One application of the σ-local formations of finite groups

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Abstract

Throughout this paper, all groups are finite. Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. If n is an integer, the symbol $\sigma(n)$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$. The integers n and m are called σ-coprime if $\sigma(n) \cap \sigma(m) = \emptyset$.

Let $t > 1$ be a natural number and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathfrak{F}$ is Σ_t^σ-closed provided $\mathfrak{F}$ contains each group G with subgroups $A_1, \ldots, A_t \in \mathfrak{F}$ whose indices $|G : A_1|$, ..., $|G : A_t|$ are pairwise σ-coprime.

In this paper, we study Σ_t^σ-closed classes of finite groups.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

Following Shemetkov [1], σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\Pi \subseteq \sigma$ and $\Pi' = \sigma \setminus \Pi$. The group G is said to be [2]: *σ-primary* if G is a σ_i-group for some i; *σ-soluble* if every chief factor of G is σ-primary.

In what follows, $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ [2, 3], $\sigma(G) = \sigma(|G|)$ and $\sigma(\mathfrak{F}) = \bigcup_{G \in \mathfrak{F}} \sigma(G)$. The the integers n and m are called *σ-coprime* if $\sigma(n) \cap \sigma(m) = \emptyset$.

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Recall also that G is called *σ-decomposable* (Shemetkov [1]) or *σ-nilpotent* (Guo and Skiba [7]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$; *meta-σ-nilpotent* [8] if G is an extension of some σ-nilpotent group be the σ-nilpotent group.

The σ-nilpotent groups have proved to be very useful in the formation theory (see, in particular, the papers [9, 10] and the books [1, Ch. IV], [6, Ch. 6]). In the recent years, the σ-nilpotent groups and various classes of meta-σ-nilpotent groups have found new and to some extent unexpected applications in the theories of permutable and generalized subnormal subgroups (see, in particular, [2, 3], [12]–[22] and the survey [8]). This circumstance make the task of further studying of σ-nilpotent and meta-σ-nilpotent groups quite actual and interesting.

In this paper, we study Σ_t^σ-closed classes of meta-σ-nilpotent groups in the sense of the following

Definition 1.1. Let $t > 1$ be a natural number and let $\mathfrak{F}$ be a class of groups. Then we say that $\mathfrak{F}$ is *Σ_t^σ-closed* provided $\mathfrak{F}$ contains each group G with subgroups $A_1, \ldots, A_t \in \mathfrak{F}$ whose indices $|G : A_1|$, ..., $|G : A_t|$ are pairwise σ-coprime.

We will omit the symbol σ in this definition in the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (we use here the notation in [3]). Thus in this case we deal with *Σ_t-closed* classes of groups, in the usual sense (see L.A. Shemetkov [1, p. 44]).

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if: (i) $G/N \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$, and (ii) $G/(N \cap R) \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$. The formation $\mathfrak{F}$ is called *saturated* or *local* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$.

We call any function f of the form

$$f : \sigma \to \{\text{formations of groups}\}$$

a formation σ-function [4], and we put

$$LF_\sigma(f) = (G \mid G = 1 \text{ or } G \neq 1 \text{ and } G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)).$$

Definition 1.2. If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then we say, following [4], that the class $\mathfrak{F}$ is *σ-local* and f is a *σ-local definition of* $\mathfrak{F}$.

Before continuing, consider some examples.

Example 1.3. (i) In view of [5, IV, 3.2], in the case when $\sigma = \sigma^1$, a formation σ-function and a σ-local formation are, respectively, a formation function and a local formation in the usual sense [5, IV, Definition 3.1] (see also [6, Chapter 2]). We use in this case instead of $LF_\sigma(f)$ the symbol $LF(f)$, as usual [5, IV, Definition 3.1].

(ii) For the formation of all identity groups $\mathfrak{I}$ we have $\mathfrak{I} = LF_\sigma(f)$, where $f(\sigma_i) = \emptyset$ for all i.

(iii) Let $\mathfrak{N}_\sigma$ be the class of all σ-nilpotent groups. Then $\mathfrak{N}_\sigma$ is a formation [2] and, clearly, $\mathfrak{N}_\sigma = LF_\sigma(f)$, where $f(\sigma_i) = \mathfrak{I}$ for all i.

(iv) Now let $\mathfrak{N}_\sigma^2$ be the class of all meta-σ-nilpotent groups. Then $\mathfrak{N}_\sigma^2 = LF_\sigma(f)$, where $f(\sigma_i) = \mathfrak{N}_\sigma$ for all i.

(v) The formation of all supersoluble groups $\mathfrak{U}$ is not σ-local for every σ with $\sigma \neq \sigma^1$. Indeed, suppose that $\mathfrak{U} = LF_\sigma(f)$ is σ-local and for some i we have $|\sigma_i| > 1$. Let $p, q \in \sigma_i$, where $p > q$. Finally, let $G = C_q \wr C_p = K \rtimes C_p$ be the regular wreath product of groups C_q and C_p with $|C_q| = q$ and $|C_p| = p$, where K is the base group of G. Then $C_G(K) = K$ and, also, $O_{\sigma_i', \sigma_i}(G) = G$ and $\sigma(G) = \{\sigma_i\}$. Since $C_p \in \mathfrak{U}$, $f(\sigma_i) \neq \emptyset$. Hence $G \in LF_\sigma(f) = \mathfrak{U}$, so $G = C_q \times C_p$ since $p > q$, a contradiction. Hence we have (iv).

The theory of Σ_t-closed classes of soluble groups and various its applications were considered by Otto-Uwe Kramer in [23] (see also [1, Chapter 1] or [11, Chapter 2]).

Our main goal here is to prove the following result.

Theorem 1.4. *Every σ-local formation of meta-σ-nilpotent groups is Σ_4^σ-closed.*

In the case when $\sigma = \sigma^1$, we get from Theorem 1.4 the following well-known facts.

Corollary 1.5 (Doerk [26]). *If G has four supersoluble subgroups A_1, A_2, A_3, A_4 whose indices $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ are pairwise coprime, then G is itself supersoluble.*

Corollary 1.6. *If G has four meta-nilpotent subgroups A_1, A_2, A_3, A_4 whose indices $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ are pairwise coprime, then G is itself meta-nilpotent.*

Corollary 1.7. *Suppose that G has four subgroups A_1, A_2, A_3, A_4 whose indices $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ are pairwise coprime. If the derived subgroup A_i' of A_i is nilpotent for all $i = 1, 2, 3, 4$, then G' is nilpotent.*

Finally, we get from Theorem 1.4 the following

Corollary 1.8 (Otto-Uwe Kramer [23]). *Every local formation of meta-nilpotent groups is Σ_4-closed.*

In fact, in the theory of the π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$) we deal with the partition $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$ [3]. Note that G is: $\sigma^{1\pi}$-soluble if and only if G is π-soluble; $\sigma^{1\pi}$-nilpotent if and only if G is *π-special* [24], that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$. Hence in this case we get from Theorem 1.4 the following results.

Corollary 1.9. *Suppose that G has four meta-π-special subgroups A_1, A_2, A_3, A_4 whose indices $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ are pairwise coprime and each of them is either a π-number or a π'-number. Suppose also that at most one of the numbers $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ is a π'-number. Then G is meta-π-special.*

Corollary 1.10. *Suppose that G has subgroups $A_1, \ldots, A_4$ such that the indices $|G : A_1|, \ldots, |G : A_4|$ are pairwise coprime and each of them is either a π-number or a π'-number. Suppose also that at most one of the numbers $|G : A_1|, |G : A_2|, |G : A_3|, |G : A_4|$ is a π'-number. If the derived subgroup A_i' of A_i is π-special for all i, then G' is π-special.*

If for a subgroup A of G we have $\sigma(|A|) \subseteq \Pi$ and $\sigma(|G : A|) \subseteq \Pi'$, then A is said to be a *Hall* Π-*subgroup* [8] of G. We say that G is Π-*closed* if G has a normal Hall Π-subgroup.

The proof of Theorem 1.4 is preceded by a large number of auxiliary results. The following theorem is one of them.

Theorem 1.11. (i) *The class of all σ-soluble Π-closed groups $\mathfrak{F}$ is Σ_3^{σ}-closed.*

(ii) *Every formation of σ-nilpotent groups $\mathfrak{M}$ is Σ_3^{σ}-closed.*

Corollary 1.12. (i) *The classes of all σ-soluble groups and of all σ-nilpotent groups are Σ_3^{σ}-closed.*

In the case when $\sigma = \sigma^1$, we get from Corollary 1.12 the following well-known results.

Corollary 1.13 (Wielandt [5, Ch. I, Theorem 3.4]). *If G has three soluble subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself soluble.*

Corollary 1.14 (Kegel [25]). *If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself nilpotent.*

Corollary 1.15 (Doerk [26]). *If G has three abelian subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself abelian.*

In the case when $\sigma = \sigma^{1\pi}$, we get from Theorem 1.11 the following facts.

Corollary 1.16. *Suppose that G has three π-soluble subgroups A_1, A_2, A_3 whose indices $|G : A_1|, |G : A_2|, |G : A_3|$ are pairwise coprime and each of them is either a π-number or a π'-number. Suppose also that at most one of the numbers $|G : A_1|, |G : A_2|, |G : A_3|$ is a π'-number. Then G is π-soluble.*

Corollary 1.17. *Suppose that G has three π-special subgroups A_1, A_2, A_3 whose indices $|G : A_1|, |G : A_2|, |G : A_3|$ are pairwise coprime and each of them is either a π-number or a π'-number. Suppose also that at most one of the numbers $|G : A_1|, |G : A_2|, |G : A_3|$ is a π'-number. Then G is π-spesial.*

2 General properties of σ-local formations

If $\mathfrak{M}$ and $\mathfrak{H}$ are classes of groups, then $\mathfrak{M}\mathfrak{H}$ is the class of groups G such that for some normal subgroup N of G we have $G/N \in \mathfrak{H}$ and $N \in \mathfrak{M}$. The *Gaschütz product* $\mathfrak{M} \circ \mathfrak{H}$ of $\mathfrak{M}$ and $\mathfrak{H}$ is defined as folows: $G \in \mathfrak{M} \circ \mathfrak{H}$ if and only if $G^{\mathfrak{H}} \in \mathfrak{M}$. The class $\mathfrak{F}$ is called *hereditary* in the sense of Mal'cev [27] if $G \in \mathfrak{F}$ whenever $G \leq A \in \mathfrak{F}$.

All statements of the following lemma are well-known (see, [28, Chapter II] or [5, Chapter IV]) and, in fact, each of them may be proved by the direct calculations.

Lemma 2.1. *Let $\mathfrak{M}$, $\mathfrak{H}$ and $\mathfrak{F}$ be formations.*

(1) *$\mathfrak{M} \circ \mathfrak{H}$ is a formation.*

(2) *If $\mathfrak{M}$ is hereditary, then $\mathfrak{M}\mathfrak{H} = \mathfrak{M} \circ \mathfrak{H}$.*

(3) $(\mathfrak{M} \circ \mathfrak{H}) \circ \mathfrak{F} = \mathfrak{M} \circ (\mathfrak{H} \circ \mathfrak{F})$.

(4) *If $\mathfrak{M}$ and $\mathfrak{H}$ are hereditary, then $\mathfrak{M}\mathfrak{H}$ is hereditary.*

(5) *If $\mathfrak{M}$ is saturated and $\pi(\mathfrak{H}) \subseteq \pi(\mathfrak{M})$, then $\mathfrak{M} \circ \mathfrak{H}$ is saturated.*

We write $\mathfrak{G}_{\Pi}$ (respectively $\mathfrak{S}_{\Pi}$) to denote the class of all Π-groups (respectively the class of all σ-soluble Π-groups). In particular, $\mathfrak{G}_{\sigma_i'}$ is the class of all σ_i'-groups and $\mathfrak{G}_{\sigma_i}$ is the class of all σ_i-groups and $\mathfrak{S}_{\sigma_i'}$ is the class of all σ-soluble σ_i'-groups.

We use $F_{\Pi}(G)$ to denote the product of all normal Π'-closed subgroups of G; we write also $F_{\sigma_i}(G)$ instead of $F_{\{\sigma_i\}}(G)$.

Lemma 2.2. (1) *The class of all (σ-soluble) Π-closed groups $\mathfrak{F}$ is a hereditary formation. Moreover,*

(2) *If E is a normal subgroup of G and $E/E \cap \Phi(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$. Hence the formation $\mathfrak{F}$ is saturated.*

(3) *If $A, B \in \mathfrak{F}$ are normal subgroups of G and $G = AB$, then $G \in \mathfrak{F}$.*

(4) *If E is a subnormal subgroup of G, then $F_{\Pi}(G) \cap E = F_{\Pi}(E)$.*

Proof. (1) It is clear that $\mathfrak{F} = \mathfrak{G}_{\Pi}\mathfrak{G}_{\Pi'}$. Hence $\mathfrak{F}$ is a hereditary formation by Lemma 2.1(1, 2, 4).

(2) Let $H/E \cap \Phi(G)$ be the normal Hall Π-subgroup of $E/E \cap \Phi(G)$. Then $H/E \cap \Phi(G)$ is characteristic in $E/E \cap \Phi(G) \trianglelefteq G/E \cap \Phi(G)$, so H is normal in G. Let $D = O_{\Pi'}(E \cap \Phi(G))$. Then, since $E \cap \Phi(G)$ is nilpotent, D is a Hall Π'-subgroup of H. Hence by the Schur-Zassenhaus theorem, H has a Hall Π-subgroup, say V, and any two Hall Π-subgroups of H are conjugated in H. Therefore, $G = HN_G(V) = (VD))N_G(V) = N_G(V)$ by the Frattini argument. Thus V is normal in G. Finally, V is a Hall Π-subgroup of E since $\sigma(|E/E \cap \Phi(G) : H/E \cap \Phi(G)|) \cap \Pi = \emptyset$, so $E \in \mathfrak{F}$.

(3) If V is a Hall Π-subgroup of A, then V is characteristic in A and so V is normal in G. Similarly, a Hall Π-subgroup W of B is normal in G. Moreover,

$$G/VW = AB/VW = (AVW/VW)(BVW/VW),$$

where

$$AVW/VW \simeq A/A \cap VW = A/V(A \cap W) \simeq (A/V)/(V(A \cap W)/V$$

and BVW/VW are Π'-groups. Hence VW is a Hall Π-subgroup of G, so $G \in \mathfrak{F}$.

(4) Since the group A is Π'-closed if and only if $A \in \mathfrak{G}_{\Pi'}\mathfrak{G}_{\Pi}$, we have (4) by [5, VIII, Proposition 2.4(d)].

The lemma is proved.

If f is a formation σ-function, then the symbol $\mathrm{Supp}(f)$ denotes the *support* of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \emptyset$.

Lemma 2.3. *Let $\mathfrak{F} = LF_\sigma(f)$ and $\Pi = Supp(f)$.*

(1) $\Pi = \sigma(\mathfrak{F})$.

(2) *$G \in \mathfrak{F}$ if and only if $G \in \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all $\sigma_i \in \sigma(G)$.*

(3) *$\mathfrak{F} = (\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{G}_\Pi$. Hence $\mathfrak{F}$ is a saturated formation.*

(4) *If every group in $\mathfrak{F}$ is σ-soluble, then $\mathfrak{F} = (\bigcap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{S}_\Pi$.*

Proof. (1) If $\sigma_i \in \Pi$, then $1 \in f(\sigma_i)$ and for every σ_i-group $G \neq 1$ we have $\sigma(G) = \{\sigma_i\}$ and $O_{\sigma_i', \sigma_i}(G) = G$. Hence $G \in LF_\sigma(f) = \mathfrak{F}$, so $\sigma_i \in \sigma(\mathfrak{F})$. Therefore $\Pi \subseteq \sigma(\mathfrak{F})$. On the other hand, if $\sigma_i \in \sigma(\mathfrak{F})$, then for some group $G \in \mathfrak{F}$ we have $\sigma_i \in \sigma(G)$ and $G/F_{\sigma_i}(G) \in f(\sigma_i)$. Thus $\sigma_i \in \Pi$, so $\Pi = \sigma(\mathfrak{F})$.

(2) If $G \in \mathfrak{F}$ and $\sigma_i \in \sigma(G)$, then $G/F_{\sigma_i}(G) \in f(\sigma_i)$, where $F_{\sigma_i}(G)$ is σ_i'-closed by Lemma 2.2(3). Hence $G \in \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)$ by Lemma 2.2(1). Similarly, if for any $\sigma_i \in \sigma(G)$ we have $G \in \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)$, then $G/F_{\sigma_i}(G) \in f(\sigma_i)$ and so $G \in \mathfrak{F}$.

(3) If $G \in \mathfrak{F}$, then $\sigma(G) \subseteq \Pi$ and so $G \in \mathfrak{G}_\Pi$. Moreover, in this case for every $\sigma_i \in \sigma(G)$ we have $G \in \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)$ by Part (2). Finally, if $\sigma_i \in \Pi \setminus \sigma(G)$, then $G \in \mathfrak{G}_{\sigma_i'} \subseteq \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)$ since the class $\mathfrak{G}_{\sigma_i'}$ is hereditary. Therefore $\mathfrak{F} \subseteq (\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{G}_\Pi$. Hence $\mathfrak{F} = (\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{G}_\Pi$ is a saturated formation by Lemmas 2.1(5) and 2.2(1, 2). Hence we have (3).

(4) See the proof of (3).

The lemma is proved.

Lemma 2.4. *If $\mathfrak{F} = LF_\sigma(f)$, then $\mathfrak{F} = LF_\sigma(t)$, where $t(\sigma_i) = f(\sigma_i) \cap \mathfrak{F}$ for all $\sigma_i \in \sigma$.*

Proof. First note that in view of Lemma 2.3(3), t is a formation σ-function and $LF_\sigma(t) \subseteq \mathfrak{F}$. On the other hand, if $G \in \mathfrak{F}$, then $G/F_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{F} = t(\sigma_i)$ for every $\sigma_i \in \sigma(G)$ and so $G \in LF_\sigma(t)$. Hence $\mathfrak{F} = LF_\sigma(t)$. The lemma is proved.

Proposition 2.5. *Let f and h be formation σ-functions and let* $\Pi = \mathrm{Supp}(f)$. *Suppose that $\mathfrak{F} = LF_\sigma(f) = LF_\sigma(h)$.*

(1) *If $\sigma_i \in \Pi$, then $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$.*

(2) *$\mathfrak{F} = LF_\sigma(F)$, where F is a formation σ-function such that*

$$F(\sigma_i) = \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i} F(\sigma_i)$$

for all $\sigma_i \in \Pi$.

Proof. (1) First suppose that $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \nsubseteq \mathfrak{F}$ and let G be a group of minimal order in $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \setminus \mathfrak{F}$. Note that $f(\sigma_i) \cap \mathfrak{F}$ is a formation by Lemma 2.3(3), so $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})$ is a formation by Lemma 2.1(1, 2). Hence $R = G^{\mathfrak{F}} \leq G^{f(\sigma_i) \cap \mathfrak{F}}$ is a unique minimal normal subgroup of G, so R is a σ_i-group.

Moreover, $F_{\sigma_i}(G) = O_{\sigma_i}(G)$ and $F_{\sigma_j}(G/R) = F_{\sigma_j}(G)/R$ for all $j \neq i$. Therefore, since $G/R \in \mathfrak{F}$

we have

$$(G/R)/F_{\sigma_j}(G/R) \simeq G/F_{\sigma_j}(G) \in f(\sigma_j)$$

for all $\sigma_j \in \sigma(G) \setminus \{\sigma_i\}$. Finally, we have

$$G/F_{\sigma_i}(G) = G/O_{\sigma_i}(G) \in f(\sigma_i)$$

since $G \in \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})$ and the class $\mathfrak{G}_{\sigma_i}$ is hereditary. But then $G \in \mathfrak{F}$, a contradiction. Hence $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$.

Now suppose that $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \not\subseteq \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})$ and let G be a group of minimal order in $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \setminus \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})$. Then G has a unique minimal normal subgroup R, $R = G^{\mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})}$ and $R \not\leq O_{\sigma_i}(G)$. Hence $O_{\sigma_i}(G) = 1$.

Let A be any non-identity σ_i-group and let $E = A \wr G = K \rtimes G$ be the regular wreath product of A and G, where K is the base group of E. Then $O_{\sigma_i'}(E) = 1$, so $F_{\sigma_i}(E) = O_{\sigma_i}(E) = K(O_{\sigma_i}(E) \cap G) = K$ since $O_{\sigma_i}(G) = 1$. Moreover, since $G \in \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$ we have $E \in \mathfrak{F}$ and so $E/F_{\sigma_i}(E) = E/K \simeq G \in h(\sigma_i) \cap \mathfrak{F} \subseteq \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})$. Thus $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})$, so $\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i}(h(\sigma_i) \cap \mathfrak{F})$.

(2) Let $\mathfrak{M} = LF_\sigma(F)$. Then

$$\mathfrak{M} = (\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i}(\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}))) \cap \mathfrak{G}_\Pi$$

$$= (\bigcap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})) \cap \mathfrak{G}_\Pi = \mathfrak{F}$$

by Lemmas 2.3(3) and 2.4. Hence we have (2).

The proposition is proved.

Corollary 2.6. (1) *For every formation σ-function f the class $LF_\sigma(f)$ is a non-empty saturated formation.*

(2) *Every σ-local formation $\mathfrak{F}$ possesses a unique σ-local definition F such that for every σ-local definition f of $\mathfrak{F}$ and for every $\sigma_i \in \sigma(\mathfrak{F})$ the following holds:*

$$F(\sigma_i) = \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) = \mathfrak{G}_{\sigma_i}F(\sigma_i).$$

Proof. (1) First note that every identity group, by definition, belongs to $LF_\sigma(f)$, so this class is non-empty. On the other hand, the class is a saturated formation by Lemma 2.3(3).

(2) This assertion directly follows from Proposition 2.5(2).

The corollary is proved.

Recall that form($\mathfrak{X}$) denotes the intersection of all formations containing the collection of groups $\mathfrak{X}$.

Proposition 2.7. *Let $\mathfrak{F} = LF_\sigma(f)$ be a σ-local formation and $\Pi = \sigma(\mathfrak{F})$. Let m be the formation σ-function such that $m(\sigma_i) = \mathrm{form}(G/F_{\sigma_i}(G)|\ G \in \mathfrak{F})$ for all $\sigma_i \in \Pi$ and $m(\sigma_i) = \emptyset$ for all $\sigma_i \in \Pi'$. Then:*

(i) *$\mathfrak{F} = LF_\sigma(m)$, and*

(ii) *$m(\sigma_i) \subseteq h(\sigma_i) \cap \mathfrak{F}$ for every formation σ-function h of $\mathfrak{F}$ and for every $\sigma_i \in \sigma$.*

Proof. Let $\mathfrak{F}(\sigma_i) = (G/F_{\sigma_i}(G)|\ G \in \mathfrak{F})$ for all $\sigma_i \in \Pi$, and let $\mathfrak{M} = LF_\sigma(m)$. Then $\mathfrak{F} \subseteq \mathfrak{M}$. On the other hand, $\mathfrak{F}(\sigma_i) \subseteq f(\sigma_i)$ and so $m(\sigma_i) \subseteq f(\sigma_i)$ for all $\sigma_i \in \Pi$. Also, we have $m(\sigma_i) = \emptyset \subseteq f(\sigma_i)$ for all $\sigma_i \in \Pi'$. Hence $\mathfrak{M} \subseteq \mathfrak{F}$, so $\mathfrak{M} = \mathfrak{F}$. The proposition is proved.

We call the σ-local definition m of the formation $\mathfrak{F}$ in Proposition 2.7 the *smallest σ-local definition* of $\mathfrak{F}$.

3 Proofs of Theorems 1.4 an 1.11

Proof of Theorem 1.11. (i) Suppose that $\mathfrak{F}$ is not Σ_3^σ-closed and let G be a group of minimal order among the groups G such that $G \notin \mathfrak{F}$ but G has subgroups A_1, A_2 and $A_3 \in \mathfrak{F}$ such that the indices $|G : A_1|$, $|G : A_2|$ and $|G : A_3|$ are pairwise σ-coprime. Then $G = A_iA_j$ for all $i \neq j$. Let R is a minimal normal subgroup of G.

(1) *G/R is σ-soluble Π-closed. Hence R is not a σ-primary Π-group.*

If for some i we have $A_i \leq R$, then for any $j \neq i$ we have $G/R = A_iA_j/R = A_jR/R \simeq A_j/(A_j \cap R) \in \mathfrak{F}$ since $\mathfrak{F}$ is a formation by Lemma 2.2. Now assume that $A_i \nleq R$ for all i. Then the hypothesis holds for G/R, so G/R is σ-soluble Π-closed by the choice of G. Therefore, R is a not σ-primary Π-group since $G \notin \mathfrak{F}$. Hence we have (1).

(2) *G is σ-soluble.*

Let L be a minimal normal subgroup of A_1. Since A_1 is σ-soluble, L is a σ_i-group for some i. Moreover, since $|G : A_2| = |A_1 : A_1 \cap A_2|$ and $|G : A_3| = |A_1 : A_1 \cap A_3|$ are σ-coprime by hypothesis, we have either $L \leq A_1 \cap A_2$ or $L \leq A_1 \cap A_3$. Therefore we can assume without loss of generality that $L \leq A_2$, so $L^G = L^{A_1A_2} = L^{A_2} \leq A_2$. Hence $1 < L^G$ is σ-soluble and so we have (2) by Claim (1).

(3) *R is a unique minimal normal subgroups of G, $R \nleq \Phi(G)$ and R is a σ_i-group for some $\sigma_i \in \Pi'$. Hence $C_G(R) \leq R$.*

Since G is σ-soluble by Claim (2), R is a σ_i-group for some i. Moreover, Claim (2) and Lemma 2.2 imply that R is a unique minimal normal subgroups of G, R is a Π'-group and $R \nleq \Phi(G)$. Hence $C_G(R) \leq R$ by [5, A, 17.2].

(4) *There are $j \neq k$ such that $R \leq A_j \cap A_k$* (Since $|G : A_j|$ and $|G : A_k|$ are σ-coprime by hypothesis, this follows from Claim (3)).

Final contradiction for (i) Since $O_\Pi(A_j)$ is normal in A_j and $R \leq O_{\Pi'}(A_j)$ by Claims (3) and

(4), we get that $O_\Pi(A_j) \leq C_G(R) \leq R \leq O_{\Pi'}(A_j)$ by Claim (3). Hence $O_\Pi(A_j) = 1$. But A_j is Π-closed by hypothesis and so A_j is a Π'-group. Similarly, one can show that A_k is a Π'-group and so $G = A_jA_k$ is a Π'-group. But then G is is Π-closed. This contradiction completes the proof of (i).

(ii) Suppose that $\mathfrak{M}$ is not Σ_3^σ-closed and let G be a group of minimal order among the groups G such that $G \notin \mathfrak{M}$ but G has subgroups A_1, A_2 and $A_3 \in \mathfrak{M}$ such that the indices $|G : A_1|$, $|G : A_2|$ and $|G : A_3|$ are pairwise σ-coprime. Then $G \neq A_i$ for all i and G is σ-nilpotent by Part (i). Moreover, the choice of G implies that $G/R \in \mathfrak{M}$ for every minimal normal subgroup R of G. Therefore R is a unique minimal normal subgroups of G since $\mathfrak{M}$ is a formation. Hence, in fact, G is σ_i-group for some i. But then from $A_1 < G$ and $A_2 < G$ we get that the indices $|G : A_1|$ and $|G : A_2|$ are not σ-coprime. This contradiction shows that we have (ii). The theorem is proved.

Lemma 3.1. *If G is σ-soluble, then $C_G(F_\sigma(G)) \leq F_\sigma(G)$.*

Proof. Let $C = C_G(F_\sigma(G))$. Assume that $C \nleq F_\sigma(G)$ and let $H/F_\sigma(G)$ be a chief factor of G such that $H \leq F_\sigma(G)C$. Then $H = F_\sigma(G)(H \cap C)$. Since G is σ-soluble,

$$H/F_\sigma(G) = F_\sigma(G)(H \cap C)/F_\sigma(G) \simeq (H \cap C)/((H \cap C) \cap F_\sigma(G))$$

is a σ_i-group.

Now let U be a minimal supplement to $(H \cap C) \cap F_\sigma(G)$ in $H \cap C$. Then $((H \cap C) \cap F_\sigma(G)) \cap U \leq \Phi(U)$, so U is σ_i-group. Moreover, $(H \cap C) \cap F_\sigma(G) \leq Z(H \cap C)$ and so $H \cap C$ is a normal σ-nilpotent subgroup of G. Hence $H \cap C \leq F_\sigma(G)$ and so $H = F_\sigma(G)$. This contradiction completes the proof of the lemma.

Lemma 3.2. *Let $\mathfrak{F} = \mathfrak{S}_\Pi\mathfrak{X}$, where $\mathfrak{X} \subseteq \mathfrak{S}_\sigma$. If the formation $\mathfrak{X}$ is Σ_t^σ-closed, then $\mathfrak{F}$ is Σ_{t+1}^σ-closed.*

Proof. Suppose that this lemma is false and let G be a group of minimal order among the groups G such that $G \notin \mathfrak{F}$ but G has subgroups $A_1, \ldots, A_{t+1} \in \mathfrak{F}$ such that the indices $|G : A_1|, \ldots, |G : A_{t+1}|$ are pairwise σ-coprime. Then G is σ-soluble by Theorem 1.11.

Let R be a minimal normal subgroup of G, so R is a σ_i-group for some i. Moreover, the hypothesis holds for G/R since $\mathfrak{F}$ is a formation by Lemma 2.1(1, 2) and so $G/R \in \mathfrak{F}$ by the choice of G. Hence R is a unique minimal normal subgroup of G by the choice of G. Therefore, $\sigma_i \in \Pi'$ and $R \leq O_{\sigma_i}(G) = F_\sigma(G)$. Hence $C_G(F_\sigma(G) = C_G(O_{\sigma_i}(G)) \leq O_{\sigma_i}(G)$ By Lemma 3.1.

By hypothesis, there are numbers $i_1, \ldots, i_t$ such that $O_{\sigma_i}(G) \leq A_{i_1} \cap \cdots \cap A_{i_t}$. Then $O_\Pi(A_{i_j}) \leq C_G(O_{\sigma_i}(G)) \leq O_{\sigma_i}(G)$. Hence $O_\Pi(A_{i_j}) = 1$ and so $A_{i_j} \in \mathfrak{X}$ for all $j = 1, \ldots, t$. Therefore $G \in \mathfrak{X} \subseteq \mathfrak{F}$ since the formation $\mathfrak{X}$ is Σ_t^σ-closed. This contradiction completes the prove of the lemma.

Lemma 3.3. *Let $\mathfrak{M}$ be a formation of σ-soluble Π-closed groups and let $\mathfrak{F} = \mathfrak{S}_\Pi\mathfrak{M}$. If $\mathfrak{M}$ is Σ_3^σ-closed, then $\mathfrak{F}$ is Σ_3^σ-closed.*

Proof. Suppose that G has subgroups $A_1, A_2, A_3 \in \mathfrak{F}$ such that the indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise σ-coprime. Then G has a normal Hall Π-subgroup V by Theorem 1.11. Hence

$V \cap A_i$ is a normal Hall Π-subgroup of A_i and so from the isomorphism $VA_i/V \simeq A_i/A_i \cap V$ we get that $VA_i/V \in \mathfrak{M}$ and the indices $|(G/V) : (A_1V/V)|$, $|(G/V) : (A_2V/V)|$, $|(G/V) : (A_3V/V)|$ are pairwise σ-coprime. But then $G/V \in \mathfrak{M}$ since $\mathfrak{M}$ is Σ_3^σ-closed by hypothesis. Hence $G \in \mathfrak{F}$. The lemma is proved.

The following lemma is evident.

Lemma 3.4. *If the class of groups $\mathfrak{F}_j$ is Σ_t^σ-closed for all $j \in J$, then the class $\bigcap_{j\in J} \mathfrak{F}_j$ is also Σ_t^σ-closed.*

A formation σ-function f is said to be: *integrated* if $f(\sigma_i) \subseteq LF_\sigma(f)$ for all i; *full* if $f(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all i.

In view of Corollary 2.6, every σ-local formation $\mathfrak{F}$ possesses a unique integrated and full σ-local definition F. We call such a function F the *canonical σ-local definition* of $\mathfrak{F}$.

Theorem 3.5. *Let $\mathfrak{F} = LF_\sigma(F)$ be a σ-local formation of σ-soluble groups, where F is the canonical σ-local definition of $\mathfrak{F}$. If the formation $F(\sigma_i)$ is Σ_t^σ-closed for every i, then the formation $\mathfrak{F}$ is Σ_{t+1}^σ-closed.*

Proof. Let $\Pi = \mathrm{Supp}(\mathfrak{F})$. Then, by Lemma 2.3(4) and Corollary 2.6,

$$\mathfrak{F} = (\bigcap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)) \cap \mathfrak{S}_\Pi = (\bigcap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'} F(\sigma_i)) \cap \mathfrak{S}_\Pi.$$

By Lemma 3.2, the formation $\mathfrak{S}_{\sigma_i'} F(\sigma_i)$ is Σ_{t+1}^σ-closed. On the other hand, the class $\mathfrak{S}_\Pi$ is Σ_2^σ-closed and so Σ_{t+1}^σ-closed. Hence $\mathfrak{F}$ is Σ_{t+1}^σ-closed by Lemma 3.4. The theorem is proved.

Proof of Theorem 1.4. Let $\mathfrak{F} = LF_\sigma(f)$ be any σ-local formation of meta-σ-nilpotent groups, where f is the smallest σ-local definition of $\mathfrak{F}$. Then the formation $f(\sigma_i)$ is contained in $\mathfrak{N}_\sigma$ for all σ_i by Proposition 2.7. Hence $f(\sigma_i)$ is Σ_3^σ-closed by Theorem 1.11.

Let F be the canonical σ-local definition of $\mathfrak{F}$. Then $F(\sigma_i) = \mathfrak{G}_{\sigma_i} f(\sigma_i)$ for all $\sigma_i \in \sigma$ by Propositions 2.5 and 2.7. Hence, $F(\sigma_i)$ is Σ_3^σ-closed by Lemma 3.3. Therefore $\mathfrak{F}$ is Σ_4^σ-closed by Theorem 3.5. The theorem is proved.

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