unique minimal normal subgroup of G, M is a maximal subgroup of G. In view of Lemma 7, N is a Sylow p-subgroup of G.

Let $\pi = \pi(M) = \pi(G) \setminus \{p\}$, $r \in \pi$, and let R be a Sylow r-subgroup of G. Since $N = C_G(N)$, we obtain from Lemma 2 (1) that in $[N]R$ there is an $S_{\langle p,r\rangle}$-subgroup $[P_1]R_1$. By the hypothesis, $[P_1]R_1$ is a Hall subgroup of $([P_1]R_1)^G$, therefore, P_1 is a Sylow p-subgroup of $([P_1]R_1)^G$. Since $N \leqslant ([P_1]R_1)^G$ and N is a Sylow p-subgroup of G, it follows that $N = P_1$. By Lemma 1, n is the order of p modulo r. But r is an arbitrary number from π, so n is the order of p modulo q for all $q \in \pi$. The group $M \simeq G/N$ is isomorphic to a subgroup of $\mathrm{GL}(n,p)$, which contains a cyclic Hall π-subgroup H by Lemma 9. In view of Theorem 5.3.2 [15], M is contained in a subgroup H^x, $x \in \mathrm{GL}(n,p)$. Therefore, M is cyclic. □

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Received by the editors: 20.04.2018.