A Robinson characterization of finite $P\sigma T$-groups

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and let G be a finite group. Then G is said to be *σ-full* if G has a Hall σ_i-subgroup for all i. A subgroup A of G is said to be *σ-permutable* in G provided G is σ-full and A permutes with all Hall σ_i-subgroups H of G (that is, $AH = HA$) for all i.

We obtain a characterization of finite groups G in which σ-permutability is a transitive relation in G, that is, if K is a σ-permutable subgroup of H and H is a σ-permutable subgroup of G, then K is a σ-permutable subgroup of G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, p_2, \ldots\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

If $1 \in \mathfrak{F}$ is a class of groups, then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-*residual of* G, that is, intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$; $G_{\mathfrak{F}}$ denotes the $\mathfrak{F}$-*radical of* G, that is, the product of all normal subgroups N of G with $N \in \mathfrak{F}$.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; G is said to be *σ-full* [1, 2] if G has a Hall σ_i-subgroup for all i.

Definition 1.1. We say that a subgroup A of G is *σ-permutable* in G [3] provided G is σ-full and H permutes with all Hall σ_i-subgroups H of G (that is, $AH = HA$) for all i.

Remark 1.2. A set $\mathcal{H}$ of subgroups of G is a *complete Hall σ-set* of G [1, 2] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i. By Proposition 3.1 in [4], a subgroup A of G is σ-permutable in G if and only if G possesses a a complete Hall σ-set $\mathcal{H}$ such that $AL^x = L^x A$ for all $L \in \mathcal{H}$ and all $x \in G$.

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Recall that G is said to be: *σ-primary* [3] if G is a σ_i-group for some i, *σ-decomposable* (Shemetkov [5]) or *σ-nilpotent* (Guo and Skiba [6]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$.

The usefulness of σ-permutable subgroups is connected mostly with the following their property.

Theorem A. (See Theorem B in [3]). *If A is a σ-permutable subgroup of G, then A^G/A_G is σ-nilpotent.*

Example 1.3. (i) In the classical case, when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\}$, the subgroup A of G is σ^0-permutable in G if and only if A permutes with all Sylow subgroups of G. Note that a σ^0-permutable subgroup is also called *S-permutable* [7]. Note also that for every S-permutable subgroup A of G the quotient A^G/A_G is nilpotent (Kegel, Deskins) by Theorem A.

(ii) In the other classical case, when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, a subgroup A of G is σ^π-permutable in G if and only if G has a Hall π-subgroup and a Hall π'-subgroup and A permutes with all Hall π-subgroups and with all Hall π'-subgroups of G. For every σ^π-permutable subgroup A of G the quotient A^G/A_G is π-decomposable, that is, $A^G/A_G = O_\pi(A^G/A_G) \times O_{\pi'}(A^G/A_G)$ by Theorem A.

(iii) In fact, in the theory of π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{0\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. The subgroup A of G is $\sigma^{0\pi}$-permutable in G if and only if G has a Hall π'-subgroup and A permutes with all Hall π'-subgroups and with all Sylow p-subgroups of G for all $p \in \pi$. For every $\sigma^{0\pi}$-permutable subgroup A of G the quotient A^G/A_G is π-nilpotent, that is, $A^G/A_G = O_\pi(F(A^G/A_G)) \times O_{\pi'}(A^G/A_G)$ by Theorem A.

We say, following [3], that G is a *$P\sigma T$-group* if σ-permutability is a transitive relation in G, that is, if K is a σ-permutable subgroup of H and H is a σ-permutable subgroup of G, then K is a σ-permutable subgroup of G. In the case when $\sigma = \sigma^0$, a $P\sigma T$-group is also called a *PST-group* [7].

Note that if $G = (Q_8 \rtimes C_3) \curlywedge (C_7 \rtimes C_3)$ (see [8, p. 50]), where $Q_8 \rtimes C_3 = SL(2,3)$ and $C_7 \rtimes C_3$ is a non-abelian group of order 21, then G is not a PST-group but G is a $P\sigma T$-group, where $\sigma = \{\{2,3\}, \{2,3\}'\}$

The description of PST-groups was first obtained by Agrawal [9], for the soluble case, and by Robinson in [10], for the general case. In the further publications, authors (see, for example, the recent papers [11]–[21] and Chapter 2 in [7]) have found out and described many other interesting characterizations of PST-groups.

In the case when G is *σ-soluble* (that is, every chief factor of G is σ-primary) the description of $P\sigma T$-groups was obtained in the paper [22] on the base of the results and methods in [3, 23, 24, 25].

Theorem B (See Theorem A in [22]). *If G is a σ-soluble $P\sigma T$-group and $D = G^{\mathfrak{N}_\sigma}$ is the σ-nilpotent residual of G, then the following conditions hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent and every element of G induces a power automorphism in D;*

(ii) *$O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.*

Conversely, if Conditions (i) and (ii) hold for some subgroups D and M of G, then G is a $P\sigma T$-group.

Before continuing, we give some further definitions.

Definition 1.4. We say that G is:

(i) *σ-supersoluble* if every chief factor of G below $G^{\mathfrak{N}_\sigma}$ is cyclic;

(ii) a *σ-SC-group* if every chief factor of G below $G^{\mathfrak{N}_\sigma}$ is simple.

Example 1.5. (i) G is supersoluble if and only if G is σ-supersoluble where $\sigma = \sigma^0$ (see Example 1.3(i)).

(ii) The group G is called an *SC-group* (Robinson [10]) or a *c-supersoluble group* (Vedernikov [26]) if every chief factor of G is a simple group. Note that G is an SC-group if and only if G is σ-SC-group where $\sigma = \sigma^0$.

(iii) Let $G = A_5 \times B$, where A_5 is the alternating group of degree 5 and $B = C_{29} \rtimes C_7$ is a non-abelian group of order 203, and let $\sigma = \{\{7\}, \{29\}, \{2, 3, 5\}, \{2, 3, 5, 7, 29\}'\}$. Then $G^{\mathfrak{N}_\sigma} = C_{29}$, so G is a σ-supersoluble group but it is neither soluble nor σ-nilpotent.

(iv) Let $G = SL(2, 7) \times A_7 \times A_5 \times B$, where $B = C_{43} \rtimes C_7$ is a non-abelian group of order 301, and let $\sigma = \{\{2, 3, 5\}, \{7, 43\}, \{2, 3, 5, , 7, 43\}'\}$. Then $G^{\mathfrak{N}_\sigma} = SL(2, 7) \times A_7$, so G is a σ-SC-group but it is not a σ-supersoluble group.

In what follows, $\mathfrak{U}_\sigma$ is the class of all σ-supersoluble groups; $\mathfrak{U}_{c\sigma}$ is the class of all σ-SC-groups.

We say that G is *σ-perfect* if $G^{\mathfrak{N}_\sigma} = G$, that is, $O^{\sigma_i}(G) = G$ for all i.

From Theorem B it follows that every σ-soluble $P\sigma T$-group is σ-supersoluble. Our first observation shows that in general case every $P\sigma T$-group is a σ-SC-group.

Proposition A. *Let G be a $P\sigma T$-group and let $D = G^{\mathfrak{S}_\sigma}$ be the σ-soluble residual of G. Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ whose members are PST-groups. Then the following conditions hold:*

(i) *G is a σ-SC-group.*

(ii) *$D = G^{\mathfrak{U}_\sigma}$ is σ-perfect and G/D is a σ-soluble $P\sigma T$-group.*

(iii) *G satisfies N_{σ_i} for all i.*

In this proposition we say that G *satisfies* N_{σ_i} if whenever N is a σ-soluble normal subgroup of G, σ_i'-elements of G induce power automorphisms in $O_{\sigma_i}(G/N)$. We say also, following [7, 2.1.18], that G *satisfies* N_p if whenever N is a soluble normal subgroup of G, p'-elements of G induce power automorphisms in $O_p(G/N)$.

Corollary 1.6 (See Proposition 2.1.1 in [7]). *Let G be a PST-group. Then:*

(i) *G is an SC-group, and*

(ii) *G of satisfies N_p for every prime p.*

Definition 1.7. We say that $(D, Z(D); U_1, \dots, U_k)$ is a *Robinson σ-complex* (a *Robinson complex* in the case $\sigma = \sigma^0$) of G if the following fold:

(i) D is a σ-perfect normal subgroup of G,

(ii) $D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$, where $U_i/Z(D)$ is a non-abelian simple chief factor of G for all i,

(iii) every chief factor of G below $Z(D)$ is cyclic, and

(iv) $D^0 \leq D$ for every normal subgroup D^0 of G satisfying Conditions (i), (ii) and (iii).

Example 1.8. Let $G = SL(2,7) \times A_7 \times A_5 \times B$ be the group in Example 1.5(iv) and $\sigma = \{\{2,3,5\}, \{7,43\}, \{2,3,5,7,43\}'\}$. Then

$$(SL(2,7) \times A_7, Z(SL(2,7)); SL(2,7), A_7 Z(SL(2,7)))$$

is a Robinson σ-complex of G and

$$(SL(2,7) \times A_7 \times A_5, Z(SL(2,7)); SL(2,7), A_7 Z(SL(2,7)), A_5 Z(SL(2,7))))$$

is a Robinson complex of G.

Being based on Theorems A and B and using some ideas in [10, 23, 24, 25], in the given paper we prove the following

Theorem C. *Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ whose members are PST-groups. Then G is a $P\sigma T$-group if and only if G has a σ-perfect normal subgroup D such that:*

(i) *G/D is a σ-soluble $P\sigma T$-group.*

(ii) *If $D \neq 1$, then G has a Robinson σ-complex of the form $(D, Z(D); U_1, \dots U_k)$, and*

(iii) *If $\{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$, where $1 \leq r < k$, then G and $G/U'_{i_1} \cdots U'_{i_r}$ satisfy N_{σ_i} for all i such that $\sigma_i \cap \pi(Z(D)) \neq \emptyset\}$.*

Corollary 1.9 (Robinson [10]). *A group G is a PST-group if and only if G has a perfect normal subgroup D such that:*

(i) *G/D is a soluble PST-group.*

(ii) *If $D \neq 1$, then G has a Robinson complex of the form $(D, Z(D); U_1, \dots U_k)$, and*

(iii) *If $\{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$, where $1 \leq r < k$, then G and $G/U'_{i_1} \cdots U'_{i_r}$ satisfy N_p for all $p \in \pi(Z(D))$.*

The class $1 \in \mathfrak{F}$ is said to be a *formation* if every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for every group G, that is, if $G \in \mathfrak{F}$, then also every homomorphic image of G belongs to $\mathfrak{F}$ and $G/N \cap R \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$. The formation $\mathfrak{F}$ is said to *(normally) hereditary* if $H \in \mathfrak{F}$ whenever $G \in \mathcal{F}$ and H is a (normal) subgroup of G.

We prove Proposition A and Theorem C in Section 3. But before, in Section 2, we study properties of σ-supersoluble groups and σ-SC-groups. In particular, we prove the following two results.

Proposition B. *For any partition σ of $\mathbb{P}$ the following hold:*

(i) *The class $\mathfrak{U}_{c\sigma}$ is a normally hereditary formation.*

(ii) *The class $\mathfrak{U}_{\sigma}$ is a hereditary formation.*

Theorem D *Let $N = G^{\mathfrak{N}_\sigma}$ and let $D = N^{\mathfrak{S}}$ be the soluble residual of N. Then G is a σ-SC-group if and only if the following hold:*

(i) *$D = G^{\mathfrak{U}_\sigma}$, and*

(ii) *if $D \neq 1$, then G has a Robinson complex of the form $(D, Z(D); U_1, \ldots, U_k)$, where $Z(D) = D_{\mathfrak{S}}$ is the soluble radical of D.*

Corollary 1.10 (Robinson [10]). *A group G is an SC-group if and only if G satisfies:*

(i) *$G/G^{\mathfrak{S}}$ is supersoluble.*

(ii) *If $D = G^{\mathfrak{S}} \neq 1$, then G has a Robinson complex of the form $(D, Z(D); U_1, \ldots, U_k)$.*

2 Proofs of Proposition B and Theorem B

The following lemma collects the properties of σ-nilpotent groups which we use in our proofs.

Lemma 2.1 (See Corollary 2.4 and Lemma 2.5 in [3]). *The class of all σ-nilpotent groups $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups.*

Lemma 2.2 (See [27, 2.2.8]). *If $\mathfrak{F}$ is a formation and N, R are subgroups of G, where N is normal in G, then*

(i) *$(G/N)^{\mathfrak{F}} = G^{\mathfrak{N}}N/N$, and*

(ii) *$G^{\mathfrak{N}}N = R^{\mathfrak{N}}N$ provided $G = RN$.*

Proof of Proposition B. (i) Let $D = G^{\mathfrak{N}_\sigma}$. First note that if R is a normal subgroup of G, then $(G/R)^{\mathfrak{N}_\sigma} = DR/R$ by Lemmas 2.1 and 2.2 and so from the G-isomorphism $DR/R \simeq D/(D \cap R)$ we get that every chief factor of G/R below $(G/R)^{\mathfrak{N}_\sigma}$ is simple if and only if every chief factor of G between D and $D \cap R$ is simple. Therefore if $G \in \mathfrak{U}_{c\sigma}$, then $G/R \in \mathfrak{U}_{c\sigma}$. Hence the class $\mathfrak{U}_{c\sigma}$ is closed under taking homomorphic images.

Now we show that if G/R, $G/N \in \mathfrak{U}_{c\sigma}$, then $G/(R \cap N) \in \mathfrak{U}_{c\sigma}$. We can assume without loss of generality that $R \cap N = 1$. Since $G/R \in \mathfrak{U}_{c\sigma}$, every chief factor of G between D and $D \cap R$ is simple. Also, every chief factor of G between D and $D \cap N$ is simple. Now let H/K be any chief factor of G below $D \cap R$. Then $H \cap D \cap N = 1$ and hence from the G-isomorphism

$$H(D \cap N)/K(D \cap N) \simeq H/(H \cap K(D \cap N)) = H/K(H \cap D \cap N) = H/K$$

we get that H/K is simple since $D \cap N \leq K(D \cap N) \leq D$. On the other hand, every chief factor of G between D and $D \cap R$ is also simple. Therefore the Jordan-Hölder theorem for groups with operators [28, A, 3.2] implies that every chief factor of G below D is simple. Hence $G \in \mathfrak{U}_{c\sigma}$, so the class $\mathfrak{U}_{c\sigma}$ is closed under taking subdirect products.

Finally, if $H \trianglelefteq G \in \mathfrak{U}_{c\sigma}$, then from Lemmas 2.1 and 2.2 and the isomorphism

$$H/(H \cap D) \simeq HD/D \in \mathfrak{N}_\sigma$$

we get that $H^{\mathfrak{N}_\sigma} \leq H \cap D$ and so every chief factor of H below $H^{\mathfrak{N}_\sigma}$ is simple since every chief factor of G below D is simple. Hence $H \in \mathfrak{U}_{c\sigma}$, so the class $\mathfrak{U}_{c\sigma}$ is closed under taking normal subgroups.

(ii) See the proof of (i).

The proposition is proved.

Lemma 2.3. *Let H/K be a non-abelian chief factor of G. If H/K is simple, then $G/HC_G(H/K)$ is soluble.*

Proof. Since $C_G(H/K)/K = C_{G/K}(H/K)$, we can assume without loss of generality that $K = 1$. Then

$$G/C_G(H) \simeq V \leq \mathrm{Aut}(H)$$

and

$$H/(H \cap C_G(H)) \simeq HC_G(H)/C_G(H) \simeq \mathrm{Inn}(H)$$

since $C_G(H) \cap H = 1$. Hence

$$G/HC_G(H) \simeq (G/C_G(H))/(HC_G(H)/C_G(H)) \simeq W \leq \mathrm{Aut}(H)/\mathrm{Inn}(H).$$

From the validity of the Schreier conjecture, it follows that $G/HC_G(H/K)$ is soluble. The lemma is proved.

Proof of Theorem D. First note that D is characteristic in N and $R = D_{\mathfrak{S}}$ is a characteristic subgroup of D, so both these subgroups are normal in G.

Necessity. In view of Proposition B(ii), $G/G^{\mathfrak{U}_\sigma}$ is σ-supersoluble and $G^{\mathfrak{U}_\sigma}$ is contained in every normal subgroup E of G with σ-supersoluble quotient G/E. By Lemmas 2.1 and 2.2, $N/D = (G/N)^{\mathfrak{N}_\sigma}$. On the other hand, every chief factor of G between N and D is abelian and so cyclic and hence G/D is σ-supersoluble. Therefore $G^{\mathfrak{U}_\sigma} \leq D$. Moreover, from Lemma 2.2 and Proposition B(ii) we also get that

$$N/G^{\mathfrak{U}_\sigma} = (G/G^{\mathfrak{U}_\sigma})^{\mathfrak{N}_\sigma},$$

so every chief factor of G between N and $G^{\mathfrak{U}_\sigma}$ is cyclic and hence $D \leq G^{\mathfrak{U}_\sigma}$. Thus $D = G^{\mathfrak{U}_\sigma}$, so if $D = 1$, then G is σ-supersoluble.

Now suppose that $D \neq 1$. We show that in this case G has a Robinson complex of the form $(D, Z(D); U_1, \ldots, U_k)$, where $Z(D) = R$. It is clear that every chief factor of G below R is cyclic, so $G/C_G(R)$ is supersoluble by [28, IV, 6.10]. Hence $D = G^{\mathfrak{U}_\sigma} \leq C_G(R)$, so $R \leq Z(D) \leq D_{\mathfrak{S}} = R$ and therefore we have $Z(D) = R$.

Now let H/K be any chief factor of G below D. Then $H \leq N$ and so in the case when H/K is abelian, this factor is cyclic, which implies that $D = G^{\mathfrak{U}_\sigma} \leq C_G(H/K)$. On the other hand, if H/K is a non-abelian simple group, then Lemma 2.3 implies that $G/HC_G(H/K)$ is soluble. Then

$$DHC_G(H/K)/HC_G(H/K) \simeq D/(D \cap HC_G(H/K)) = D/HC_D(H/K)$$

is soluble, so $D = HC_D(H/K)$ since D is evidently perfect. Therefore, in both cases, every element of D induces an inner automorphism on H/K. Therefore D is quasinilpotent. Hence in view of [29, X, 13.6], G has a Robinson complex of the form $(D, Z(D), U_1, \ldots, U_k)$.

Sufficiency. From Conditions (i), (ii) and (iii), it follows that all factors below N of any chief series of G passing through N are simple. Therefore the Jordan-Hölder theorem for groups with operators [28, A, 3.2] implies that every chief factor of G below N is simple. Therefore G is a σ-SC-group.

The theorem is proved. ‘

3 Proofs of Proposition A and Theorem A

Recall that a subgroup A of G is called *σ-subnormal* in G [3] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

Lemma 3.1 (See Remark 1.1 and [Proposition 2.6]arivII). *G is σ-nilpotent if and only if every subgroup of G σ-subnormal in G.*

Lemma 3.2. *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *AN/N is σ-subnormal in G/N.*

(3) *If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

(4) *If $H \neq 1$ is a Hall σ_i-subgroup of G and A is not a σ_i'-group, then $A \cap H \neq 1$ is a Hall σ_i-subgroup of A.*

(5) *If A is a σ_i-group, then $A \leq O_{\sigma_i}(G)$.*

(6) *If A is a Hall σ_i-subgroup of G, then A is normal in G.*

(7) *If $|G : A|$ is a σ_i-number, then $O^{\sigma_i}(A) = O^{\sigma_i}(G)$.*

(8) *If G is σ-perfect, then A is subnormal in G.*

(9) *$A^{\mathfrak{N}_\sigma}$ is subnormal in G.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either $A_{i-1} \trianglelefteq A_i$

or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

(1)–(7) See Lemma 2.6 in [3].

(8) A is subnormal in M by the choice of G. On the other hand, since G is σ-perfect, G/M_G is not σ-primary. Hence M is normal in G and so A is subnormal in G.

(9) A is σ-subnormal in $AM_G \leq M$ by Part (1), so the choice of G implies that $A^{\mathfrak{N}_\sigma}$ is subnormal in AM_G. Hence G/M_G is a σ_i-group for some i, so $M_GA/M_G \simeq A/A \cap M_G$ is a σ_i-group. Hence $A^{\mathfrak{N}_\sigma} \leq M_G$, so $A^{\mathfrak{N}_\sigma}$ is subnormal in M_G and hence $A^{\mathfrak{N}_\sigma}$ is subnormal in G.

Lemma is proved.

The following lemma, in fact, is a corollary of Theorem A and Lemmas 3.1 and 3.2(3).

Lemma 3.3. *The following statements hold:*

(i) *G is a $P\sigma T$-group if and only if every σ-subnormal subgroup of G is σ-permutable in G.*

(ii) *If G is a $P\sigma T$-group, then every quotient G/N of G is also a $P\sigma T$-group.*

Lemma 3.4. *Let A and B be subgroups of G, where A is σ-permutable G.*

(1) *If $A \leq B$ and B is σ-subnormal in G, then A is σ-permutable B.*

(2) *Suppose that B is a σ_i-group. Then B is σ-permutable in G if and only if $O^{\sigma_i}(G) \leq N_G(B)$.*

Proof. (1) By hypothesis, G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$. Then $\mathcal{H}_0 = \{H_1 \cap B, \ldots, H_t \cap B\}$ is a complete Hall σ-set of B by Lemma 3.2(4). Moreover, for every $x \in B$ and $H \in \mathcal{H}$ we have $AH^x = H^xA$, so

$$AH^x \cap B = A(H^x \cap B) = A(H \cap B)^x = (H \cap B)^x A.$$

Hence A is σ-permutable in B by Remark 1.2.

(2) See Lemma 3.1 in [3].

The lemma is proved.

Proof of Proposition A. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ and $N = G^{\mathfrak{N}_\sigma}$ be the σ-nilpotent residual of G. Then $D \leq N$.

(1) *Statement (i) holds for G.*

Suppose that this is false and let G be a counterexample of minimal order. If $D = 1$, then G is σ-soluble and so G is a σ-SC-group by Theorem B. Therefore $D \neq 1$. Let R be a minimal normal subgroup of G contained in D. Then G/R is a $P\sigma T$-group by Lemma 3.3(ii). Therefore the choice of G implies that G/R is a σ-SC-group. Since $(G/R)^{\mathfrak{N}_\sigma} = N/R$ by Lemmas 2.1 and 2.2, every chief factor of G/R below N/R is simple. Hence every chief factor of G between $G^{\mathfrak{N}_\sigma}$ and R is simple. Therefore, in view of the Jordan-Hölder theorem for groups with operators [28, A, 3.2], it is enough to show that R is simple. Suppose that this is false. Let L be a minimal normal subgroup of R.

Then $1 < L < R$ and L is σ-permutable in G by Lemma 3.3(i) since G is a $P\sigma T$-group. Moreover, $L_G = 1$ and so L is σ-nilpotent by Theorem A. Therefore R is a σ_i-group for some i, so for some k we have $R \leq H_k$. Now let V be a maximal subgroup of R. Then V is σ-subnormal in G, so V is σ-permutable in G and hence

$$R \leq D \leq O^{\sigma_i}(G) \leq N_G(V)$$

by Lemma 3.4(2). Thus R is nilpotent, so R is a p-group for some $p \in \sigma_i$. Now let V be a maximal subgroup of R such that V is normal in a Sylow p-subgroup of P of H_k. By hypothesis, H_k is a PST-group and so V is S-permutable in H_k since it is subnormal in H_k. Then, by Lemma 3.4(2) (taking in the case $\sigma = \{\{2\}, \{3\}, \ldots\}$), we have $H_k = PO^p(H_k) \leq N_G(V)$. Therefore, in view of Lemma 3.4(2), we have

$$G = H_k O^{\sigma_i}(G) \leq N_G(V).$$

Hence $V = 1$ and so $|R| = p$, a contradiction. Thus we have (1).

(2) *Statement (ii) holds for G.*

It is clear that D is σ-perfect and G/D is σ-soluble. In view of Lemma 3.3(ii), G/D is a $P\sigma T$-group. It is also cleat that $D \leq G^{\mathfrak{U}_\sigma}$. On the other hand, G/D is σ-supersoluble by Theorem B. Therefore $G^{\mathfrak{U}_\sigma} \leq D$ and so we have $D = G^{\mathfrak{U}_\sigma}$. Hence we have (2).

(3) *Statement (iii) holds for G.*

Let L be a σ-soluble normal subgroup of G and let x be a σ_i'-element of G. Let $V/L \leq O_{\sigma_i}(G/L)$. Then V/L is σ-subnormal in G/L, so V/L is σ-permutable in G/L by Lemma 3.3(i) since G/L is a $P\sigma T$-group by Lemma 3.3(ii). Therefore

$$xL \in O^{\sigma_i}(G/L) \leq N_{G/L}(V/L)$$

by Lemma 3.4(2). Hence Statement (iii) holds for G.

The proposition is proved.

Lemma 3.5. *Let G be a non-σ-supersoluble σ-full σ-SC-group and let $(D, Z(D); U_1, \ldots, U_k)$ be a Robinson complex G, where $D = G^{\mathfrak{U}_\sigma}$. Let U be a non-σ-permutable σ-subnormal subgroup of G of minimal order. Suppose that $S/Z(S)$ is σ-perfect. Then:*

(i) *If US_i'/U_i' is σ-permutable in G/U_i' for all i, then U is σ-supersoluble.*

(ii) *If U is σ-supersoluble and UL/L is σ-permutable in G/L for all non-trivial nilpotent normal subgroups L of G, then U is a cyclic p-group for some prime p.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. By hypothesis, for some i and for some Hall σ_i-subgroup H of G we have $UH \neq HU$.

(i) Assume that this is false. Then $U \cap D \neq 1$ since $UD/D \simeq U/(U \cap D)$ is σ-supersoluble by Proposition B(ii). Moreover, Lemma 3.2(1)(2), implies that $(U \cap D)Z(D)/Z(D)$ is σ-subnormal in $D/Z(D)$ and so $(U \cap D)Z(D)/Z(D)$ is a non-trivial subnormal subgroup of $D/Z(D)$ by Lemma

3.2(8) since $D/Z(D)$ is σ-perfect by hypothesis. Hence for some i we have

$$U_i/Z(D) \leq (U \cap D)Z(D)/Z(D),$$

so $U_i \leq (U \cap D)Z(D)$. But then

$$U_i' \leq ((U \cap D)Z(D))' \leq U \cap D.$$

By hypothesis, $UU_i'/U_i' = U/U_i'$ is σ-permutable in G/U_i' and so

$$UH/U_i' = (U/U_i')(HU_i'/U_i') = (HU_i'/U_i')(U/U_i') = HU/U_i'.$$

Hence $UH = HU$, a contradiction. Therefore Statement (i) holds.

(ii) Let $N = U^{\mathfrak{N}_\sigma}$. Then D is subnormal in G by Lemma 3.2(9). Since U is σ-supersoluble by hypothesis, $N < U$. By Lemmas 2.1, 2.2 and 3.2(3), every proper subgroup V of U with $N \leq V$ is σ-subnormal in G, so the minimality of U implies that $VH = HV$. Therefore, if U has at least two distinct maximal subgroups V and W such that $N \leq V \cap W$, then $U = \langle V, W \rangle$ is permutes with H by [28, A, 1.6], contrary to our assumption on U and H. Hence U/N is a cyclic p-group for some prime p.

First assume that $p \in \sigma_i$. Lemma 3.2(4) implies that $H \cap U$ is a Hall σ_i-subgroup of U, so $U = N(H \cap U) = (H \cap U)N$. Hence

$$UH = (H \cap U)NH = H(H \cap U)N = HU,$$

a contradiction. Thus $p \in \sigma_j$ for some $j \neq i$.

Now we show that U is a $P\sigma T$-group. Let V be a proper σ-subnormal subgroup of U. Then V is σ-subnormal in G since U is σ-subnormal in G. The minimality of U implies that V is σ-permutable in G, so V is σ-permutable in U by Lemma 3.4(1). Hence U is a σ-soluble $P\sigma T$-group by Lemma 3.3(i), so N is abelian by Theorem B.

Therefore N is a σ_j'-group, so $N \leq O = O_{\sigma_j'}(F(G))$ by Lemma 3.2(5) (taking in the case $\sigma = \{\{2\}, \{3\}, \ldots\}$). By hypothesis, OU/O permutes with OH/O. By Lemma 3.2(1)(2), OU/O is σ-subnormal in

$$(OU/O)(OH/O) = (OH/O)(OU/O) = OHU/O,$$

where $OU/O \simeq U/U \cap O$ is a σ_j-group and $OH/O \simeq H/H \cap O$ is a σ_i-group. Hence UO/O is normal in OHU/O by Lemma 3.2(6). Hence $H \leq N_G(OU)$

$$H \leq N_G(O^{\sigma_j'}(OU)) = N_G(O^{\sigma_j'}(U))$$

by Lemma 3.2(7) since $p \in \sigma_j$ implies that $O^{\sigma_j'}(U) = U$. But then $HU = UH$, a contradiction. Therefore Statement (ii) holds.

The lemma is proved.

Lemma 3.6. *Suppose that G has a Robinson σ-complex $(D, Z(D); U_1, \ldots, U_k)$, and let N be a normal subgroup of G.*

(i) *If $N = U_i'$, then*

$$(D/N, Z(D/N) = U_i/N; U_1N/N, \ldots, U_{i-1}N/N, U_{i+1}N/N, \ldots U_kN/N, U_i/N)$$

is a Robinson σ-complex of G/N, where $U_i/U_i' \simeq Z(D)/(Z(D) \cap U_i')$.

(ii) *If N is nilpotent, then*

$$(DN/N, Z(DN/N) = Z(D)N/N; U_1N/N, \ldots, U_kN/N)$$

is a Robinson σ-complex of G/N.

Proof. See Remark 1.6.8 in [7].

Lemma 3.7 (See Knyagina and Monakhov [31]). *Let H, K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Lemma 3.8. *If G satisfies N_{σ_i}, then G/R satisfies N_{σ_i} for every normal σ-soluble subgroup R of G.*

Proof. Let N/R be a normal σ-soluble subgroup of G/R and let

$$(V/R)/(N/R) \leq O_{\sigma_i}((G/R)/(N/R)).$$

Then N is a normal σ-soluble subgroup of G and $V/N \leq O_{\sigma_i}(G/N)$. Moreover, for every σ_i'-element $xR \in G/R$ there is a σ_i'-element $y \in G$ such that $xR = yR$ and so $yN \leq N_{G/N}(V/N)$, which implies that

$$xR(N/R) \in N_{(G/R)/(N/R)}((V/R)/(N/R)).$$

Hence G/R satisfies N_{σ_i}, as required.

By the analogy with the notation $\pi(n)$, we will write $\sigma(n)$ to denote the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$.

Proof of Theorem C. First assume that G is a $P\sigma T$-group and let $D = G^{\mathfrak{S}_\sigma}$ be the σ-soluble residual of G. Then D is clearly σ-perfect and, by Proposition A, G is a σ-SC-group and Statements (i) and (iii) hold for G. Moreover, Theorem B implies that D coincides with the σ-supersoluble residual $G^{\mathfrak{U}_\sigma}$ of G and if $D \neq 1$, then G possesses a Robinson σ-complex of the form $(D, Z(D); U_1, \ldots U_k)$. Therefore the necessity of the condition of the theorem holds for G.

Now assume that G has a normal σ-perfect subgroup D and D satisfies Conditions (i), (ii) and (iii). We show that G is a $P\sigma T$-group. Suppose that this is false and let G be a counterexample of minimal order. Then $D \neq 1$ and G has a σ-subnormal subgroup U such that $UH \neq HU$ for some i and some Hall σ_i-subgroup H of G and also every σ-subnormal subgroup U_0 of G with $U_0 < U$ is σ-permutable in G. Finally, note that $D = G^{\mathfrak{U}_\sigma}$ by Condition (i) and Theorem B.

(1) *U is σ-supersoluble.*

In view of Lemma 3.5(i), it is enough to show that the hypothesis holds on G/U_i' for all $i = 1, \ldots, k$. Let $N = U_i'$. We can assume without loss of generality that $i = 1$. Then

$$(D/N)^{\mathfrak{N}_\sigma} = D^{\mathfrak{N}_\sigma}N/N = D/N$$

by Lemmas 2.1 and 2.2, so D/N is a normal σ-perfect subgroup of G/N. Moreover, $(G/N)/(D/N) \simeq D/D$ is a σ-soluble $P\sigma T$-group. Now assume that $D/N \neq 1$. Then, by Lemma 3.6(i),

$$(D/N, Z(D/N); U_2N/N, \ldots U_kN/N)$$

is a Robinson σ-complex of G/N, where $Z(D/N) = U_1/N$. Moreover, if $\{i_1, \ldots, i_r\} \subseteq \{2, \ldots, k\}$, where $2 \leq r < k$, then the quotients $G/N = G/U_1'$ and

$$(G/N)/(U_{i_1}N/N)' \cdots (U_{i_r}N/N)' = (G/N)/(U_{i_1}' \cdots U_{i_r}'U_1'/N) \simeq G/U_{i_1}' \cdots U_{i_r}'U_1'$$

satisfy N_{σ_l} for all

$$\sigma_l \in \sigma(U_1/N) = \sigma(Z(D/N)) \subseteq \sigma(Z(D)/(Z(D) \cap U_1')).$$

Therefore the hypothesis holds for G/R, so we have (1).

(2) *U is a cyclic p-group for some prime $p \in \sigma_j$, where $j \neq i$.*

First we show that U is a cyclic p-group for some prime. In view of Claim (1) and Lemma 3.5(ii), it is enough to show that the hypothesis holds on G/N for every normal nilpotent subgroup N of G. First note that

$$(DN/N)^{\mathfrak{N}_\sigma} = D^{\mathfrak{N}_\sigma}N/N = DN/N$$

y Lemma 2.2(ii), so D/N is a normal σ-perfect subgroup of G/N. Moreover,

$$(DN/N, Z(DN/N) = Z(D)N/N; U_1N/N, \ldots, U_kN/N)$$

is a Robinson σ-complex of G/N by Lemma 3.6(ii). Finally, if V/N is a normal σ-soluble subgroup of G/N, then V is a normal σ-soluble subgroup of G and so for $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, t\}$, where $1 \leq r < k$, the quotient G/N and, by Lemma 3.8, the quotient

$$(G/N)/(U_{i_1}N/N)' \cdots (U_{i_r}N/N)' = (G/N)/(U_{i_1}' \cdots U_{i_r}'N/N)$$

$$\simeq G/U_{i_1}' \cdots U_{i_r}'N \simeq (G/U_{i_1}' \cdots U_{i_r}')/(U_{i_1}' \cdots U_{i_r}'N/U_{i_1}' \cdots U_{i_r}')$$

satisfy N_{σ_l} for all for all

$$\sigma_l \in \sigma(Z(DN/N)) = \sigma(Z(D)N/N) \subseteq \sigma(Z(D))$$

since $U'_{i_1} \cdots U'_{i_r} N/U'_{i_1} \cdots U'_{i_r} \simeq N/(N \cap U'_{i_1} \cdots U'_{i_r})$ is σ-soluble.

Therefore the hypothesis holds on G/N, so U is a cyclic p-group for some prime $p \in \sigma_j$. Finally, Lemma 3.2(4) implies that in the case $i = j$ we have $U \leq H$, so $UH = H = HU$. Therefore $j \neq i$. Finally, again by Lemma 3.2(4), $U \leq O_{\sigma_j}(G)$.

(3) $O_{\sigma_j}(G) \cap D = 1$.

Suppose that $L = O_{\sigma_j}(G) \cap D \neq 1$. Then, since $D/Z(D)$ is σ-perfect, $L \leq Z(D)$ and so G satisfies N_{σ_j} by Condition (iii). Therefore $H \leq N_G(U)$ since $i \neq j$, $U \leq O_{\sigma_j}(G)$ and H is a σ_i-group. But then $HU = UH$, a contradiction. Hence we have (3).

Final contradiction for the sufficiency. By Lemma 3.2(2), UD/D is σ-subnormal in G/D. On the other hand, HD/D is a Hall σ_i-subgroup of G/D. Hence

$$(UD/D)(HD/D) = (HD/D)(UD/D) = HUD/D$$

by Condition (i) and Lemma 3.3(i), so HUD is a subgroup of G. Therefore, by Claims (2), (3) and Lemma 3.7,

$$UHD \cap HO_{\sigma_j}(G) = UH(D \cap HO_{\sigma_j}(G)) = UH(D \cap H)(D \cap O_{\sigma_j}(G)))$$
$$= UH(D \cap H) = UH$$

is a subgroup of G and so $HU = UH$, a contradiction.

The theorem is proved.

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