On one generalization of finite $\mathfrak{U}$-critical groups

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Abstract

A proper subgroup H of a group G is said to be: $\mathbb{P}$-*subnormal* in G if there exists a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $i = 1, \ldots, n$; $\mathbb{P}$-*abnormal* in G if for every two subgroups $K \leq L$ of G, where $H \leq K$, $|L : K|$ is not a prime. In this paper we describe finite groups in which every non-identity subgroup is either $\mathbb{P}$-subnormal or $\mathbb{P}$-abnormal.

1 Introduction

Throughout this paper, all groups are finite, G denotes a finite group and p is a prime. We use $\mathfrak{N}$ and $\mathfrak{U}$ to denote the classes of all nilpotent and of all supersoluble groups, respectively. A sugroup H of G is said to be a *Gaschütz* subgroup of G (Shemetkov [1, p. 170]) if H is supersoluble and $|L : K|$ is not a prime whenever $H \leq K \leq L \leq G$.

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be: *saturated* if $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \leq \Phi(G)$; *hereditary* if $H \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$ and H is a subgroup of G.

A group G is said to be $\mathfrak{F}$-*critical* if G is not in $\mathfrak{F}$ but all proper subgroups of G are in $\mathfrak{F}$ [2, p. 517]. An $\mathfrak{N}$-critical group is also called a *Schmidt group*.

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A proper subgroup H of G is said to be: $\mathfrak{F}$-*subnormal* in G if there exists a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that H_{i-1} is a maximal subgroup of H_i and $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all $i = 1, \ldots, n$; $\mathfrak{F}$-*abnormal* in G if $L/K_L \notin \mathfrak{F}$ whenever $H \leq K < L \leq G$ and K is a maximal subgroup of L. A group $G \notin \mathfrak{F}$ is said to be an $E_{\mathfrak{F}}$-*group* [3] if every non-identity subgroup of G is either $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal in G.

In [4], Fattahi described groups in which every subgroup is either normal or abnormal. As a generalization of this result, Ebert and Bauman classified the $E_{\mathfrak{F}}$-groups in the case when $\mathfrak{F} = \mathfrak{N}$ (in this case G is a group in which every subgroup is either subnormal or abnormal), and in the case when $\mathfrak{F}$ is the class of all soluble p-nilpotent groups, for odd prime p [3]. In the future, the $E_{\mathfrak{F}}$-groups were studied for some other $\mathfrak{F}$ (see for example [5, 6, 7, 8]). Nevertheless, it should be noted that a complete description of the $E_{\mathfrak{F}}$-groups was obtained only for such cases $\mathfrak{F}$ when every $\mathfrak{F}$-critical group is a Schmidt group [4, 6, 7, 8]). Thus, for example in the case, where $\mathfrak{F} = \mathfrak{U}$, the structure of $E_{\mathfrak{F}}$-groups has not been known since the methods in [4, 5, 6, 7, 8] could not be used in the analysis of this case.

Note, in passing, that if G is soluble and H is a subgroup of G, then H: is $\mathfrak{U}$-subnormal in G if and only if there exists a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $i = 1, \ldots, n$; $\mathfrak{U}$-abnormal in G if and only if $|L : K|$ is not a prime whenever $H \leq K \leq L \leq G$.

If G is supersoluble, then clearly every subgroup of G is $\mathfrak{U}$-subnormal in G. A full description of $E_{\mathfrak{U}}$-groups, for the non-supersoluble case, gives the following our result.

Theorem A. *Let G be an $E_{\mathfrak{U}}$-group and $D = G^{\mathfrak{U}}$ the supersoluble residual of G. Then $G = D \rtimes H$, where:*

(i) *H is a Hall Gaschütz subgroup of G. Hence if H is nilpotent, then it is a Carter subgroup of G.*

(ii) *Every chief factor of G below D is non-cyclic. Hence H is a supersoluble normalizer ($\mathfrak{U}$-normalizer, in other words) of G in the sence of* [9].

(iii) *$|G : DG'|$ is a prime power number.*

(iv) *If H is not a cyclic group of prime power order p^n, where $n > 1$, then D is nilpotent.*

(v) *$H\Phi(G)/\Phi(G)$ is either a Miller-Moreno group or an abelian group of prime power order.*

(vi) *Every proper subgroup of G containing D is supersoluble.*

Conversely, any group satisfying the above conditions is an $E_{\mathfrak{U}}$-group.

Corollary 1.1. *Let $G = D \rtimes H$ be an $E_{\mathfrak{U}}$-group, where $D = G^{\mathfrak{U}}$. If H is nilpotent, then it is a system normalizer of G.*

From the describtion of $\mathfrak{U}$-critical groups G [10, 11] it follows that every subgroup of G contaning $\Phi(G) \cap G^{\mathfrak{U}}$ is either $\mathfrak{U}$-subnormal or $\mathfrak{U}$-abnormal in G (see Lemma 2.6 below). Another application of Theorem A is the following result, which classifies all groups with such a property.

Theorem B. *Let G be a non-supersoluble group and $\Phi = \Phi(G) \cap G^{\mathfrak{U}}$. Then every non-identity subgroup of G contaning Φ is either $\mathfrak{U}$-subnormal or $\mathfrak{U}$-abnormal in G if and only if $G = D \rtimes H$ is a soluble group, where H is a Hall subgroup of G such that $H\Phi/\Phi$ is a Gaschütz subgroup of G, and with respect to G Assertions (iii)–(vi) in Theorem A hold.*

All unexplained notation and terminology are standard. The reader is referred to [12], [2], [13], or [14] if necessary.

2 Preliminaries

The following lemma collects some well-known properties of $\mathfrak{F}$-subnormal subgroups which will be used in our proofs.

Lemma 2.1. *Let $\mathfrak{F}$ be a hereditary saturated formation, H and K the subgroups of G and H is $\mathfrak{F}$-subnormal in G.*

(1) *$H \cap K$ is $\mathfrak{F}$-subnormal in K* [14, 6.1.7(2)].

(2) *If N is a normal subgroup in G, then HN/N is $\mathfrak{F}$-subnormal in G/N.* [14, 6.1.6(3)].

(3) *If K is an $\mathfrak{F}$-subnormal subgroup of H, then K is $\mathfrak{F}$-subnormal in G* [14, 6.1.6(1)].

(4) *If $G^{\mathfrak{F}} \leq K$, then K is $\mathfrak{F}$-subnormal in G* [14, 6.1.7(1)].

(5) *If $K \leq H$ and $H \in \mathfrak{F}$, then K is $\mathfrak{F}$-subnormal in G.*

A minimal normal subgroup R of G is called $\mathfrak{F}$-*central in* G provided $R \rtimes (G/C_G(R)) \in \mathfrak{F}$, otherwise it is called $\mathfrak{F}$-*eccentric* in G.

Lemma 2.2. *Let $\mathfrak{F}$ be a formation and M a maximal subgroup of G. Let R be a minimal normal subgroup of G such that $MR = G$. Then $G/M_G \in \mathfrak{F}$ if and only if R is $\mathfrak{F}$-central in G.*

Proof. In view of the G-isomorphism $R \simeq RM_G/M_G$ we can assume without loss of generality that $M_G = 1$. Let $C = C_G(R)$.

If $G \simeq G/M_G \in \mathfrak{F}$, then R is $\mathfrak{F}$-central in G by the Barnes-Kegel's Theorem [2, IV, 1.5]. Now let $R \rtimes (G/C) \in \mathfrak{F}$. First assume that R is non-abelian. If R is the unique minimal normal subgroup of G, then $C = 1$ and so $G \in \mathfrak{F}$. Now let G have a minimal normal subgroup $L \neq R$. Then, since $M_G = 1$, $G = R \rtimes M = L \rtimes M$ and $C = L$ by [2, A, 15.2]. Hence $M \simeq G/L \simeq G/R \in \mathfrak{F}$ and so $G \in \mathfrak{F}$ since $\mathfrak{F}$ is a formation.

Finally, if R is an abelian p-group, then $C = R$ by [2, A, 15.2] and so $G \simeq G/M_G \simeq R \rtimes (G/R) \in \mathfrak{F}$. The lemma is proved.

Lemma 2.3 (See Lemma 2.15 in [15]). *Let E be a normal non-identity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.*

Lemma 2.4. *Let $\mathfrak{F}$ be a non-empty hereditary saturated formation, G an $E_{\mathfrak{F}}$-group and $D = G^{\mathfrak{F}}$.*

(i) *Every $\mathfrak{F}$-subnormal subgroup of G belongs to $\mathfrak{F}$.*

(ii) $F^*(G) \leq D\Phi(G)$.

Proof. (i) Let H be any $\mathfrak{F}$-subnormal subgroup of G and K a maximal subgroup of H. Then K is not $\mathfrak{F}$-abnormal, so it is $\mathfrak{F}$-subnormal in G by hypothesis. Hence K is $\mathfrak{F}$-subnormal in H by Lemma 2.1.(1), that is, $H/K_H \in \mathfrak{F}$. Therefore, since $\mathfrak{F}$ is a saturated formation, $H \in \mathfrak{F}$.

(ii) Without loss of generality we can assume that $\Phi(G) = 1$. In this case $F^*(G) = N_1 \times \cdots \times N_t$ for some minimal normal subgroups $N_1, \ldots, N_t$ of G by Lemma 2.3. Let $N = N_i$ and M be a maximal subgroup of G such that $G = MN$. Assume that M is $\mathfrak{F}$-subnormal in G. Then $D \leq M_G$, so N is $\mathfrak{F}$-central in G by Lemma 2.2. On the other hand, Assertion (i) implies that $M \in \mathfrak{F}$. Thus $G/N \simeq M/M \cap N \in \mathfrak{F}$ and so $G \simeq G/N \cap M_G \in \mathfrak{F}$. This contraiction shows that M is $\mathfrak{F}$-abnormal in G, so $G/M_G \notin \mathfrak{F}$. Hence $N \leq D$ by Lemma 2.2. Therefore $F^*(G) \leq D$. The lemma is proved.

Lemma 2.5. *Let $\mathfrak{F}$ be a non-empty formation, G an $\mathfrak{F}$-critical group and $D = G^{\mathfrak{F}}$.*

(i) *If G is soluble , then D is a p-group for some prime p.*

(ii) *If $\mathfrak{F}$ is saturated and D is soluble, then the following statements hold:*

(a) *D is a p-group for some prime p.*

(b) *$D/\Phi(D)$ is a chief factor of G.*

Proof. (i) Since G is soluble, $\Phi(G) < F(G)$. Hence for some prime p we have $O_p(G) \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = O_p(G)M$. Then $G/O_p(G) = O_p(G)M/O_p(G) \simeq M/M \cap O_p(G) \in \mathfrak{F}$ since G is an $\mathfrak{F}$-critical group. Thus $D \leq O_p(G)$.

(ii) See Theorem 24.2 in [1, V] or [2, VII, 6.18]. The lemma is proved.

Lemma 2.6. *Let $\mathfrak{F}$ be a hereditary saturated formation and G an $\mathfrak{F}$-critical soluble group. Then every subgroup of G contaning $\Phi(G) \cap G^{\mathfrak{F}}$ is either $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal in G.*

Proof. It is enough to consider the case when $\Phi(G) \cap G^{\mathfrak{F}} = 1$. By Lemma 2.5, D is a minimal normal subgroup of G. Let A be any non-identity subgroup of G. First assume that $DA < G$. Then $DA \in \mathfrak{F}$, and DA is $\mathfrak{F}$-subnormal in G by Lemma 2.1(4). Hence A is $\mathfrak{F}$-subnormal in G by Lemma 2.1(3). Now assume that $DA = G$. Then A is a maximal subgroup of G, so A is $\mathfrak{F}$-abnormal in G. The lemma is proved.

Lemma 2.7 (Friesen [17, 4, 3.4]). *If $G = AB$, where A and B are normal supersoluble subgroups of G and $(|G : A|, |G : B|) = 1$, then G is supersoluble.*

We shall need the following special case of Theorem C in [15].

Lemma 2.8. *Let $\mathcal{F}$ be a hereditary saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Phi(G) \in \mathcal{F}$, then $E \in \mathcal{F}$.*

A subgroup H of G is said to be: $\mathbb{P}$-*subnormal* in G [18, 19] if there exists a chain of subgroups

$H = H_0 < H_1 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $i = 1, \ldots, n$; $\mathbb{P}$-*abnormal* in G if $|L : K|$ is not a prime whenever $H \leq K \leq L \leq G$. We say that H satisfies the $\mathbb{P}$-*property* in G if H is either $\mathbb{P}$-subnormal or $\mathbb{P}$-abnormal in G.

Lemma 2.9. (i) *If every non-identity subgroup of G of prime order satisfies the $\mathbb{P}$-property in G, then G is not a simple non-abelian group.*

(ii) *If every non-identity cyclic subgroup of G of prime power order satisfies the $\mathbb{P}$-property in G, then G is soluble.*

Proof. (i) Suppose that this is false and let p be the smallest prime dividing $|G|$. Then a Sylow p-subgroup P of G is not cyclic. Let H be a subgroup of order p in P. Then $H < P$, so by hypothesis, G has a maximal subgroup M such that $H \leq M$ and $|G : M| = q$ for some prime q. Since G is a simple non-abelian group, $M_G = 1$ and by considering the permutation representation of G on the right cosets of H, we see that G is isomorphic to some subgroup of the symmetric group S_q of degree q. Hence q is the largest prime divisor of $|G|$ and $|Q| = q$, where Q is a Sylow q-subgroup Q of G. It follows that $q \neq p$. It is clear that G is not q-nilpotent, so it has a q-closed Schmidt subgroup H such that $Q \leq H$ by [16, IV, 5.4]. Since Q is normal in H, it is P-subnormal in G by hypothesis. Hence G has a maximal subgroup T such that $Q \leq T$ and $|G : T| = r$ is a prime. But then r is the largest prime dividing $|G|$ and so $r = q$, a contradiction. Hence we have (i).

(ii) Since the hypothesis clearly holds for every quotient of G and every normal subgroup of G, this assertion is a corollary of Assertion (i). The lemma is proved.

3 Proofs of Theorems A and B

Proof of Theorem A. *Necessity.* Suppose that this is false and let G be a counterexample of minimal order. Let $\pi = \pi(D)$.

(1) *The hypothesis holds on G/R for every normal subgroup R of G not containing D.*

First note that $G/R \notin \mathfrak{U}$ since $D \not\leq R$. Therefore this claim is a corollary of Lemma 2.1(2).

(2) *Every subgroup E of G containing D is supersoluble. Hence G is soluble.*

First note that the hypothesis holds for D, so D is soluble by Lemma 2.9. On the other hand, E is $\mathfrak{U}$-subnormal in G by Lemma 2.1(4) and so E is supersoluble by Lemma 2.4(i). Hence we have (2).

(3) *D is a Hall subgroup of G.*

Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$, where G_p is a Sylow p-subgroup of G. Then $|G_p| > p$. Let R be a minimal normal subgroup of G.

(a) *R is a p-group. Hence $O_{p'}(G) = 1$.*

Since G is soluble by Claim (2), R is a q-group for some prime q. Moreover, $DR/R = (G/R)^{\mathfrak{U}}$ is a Hall subgroup of G by the choice of G since the hypothesis holds for G/R by Claim (1). Therefore every Sylow r-subgroup of D, where $r \neq q$, is a Sylow subgroup of G. Hence $q = p$ and so $O_{p'}(G) = 1$.

(b) *If $R \leq D$, then R is a Sylow p-subgroup of D. If $R \cap D = 1$, then $G_p = R \rtimes P$.*

Assume $R \leq D$. Then $R \leq P$ and P/R is a Sylow p-subgroup of D/R. If $P/R \neq 1$, then Claim (1) and the choice of G imply that $P/R = G_p/R$ and so $P = G_p$. This contradiction shows that $P = R$ is a Sylow p-subgroup of D.

Now assume that $R \nleq D$. Then $RP/R = G_p/R$ since $DR/R = (G/R)^{\mathfrak{U}}$ is a Hall subgroup of G/R by Claim (1), so $G_p = R \rtimes P$ since $R \cap D = 1$.

(c) $R \nleq \Phi(G)$. *Hence* $\Phi(G) = 1$.

Assume that $R \leq \Phi(G)$. If $R \cap D = 1$, then $G_p = R \rtimes P$ by Claim (b) and so $R \nleq \Phi(G)$ by the Gaschütz Theorem [16, I, 17.4]. This contradiction shows that $R \leq D$, so R is the Sylow p-subgroup of D by Claim (b). Hence D/R is a p'-group, so a p-complement S of D is normal in G by Lemma 2.8. But then $S \leq O_{p'}(G) = 1$. Hence $D = R$ and so G is supersoluble since $D = G^{\mathfrak{U}}$, a contrdiction. Thus we have (c).

(d) *G_p is normal in G.*

Let E be any normal maximal subgroup of G containing D with $|G : E| = q$. Then $O_{p'}(E) \leq O_{p'}(G) = 1$, so p is the largest prime dividing $|E|$ since E is supersoluble by Claim (2). If $q \neq p$, then $G_p \leq E$ and so G_p is normal in G since in this case G_p is a characteristic subgroup of E.

Finally, assume that $q = p$. Then p is the largest prime dividing $|G|$ and so DG_p is normal in G since $G/D = G/G^{\mathfrak{U}}$ is supersoluble. If $DG_p \neq G$, we can get as above that G_p is normal in G. Now assume that $DG_p = G$. Since $R \nleq \Phi(G)$ by Claim (c), it has a complement in G and so R has a complement V in G_p. It is clear that V is not $\mathfrak{U}$-abnormal in G, so for some maximal $\mathfrak{U}$-subnormal subgroup M of G we have $V \leq M$, which implies that $G = DV \leq M$. This contradiction shows that the case under consideration is impossible. Hence G_p is normal in G

Final contradiction for (3). In view of Claim (d), $\Phi(G_p) \leq \Phi(G) = 1$. Therefore, by the Maschke's Theorem, G has a minimal normal subgroup $L \nleq \Phi(G)$ such that $L \leq G_p$ and $L \nleq D$. Then $|L| = p$ and for some maximal subgroup M of G we have $G = LM$. Hence G/M_G is supersoluble by Lemma 2.3, which implies that $D \leq M$ and so M is supersoluble by Claim (2). But then G is supersoluble, a contradiction. Hence we have (3).

(4) $F(G) \leq D\Phi(G)$ (This directly follows from Lemma 2.4(ii)).

(5) $|G : DG'|$ *is a prime power number* (Since G is not supersoluble, this directly follows from Claim (2) and Lemma 2.7).

(6) *If* $\Phi(G) = 1$, *then* $O_{\pi'}(G) = 1$.

Assume that $O_{\pi'}(G) \neq 1$ and let R be a minimal normal subgroup of G contained in $O_{\pi'}(G)$. Then $R \nleq D$, so in view of the G-isomorphism $RD/D \simeq R$, R is cyclic. Since $\Phi(G) = 1$, for some maximal subgroup M of G we have $RM = G$ and so G/M_G is supersoluble by Lemma 2.3. It follows that $D \leq M$ and hence M is supersoluble by Claim (2). But then $G = RM$ is supersoluble. This contradiction shows that we have (6).

(7) *If H is a complement to D in G, then $H\Phi(G)/\Phi(G)$ is either a Miller-Moreno group or an abelian group of prime power order.*

Without loss of generality we can assume that $\Phi(G) = 1$. First we shall show that every proper subgroup A of H is abelian. Let $C = C_G(F(G))$. Then $C \leq F(G)$ since G is soluble. On the other hand, Claim (4) implies that $F(G) \leq D$, so $C \leq D$. It follows that $F(DA) = F(G)$, so $AF(G)/F(G) \simeq A$ is abelian since DA is supersoluble by Claim (2). Therefore, if H is not abelian, then H is a Miller-Moreno group.

Finally, suppose that H is abelian. Then $G' \leq D$ by Claim (3), so Claim (5) implies that $|G : DG'| = |G : D| = |H|$ is a prime power number.

(8) *H is a Gaschütz subgroup of G.*

Since $D = G^{\mathfrak{U}}$ and $G = D \rtimes H$, H is supersoluble. It is clear also that H is not $\mathfrak{U}$-subnormal in G. Hence H is $\mathfrak{U}$-abnormal in G by hypothesis. Therefore H is a Gaschütz subgroup of G since G is soluble by Claim (2).

(9) *If H is not a cyclic group of prime power order q^n, where $n > 1$, then D is nilpotent.*

Suppose that this is false and let $R \leq O_p(G)$ be a minimal normal subgroup of G.

(*) *$|H|$ is not a prime.*

Indeed, assume that $H = \langle a \rangle$, where $|a|$ is a prime. Since H is a Gaschütz subgroup of G, $N_G(H) = H$ and hence a induces a regular automorphism on D. Hence D is nilpotent by the Thompson's theorem [20, V, 8.14], a contradiction. Hence we have (*).

(**) *If $R \leq D$ or $R \leq \Phi(G)$, where R is a minimal normal subgroup of G, then DR/R is nilpotent. Hence $\Phi(G) = 1$, $R = F(D) = C_D(R)$ is the unique minimal normal subgroup of G contained in D and R is the Sylow p-subgroup of G for some prime p.*

The choice of G and Claims (1) and (*) imply that in order to prove that DR/R is nilpotent, it is enough to show that HR/R is not a cyclic group of order q^n, where $n > 1$ and q is a prime. In the case when $R \leq D$ it is evident. Now assume that $R \leq \Phi(G) \cap H$. Then $R \nleq D$ and hence $|R| = p$ for some prime p. Let G_p be a Sylow p-subgroup of H. Then G_p is a Sylow p-subgroup of G since H is a Hall subgroup of G by Claim (3). Suppose that $R \nleq \Phi(H)$. Then for some maximal subgroup M of H we have $H = R \rtimes M$, so $G_p = R \rtimes (M \cap H)$. But then R has a complement in G by Gaschütz's Theorem [16, I, 17.4]. This contradiction shows that $R \leq \Phi(H)$. Suppose that H/R is cyclic. Then H is nilpotent and so $\Phi(H)$ is a maximal subgroup of H. It follows that H is a cyclic group of order p^n, where $n > 1$, a contradiction. Therefore the hypothesis holds for G/R.

If $R \leq \Phi(G)$, then from Lemma 2.8 we deduce that D is nilpotent since $DR/R \simeq D$ is nilpotent, a contradiction. Hence $\Phi(G) = 1$. Therefore $F(D)$ is the direct product of some minimal normal subgroups of G by Lemma 2.3 since $\Phi(F(G)) \leq \Phi(G) = 1$. If G has a minimal normal subgroup $L \neq R$ such that $L \leq D$, then $D \simeq D/1 = D/R \cap L$ is nilpotent. Therefore R is the unique minimal normal subgroup of G contained in D. Therefore $R = F(D) = C_D(R)$. Finally, since D is supersoluble, a Sylow p-subgroup P of G, where p is the largest prime dividing $|D|$, is normal in D and so $P \leq C_D(R) = R$. Hence R is the Sylow p-subgroup of D, so R is the Sylow p-subgroup of G since D is a Hall subgroup of G by Claim (3).

(***) *$R = C_G(R)$. Hence $F(G) = R$.*

Let $C = C_G(R)$ and S be a p-complement of C. Then, in view of Claim (**), $C = R \times S$ is normal in G and so S and $S \cap D$ are normal in G. Therefore Claim (**) implies that $S \cap D = 1$. Therefore $S \leq O_{\pi'}(G) = 1$ by Claim (6). Hence $C = C_G(R)$.

Final contradiction for (9). First assume that H is a q-group for some prime q and V and W are different maximal subgroups of H. Then DV and DW are supersoluble by Claim (2) and $G = DVW = (DV)(DW)$. Hence G is metanilpotent and then G/R is nilpotent by Claim (***). Hence $D = R$ is nilpotent. This contradiction shows that $H = AB$, where A is a Sylow q-subgroup of H for some prime q dividing $|H|$ and $B \neq 1$ is a q-complement of H. Let S be a p-complement of D such that $SB = BS$. Then $DB/F(DB) = DB/R \simeq SB$ is abelian. Hence $|G : C_G(S)|$ is a $\{p, q\}$-number. Similarly, one can obtain that $|G : C_G(S)|$ is $(\{p\} \cup \{q'\})$-number. Hence $|G : C_G(S)|$ is a power of p. Therefore a p-complement of G is supersoluble, which implies that $D = R$, a contradiction. Hence we have (9).

(10) *Every chief factor K/L of G below D is non-cyclic.*

If $L \neq 1$, it is true by Claim (1) and the choice of G. On the other hand, in the case when $L = 1$ K is not cyclic by Claim (8).

From Claims (1)–(10) it follows that Assertions (i)–(vi) are true for G, which contradicts the choice of G. This completes the proof of the necessity.

Sufficiency. Let A be a non-identity subgroup of G. We shall show that A is either $\mathfrak{U}$-subnormal or $\mathfrak{U}$-abnormal in G. It is clear that $A = V \rtimes W$, where $V = A \cap D$ and W is a Hall π'-subgroup of H. Moreover, since G is soluble and H is a Hall π'-subgroup of G, we can assume without loss of generality that $W \leq H$ and so $A = (A \cap D)(A \cap H)$. If $H \leq A$, then A is $\mathfrak{U}$-abnormal in G since H is a Gaschütz subgroup of G by hypothesis. Assume that $A \cap H < H$ and let $E = D(A \cap H)$. Then E is $\mathfrak{U}$-subnormal in G and E is supersoluble by Assertion (vi). Hence A is $\mathfrak{U}$-subnormal in G by Lemma 2.1(3).

Proof of Theorem B. *Necessity.* Suppose that this is false and let G be a counterexample of minimal order. Then $\Phi \neq D$ and, in view of Theorem A, $\Phi \neq 1$.

(1) *Assertions (i)–(vi) in Theorem A are true for G/Φ. Moreover, Assertions (iii)–(vi) in Theorem A are true for G/R for any non-identity normal subgroup R of G not containing D.*

First note that $\Phi(G/\Phi) \cap (G/\Phi)^{\mathfrak{U}} = (\Phi(G)/\Phi) \cap (D/\Phi) = (\Phi(G) \cap D)/\Phi = 1$, so G/Φ is an $E_{\mathfrak{F}}$-group by Lemma 2.1(2). Therefore Assertions (i)–(vi) in Theorem A are true for G/Φ. In order to prove the second assertion of (1), it is enough to show that the hypothesis holds for G/R. First note that $G/R \notin \mathfrak{U}$ since $D \nleq R$ by our hypothesis on R. Let A/R be a subgroup of G/R contaning $\Phi(G/R) \cap (G/R)^{\mathfrak{U}} = \Phi(G/R) \cap (DR/R)$. Then, since $\Phi R/R \leq \Phi(G/R)$, A contains Φ and so A is either $\mathfrak{U}$-subnormal or $\mathfrak{U}$-abnormal in G. Hence A/R is either $\mathfrak{U}$-subnormal or $\mathfrak{U}$-abnormal in G/R by Lemma 2.1(2). Therefore the hypothesis holds for G/R for any normal subgroup R of G not containing D.

(2) *$G = D \rtimes H$ is a soluble group, where H is a Hall subgroup of G such that $H\Phi/\Phi$ is a Gaschütz subgroup of G/Φ.*

From Claim (1) we get that $D/\Phi = (G/\Phi)^{\mathfrak{U}}$ is a Hall subgroup of G. Therefore G is soluble. It follows also that Φ is the Sylow p-subgroup of D for some prime p and a p-complement S of D is a Hall subgroup of G.

Since $\Phi \leq \Phi(G)$, from Lemma 2.8 it follows that $D = R \times S$. Hence for some minimal normal subgroup L of G we have $L \leq S$ since $\Phi \neq D$. Then $D/L = (G/L)^{\mathfrak{U}}$ is a Hall subgroup of G by Claim (1) and hence Φ is a Sylow subgroup of G. But this is impossible since $\Phi \leq \Phi(G)$. Hence D is a Hall subgroup of G, so D has a complement H in G. It is clear that $H\Phi/\Phi$ is a complement to $D/\Phi = (G/\Phi)^{\mathfrak{U}}$ in G/Φ, so $H\Phi/\Phi$ is a Gaschütz subgroup of G/Φ since Assertion (i) in Theorem A is true for G/Φ by Claim (1).

(3) *Assertion (iii) in Theorem A is true for G.*

Indeed, from Claim (1) we get that $|(G/\Phi) : (D/\Phi)(G'\Phi/\Phi)| = |G : DG'\Phi| = |G : DG'|$ since $\Phi \leq D$.

(4) *Assertion (iv) in Theorem A is true for G.*

By Claim (1), D/Φ is nilpotent. But then D is nilpotent by Lemma 2.8.

(5) *Assertion (v) in Theorem A is true for G* (This directly follows from Claim (1)).

(6) *Assertion (vi) in Theorem A is true for G.*

Let E be any proper subgroup of G containing D. We shall show that E is supersoluble. Suppose that this is false. Let R be a minimal normal subgroup of G contained in D. Then $D/R = (G/R)^{\mathfrak{U}} \leq E/R < G/R$. Hence E/R is supersoluble by Claim (1). Moreover, E/Φ is supersoluble by the same Claim, so in the case when H is abelian E is supersoluble by Lemma 2.8. Hence H is not abelian. In this case D is nilpotent by Claim (1). If G has a mimimal normal subgroup $L \neq R$ such that $L \leq D$, then $E \simeq E/R \cap L$ is supersoluble. Therefore R is the only minimal normal subgroup of G contained in D. Hence D is a Sylow p-group of G for some prime p. Suppose that $R \leq \Phi(D)$. Then $E/\Phi(D)$ is supersoluble. Moreover, $\Phi(D) \leq \Phi(E)$ since D is normal in E and hence E is supersoluble. Therefore $\Phi(D) = 1$, so $R = D$ by the Maschke's Theorem. But then $\Phi = 1$, a contradiction. Hence we have (6).

From Claims (2)–(6) it follows that the necessity conditions of the theorem are true for G, which contradicts the choice of G. This completes the proof of the necessity.

The sufficiency condition in the theorem directly follows form Theorem A.

4 Final remarks

1. The structure of $\mathfrak{U}$-critical groups are well-known [10, 11]. In particupar, the supersoluble residual of $G^{\mathfrak{U}}$ of an $\mathfrak{U}$-critical group G is a Sylow subgroup of G. This observation and Theorem A are motivations for the following question: *Let G be an $E_{\mathfrak{U}}$-group. Is it true then that $G^{\mathfrak{U}}$ is a Sylow subgroup of G or, at least, the number $|\pi(G^{\mathfrak{U}})|$ is limited to the top?*

The following elementary example shows that the answer to this question is negative.

Example 4.1. Let $p_1 < p_2 < \cdots < p_n < p$ be a set of primes, B a group of order p and P_i a simple $\mathbb{F}_{p_i}B$-module which is faithful for B. Let $A_i = P_i \rtimes B$ and $G = (\ldots((A_1 \curlywedge A_2) \curlywedge A_3) \curlywedge \cdots) \curlywedge A_n$ (see [16, p. 50]). Then G is an $E_{\mathfrak{U}}$-group, $G^{\mathfrak{U}} = P_1 P_2 \cdots P_n$ and $|G/G^{\mathfrak{U}}| = p$.

2. The following example shows that the subgroup D in Theorem A is not necessary nilpotent.

Example 4.2. Let $H = H_2 \rtimes H_3$ is a 2-closed Schmidt group, where H_2 is a Sylow 2-subgroup of G and $H_3 = \langle a \rangle$ a cyclic sylow 3-subgroup of G. Then, by [2, B, 10.7], there exists a simple $\mathbb{F}_7 H$-module P which is faithful for H. Let $G = P \rtimes H$. It is no difficult to show that G is an $E_{\mathfrak{U}}$-group and $G^{\mathfrak{U}} = PH_2$ is non-nilpotent.

3. It is also not difficult to show that the subgroup H in Theorem A is not necessary cyclic.

4. We do not know the answer to the following question: *What is the structure of the group G provided that each nontrivial nilpotent subgroup of G is either $\mathfrak{U}$-subnormal or or $\mathfrak{U}$-abnormal in G?*

5. Partially, the results of this paper were announced in [21].

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