On finite $P\sigma T$-groups

Alexander N. Skiba
Department of Mathematics, Francisk Skorina Gomel State University,
Gomel 246019, Belarus
E-mail: alexander.skiba49@gmail.com

Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and G a finite group. G is said to be *σ-soluble* if every chief factor H/K of G is a σ_i-group for some $i = i(H/K)$.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $i \in I$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. A subgroup A of G is said to be *σ-permutable* or *σ-quasinormal* in G if G has a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$.

We obtain a characterization of finite σ-soluble groups G in which σ-quasinormality is a transitive relation in G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. G is said to be a D_π-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is always supposed to be a subset of the set σ and $\Pi' = \sigma \setminus \Pi$.

By the analogy with the notation $\pi(n)$, we write $\sigma(n)$ to denote the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$. G is said to be: *σ-primary* [1] if $|\sigma(G)| \leq 1$; *σ-decomposable* (Shemetkov [2]) or *σ-nilpotent* (Guo and Skiba [3]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$; *σ-soluble* [1] if every chief factor of G is σ-primary; a *σ-full group of Sylow type* [1] if every subgroup E of G is a D_{σ_i}-group for every $\sigma_i \in \sigma(E)$.

A natural number n is said to be a *Π-number* if $\sigma(n) \subseteq \Pi$. A subgroup A of G is said to be: a *Hall Π-subgroup* of G [1, 4] if $|A|$ is a Π-number and $|G : A|$ is a Π'-number; a *σ-Hall subgroup* of G if A is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.

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A set $\mathcal{H}$ of subgroups of G is a *complete Hall σ-set* of G [4, 5] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$.

Recall that a subgroup A of G is said to be: *σ-permutable* or *σ-quasinormal* in G [1] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^x A$ for all $H \in \mathcal{H}$ and all $x \in G$; *σ-subnormal* in G [1] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

In the classical case, when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\}$, σ-quasinormal subgroups are also called *S-quasinormal* or *S-permutable* [6, 7], and a subgroup A of G is subnormal in G if and only if it is σ^0-subnormal in G.

We say that G is a *$P\sigma T$-group* [1] if σ-quasinormality is a transitive relation in G, that is, if K is a σ-quasinormal subgroup of H and H is a σ-quasinormal subgroup of G, then K is a σ-quasinormal subgroup of G. In the case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, $P\sigma T$-groups are called *PST-groups* [6].

In view of Theorem B in [1], $P\sigma T$-groups can be characterized as the groups in which every σ-subnormal subgroup is σ-quasinormal in G.

Our first observation is the following fact, which generalizes the sufficiency condition in Theorem A of the paper [1].

Theorem A. *Let G have a normal σ-Hall subgroup D such that:*

(i) G/D is a $P\sigma T$-group, and

(ii) every σ-subnormal subgroup of D is normal in G.

If G is a σ-full group of Sylow type, then G is a $P\sigma T$-group.

Corollary 1.1 (See Theorem A in [1]). *Let G have a normal σ-Hall subgroup D such that:*

(i) *G/D is σ-nilpotent, and*

(ii) *every subgroup of D is normal in G.*

Then G is a $P\sigma T$-group.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, we get from Theorem A the following

Corollary 1.2 (See Theorem 2.4 in [8]). *Let G have a normal Hall subgroup D such that:*

(i) *G/D is a PST-group, and*

(ii) *every subnormal subgroup of D is normal in G.*

Then G is a PST-group.

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N; $G^{\mathfrak{N}}$ denotes the *nilpotent residual* of G [9].

Definition 1.3. We say that G is a *special $P\sigma T$-group* provided the σ-nilpotent residual $D = G^{\mathfrak{N}_\sigma}$ of G is contained in a Hall σ_i-subgroup E of G for some i and the following conditions hold:

(i) D is a Hall subgroup of G and every element of G induces a power automorphism in D;

(ii) D has a normal complement S in E.

Note that if $G = C_5 \times (C_3 \rtimes C_2)$, where $C_3 \rtimes C_2 \simeq S_3$ and $\sigma = \{\{3,5\},\{3,5\}'\}$, then G is a special $P\sigma T$-group with $C_3 = G^{\mathfrak{N}_\sigma}$.

The following theorem shows that every special $P\sigma T$-group is a $P\sigma T$-group.

Theorem B. *Suppose that G has a σ-nilpotent normal Hall subgroup D with σ-nilpotent quotient G/D such that $G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(D)$. Then G is a $P\sigma T$-group.*

Generalizing the concept of complete Wielandt σ-set of a group in [3], we say that a complete Hall σ-set $\mathcal{H}$ of G is a *generalized Wielandt σ-set* of G if every member H of $\mathcal{H}$ is $\pi(G^{\mathfrak{N}_\sigma})$-supersoluble.

Using Theorem B, we prove also the following revised version of Theorem A in [1].

Theorem C. *Let G be σ-soluble and $D = G^{\mathfrak{N}_\sigma}$. Suppose that G has a generalized Wielandt σ-set. Then G is a $P\sigma T$-group if and only if the following conditions hold:*

(i) *D is an abelian Hall subgroup of G of odd order and every element of G induces a power automorphism in D;*

(ii) *$G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(D)$.*

Corollary 1.4 (See Theorem 2.3 in [8]). *Let G be a soluble and $D = G^{\mathfrak{N}}$. If G is a PST-group, then D is an abelian Hall subgroup of G of odd order and every element of G induces a power automorphism in D.*

2 Some preliminary results

In view of Theorems A and B in [4], the following fact is true.

Lemma 2.1. *If G is σ-soluble, then G is a σ-full group of Sylow type.*

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.2 (See Corollary 2.4 and Lemma 2.5 in [1]). *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

In view of Proposition 2.2.8 in [9], we get from Lemma 2.2 the following

Lemma 2.3. *If N is a normal subgroup of G, then*

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N.$$

Lemma 2.4 (See Knyagina and Monakhov [10]). *Let H, K and N be pairwise permutable subgroups of G and H be a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Lemma 2.5. *The following statements hold:*

(i) *G is a $P\sigma T$-group if and only if every σ-subnormal subgroup of G is σ-quasinormal in G.*

(ii) *If G is a $P\sigma T$-group, then every quotient G/N of G is also a $P\sigma T$-group.*

(iii) *If G is a special $P\sigma T$-group, then every quotient G/N of G is also a special $P\sigma T$-group.*

Proof. (i) This follows from the fact (see Theorem B in [1]) that every σ-quasinormal subgroup of G is σ-subnormal in G.

(ii) Let H/N be a σ-subnormal subgroup of G/N. Then H is a σ-subnormal subgroup of G by Lemma 2.6(5) in [1], so H is σ-quasinormal in G by hypothesis and Part (i). Hence H/N is σ-quasinormal in G/N by Lemma 2.8(2) in [1]. Hence G/N is a $P\sigma T$-group by Part (i).

(iii) Suppose that $D = G^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G and $D \leq E$, where $E = D \times S$ is a Hall σ_i-subgroup E of G, and every element of G induces a power automorphism in D. Then EN/N is a Hall σ_i-subgroup of G/N and $DN/N = (G/N)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/N by Lemma 2.3. Moreover, $EN/N = (DN/N)(SN/N)$ and, by Lemma 2.4,

$$DN \cap SN = N(D \cap SN) = N(D \cap S)(D \cap N) = N(D \cap N) = N,$$

which implies that $(DN/N) \cap (SN/N) = 1$. Hence $EN/N = (DN/N) \times (SN/N)$.

Finally, let $H/N \leq DN/N$. Then $H = N(H \cap D)$, where $H \cap D$ is normal in G by hypothesis. But then $H/N = N(H \cap D)/N$ is normal in G/N, so every element of G/N induces a power automorphism on DN/D. Hence G/N is a special $P\sigma T$-group.

The lemma is proved.

3 Proofs of the results

Proof of Theorem A. Since G is a σ-full group of Sylow type by hypothesis, it possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup H of G is σ-quasinormal in G if and only if $HH_i^x = H_i^x H$ for all $H_i \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

Assume that this theorem is false and let G be a counterexample of minimal order. Then $D \neq 1$ and for some σ-subnormal subgroup H of G and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$.

(1) *The hypothesis holds for every quotient G/N of G.*

It is clear that G/N is a σ-full group of Sylow type and DN/N is a normal σ-Hall subgroup of G/N. On the other hand,

$$(G/N)/(DN/N) \simeq G/DN \simeq (G/D)/(DN/D),$$

so $(G/N)/(DN/N)$ is a $P\sigma T$-group by Lemma 2.5(ii). Finally, let H/N be a σ-subnormal subgroup of DN/N. Then $H = N(H \cap D)$ and, by Lemma 2.6(5) in [1], H is σ-subnormal in G. Hence $H \cap D$ is σ-subnormal in D by Lemma 2.6(1) in [1], so $H \cap D$ is normal in G by hypothesis. Thus $H/N = N(H \cap D)/N$ is normal in G/N. Therefore the hypothesis holds on G/N.

(2) $H_G = 1$.

Assume that $H_G \neq 1$. Clearly, H/H_G is σ-subnormal in G/H_G. Claim (1) implies that the hypothesis holds for G/H_G, so the choice of G implies that G/H_G is a $P\sigma T$-group. Hence

$$(H/H_G)(EH_G/H_G) = (EH_G/H_G)(H/H_G).$$

by Lemma 2.5(i). Therefore $EH = EHH_G$ is a subgroup of G and so $HE = EH$, a contradiction. Hence $H_G = 1$.

(3) $DH = D \times H$.

By Lemma 2.6(1) in [1], $H \cap D$ is σ-subnormal in D. Hence $H \cap D$ is normal in G by hypothesis, which implies that $H \cap D = 1$ by Claim (2). Lemma 2.6(1) in [1] implies also that H is σ-subnormal in DH. But H is a σ-Hall subgroup of DH since D is a σ-Hall subgroup of G and $H \cap D = 1$. Therefore H is normal in DH by Lemma 2.6(10) in [1], so $DH = D \times H$.

Final contradiction. Since D is a σ-Hall subgroup of G, then either $E \leq D$ or $E \cap D = 1$. But the former case is impossible by Claim (3) since $HE \neq EH$, so $E \cap D = 1$. Therefore E is a Π'-subgroup of G, where $\Pi = \sigma(D)$. By the Schur-Zassenhaus theorem, D has a complement M in G. Then M is a Hall Π'-subgroup of G and so for some $x \in G$ we have $E \leq M^x$ since G is a σ-full group of Sylow type. On the other hand, $H \cap M^x$ is a Hall Π'-subgroup of H by Lemma 2.6(7) in [1] and hence $H \cap M^x = H \leq M^x$ since $H \cap D = 1$ by Claim (3). Lemma 2.6(1) in [1] implies that H is σ-subnormal in M^x. But $M^x \simeq G/D$ is a $P\sigma T$-group by hypothesis, so $HE = EH$ by Lemma 2.5(i). This contradiction completes the proof of the theorem.

Lemma 3.1. *If G is a special $P\sigma T$-group, then it is a $P\sigma T$-group.*

Proof. Let $D = G^{\mathfrak{N}_\sigma}$ and E be a normal Hall σ_i-subgroup of G such that $E = D \times S$. Since G/D is σ-nilpotent, G is σ-soluble. Hence G is a σ-full group of Sylow type by Lemma 2.1. Therefore G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup H of G is σ-quasinormal in G if and only if $HH_j^x = H_j^x H$ for all $H_j \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that H_j is a σ_j-group for all $j = 1, \ldots, t$.

Assume that this lemma is false and let G be a counterexample of minimal order. Then G is not σ-nilpotent, and for some σ-subnormal subgroup H of G and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$. The subgroup S is normal in G since it is characteristic in E. Since G is not σ-nilpotent, $D \neq 1$. On the other hand, Theorem A and the choice of G imply that $S \neq 1$ since every subgroup of D is normal in G by hypothesis. Let R and N be minimal normal subgroups of G such that $R \leq D$ and $N \leq S$. Then R is a group of order p for some prime p. Hence $R \cap HN \leq O_p(HN) \leq P$, where P is a Sylow p-subgroup of H since $\pi(D) \cap \pi(S) = \emptyset$, so $R \cap HN = R \cap H$.

The hypothesis holds for G/R and G/N by Lemma 2.5(iii). Hence the choice of G and Lemma 2.5(i) imply that

$$EHR/R = (ER/R)(HR/R) = (HR/R)(EHR/R)$$

and so EHR is a subgroup of G. Similarly we get that EHN is a subgroup of G. Since $|R| = p$ and EH is not a subgroup of G, $R \cap E = 1$. Therefore from Lemma 2.4 we get that that $R \cap EHN = R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap HN$. Hence

$$EHR \cap EHN = E(HR \cap EHN) = EH(R \cap EHN) = EH(R \cap HN) =$$

$$= EH(R \cap HN) = EH(R \cap H) = EH$$

is a subgroup of G. Hence $HE = EH$, a contradiction. The lemma is proved.

Lemma 3.2. *If $\mathcal{H} = \{H_1, \ldots, H_t\}$ is a generalized Wielandt σ-set of G, then*

$$\mathcal{H}_0 = \{H_1N/N, \ldots, H_tN/N\}$$

is a generalized Wielandt σ-set of G/N.

Proof. It is clear that $\mathcal{H}_0$ is a complete Hall σ-set of G/N. Now let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(G^{\mathfrak{N}_\sigma})$. Then $(G/N)^{\mathfrak{N}_\sigma} = DN/N$ by Lemma 2.3, so

$$\pi_0 = \pi((G/N)^{\mathfrak{N}_\sigma}) = \pi(DN/N) \subseteq \pi(D) = \pi.$$

Hence every member H_i of $\mathcal{H}$ is π_0-supersoluble, so H_iN/N is π_0-supersoluble. Hence $\mathcal{H}_0$ is a generalized Wielandt σ-set of G/N. The lemma is proved.

Proof of Theorem B. Clearly, G is σ-soluble, so G is a σ-full group of Sylow type by Lemma 2.1. Therefore G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup H of G is σ-quasinormal in G if and only if $HH_i^x = H_i^x H$ for all $H_i \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

Assume that this theorem is false and let G be a counterexample of minimal order. Then $D \neq 1$ and for some σ-subnormal subgroup H of G and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$.

(1) *G is not a special $P\sigma T$-group* (This follows from Lemma 3.1 and the choice of G).

(2) $|\sigma(D)| > 1$.

Indeed, suppose that $\sigma(D) = \{\sigma_i\}$. Then $O^{\sigma_i}(D) = 1$, so $G \simeq G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group by hypothesis, contrary to Claim (1).

(3) *The hypothesis holds for every quotient G/N of G, where $N \leq D$.*

First we show that $(G/N)/O^{\sigma_i}(DN/N)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(DN/N)$. Note that $\sigma_i \in \sigma(DN/N) = \sigma(D/(D \cap N)) \subseteq \sigma(D)$, so $G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group by hypothesis. It is not difficult to show that

$$O^{\sigma_i}(D)N/N = O^{\sigma_i}(D/N).$$

Hence

$$(G/N)/(O^{\sigma_i}(D/N)) = (G/N)/(O^{\sigma_i}(D)N/N) \simeq G/NO^{\sigma_i}(D) \simeq$$

$$\simeq (G/O^{\sigma_i}(D))/(O^{\sigma_i}(D)N/O^{\sigma_i}(D))$$

is a special $P\sigma T$-group by Lemma 2.5(iii).

It is clear also that $DN/N \simeq D/D \cap N$ is a σ-nilpotent normal Hall subgroup of G/N with σ-nilpotent quotient

$$(G/N)/(DN/N) \simeq G/DN \simeq (G/D)/(DN/D)$$

by Lemma 2.2. Hence we have (3).

(4) *If N is a minimal normal subgroup of G contained in D, then EHN is a subgroup of G.*

Claim (3) and the choice of G implies that the conclusion of the theorem holds for G/N. On the other hand, EN/E is a Hall σ_k-subgroup of G/N and, by Lemma 2.6(4) in [1], HN/N is a σ-subnormal subgroup of G. Note also that G/N is σ-soluble, so every two Hall σ_k-subgroups of G/N are conjugate by Lemma 2.1. Thus,

$$(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N$$

by Lemma 2.5(i). Hence EHN is a subgroup of G.

Final contradiction. Since $|\sigma(D)| > 1$ by Claim (2) and D is σ-nilpotent, G has at least two σ-primary minimal normal subgroups R and N such that $R, N \leq D$ and $\sigma(R) \neq \sigma(N)$. Then at least one of the subgroups R or N, R say, is a σ_i-group for some $i \neq k$. Then $R \cap HN \leq O_{\sigma_i}(HN) \leq V$, where V is a Hall σ_i-subgroup of H, since N is a σ_i'-group and G is a σ-full group of Sylow type. Hence $R \cap HN = R \cap H$. Claim (4) implies that EHR and EHN are subgroups of G. Now, arguing similarly as in the proof of Lemma 3.1, one can show that $EHR \cap EHN = EH$ is a subgroup of G, so $HE = EH$. This contradiction completes the proof of the result.

Proof of Theorem C. Let $\pi = \pi(D)$ and $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a generalized Wielandt σ-set of G. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Since G is σ-soluble by hypothesis, G is a σ-full group of Sylow type by Lemma 2.1.

Necessity. Assume that this is false and let G be a counterexample of minimal order. Then $D \neq 1$.

(1) *If R is a non-identity normal subgroup of G, then the hypothesis holds for G/R. Hence the necessity condition of the theorem holds for G/R* (Since the hypothesis holds for G/R by Lemmas 2.5(ii) and 3.2, this follows from the choice of G).

(2) *If E is a proper σ-subnormal subgroup of G, then $E^{\mathfrak{N}_\sigma} \leq D$ and the necessity condition of the theorem holds for E.*

Every σ-subnormal subgroup H of E is σ-subnormal in G by Lemma 2.6(2) in [1] and hence H is σ-quasinormal in G by hypothesis and Lemma 2.5(i). Thus H is σ-quasinormal in E by Lemma 2.8(1) in [1] since G is a σ-full group of Sylow type. Thus, E is a σ-soluble $P\sigma T$-group. It is clear that E possesses a complete Hall σ-set $H_0 = \{E_1, \ldots, E_n\}$ such that $E_i \leq H_i^{x_i}$ for some $x_i \in G$ for all $i = 1, \ldots, n$. Hence every member of H_0 is π-supersoluble. Moreover, since

$$E/E \cap D \simeq ED/D \in \mathfrak{N}_\sigma$$

and $\mathfrak{N}_\sigma$ is a hereditary class by Lemma 2.2, we have $E/E \cap D \in \mathfrak{N}_\sigma$. Hence $E^{\mathfrak{N}_\sigma} \leq E \cap D$. Therefore, $\pi_0 = \pi(E^{\mathfrak{N}_\sigma}) \subseteq \pi$. Hence every member of H_0 is π_0-supersoluble. Hence H_0 is a generalized Wielandt σ-set of E.

Therefore the hypothesis holds for E, so the necessity condition of the theorem holds for E by the choice of G.

(3) *D is nilpotent.*

Assume that this is false and let R be a minimal normal subgroup of G. Then $RD/R = (G/R)^{\mathfrak{N}_\sigma}$ is abelian by Lemma 2.3 and Claim (1). Therefore $R \leq D$, R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Lemma 2.2. Let V be a maximal subgroup of R. Since G is σ-soluble by hypothesis, R is a σ_i-group for some i. Hence V is σ-subnormal in G by Lemma 2.6(6) in [1], so V is σ-quasinormal in G by hypothesis and Lemma 2.5(i). Then $R \leq D \leq O^{\sigma_i}(G) \leq N_G(V)$ by Lemma 3.1 in [1]. Hence R is abelian, so $R = C_G(R)$ is a p-group for some prime p by [11, A, 15.2].

It is clear that $R \leq H_i \cap D$ for some i. Then H_i is p-supersoluble by hypothesis, so some subgroup L of R of order p is normal in H_i. On the other hand, L is clearly σ-quasinormal in G and hence $G = H_iO^{\sigma_i}(G) \leq N_G(L)$ by Lemma 3.1 in [1], so $R = L$. Therefore $G/C_G(R) = G/R$ is a cyclic group. Hence G is supersoluble and therefore D is nilpotent.

(4) *D is a Hall subgroup of G.*

Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$, where $G_p \in \mathrm{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$.

(a) *$D = P$ is a minimal normal subgroup of G.*

Let R be a minimal normal subgroup of G contained in D. Since D is nilpotent by Claim (3), R is a q-group for some prime q. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/R by Claim (1) and Lemma 2.3. Suppose that $PR/R \neq 1$. Then $PR/R \in \mathrm{Syl}_p(G/R)$. If $q \neq p$, then $P \in \mathrm{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence $q = p$, so $R \leq P$ and therefore $P/R \in \mathrm{Syl}_p(G/R)$ and we again get that $P \in \mathrm{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G contained in D. Since D is nilpotent, a p'-complement E of D is characteristic in D and so it is normal in G. Hence $E = 1$, which implies that $R = D = P$.

(b) $D \nleq \Phi(G)$. *Hence for some maximal subgroup M of G we have $G = D \rtimes M$* (This follows from Lemma 2.2 since G is not σ-nilpotent).

(c) *If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times (L \cap G_p)$. Hence $O_{p'}(G) = 1$.*

Indeed, $DL/L \simeq D$ is a Hall subgroup of G/L by Claim (1). Hence $G_pL/L = RL/L$, so $G_p = D \times (L \cap G_p)$. Thus $O_{p'}(G) = 1$ since $D < G_p$ by Claim (a).

(d) *$V = C_G(D) \cap M$ is a normal subgroup of G and $C_G(D) = D \times V \leq H_1$.*

In view of Claim (b), $C_G(D) = D \times V$, where $V = C_G(D) \cap M$ is a normal subgroup of G. By Claim (a), $V \cap D = 1$ and hence $V \simeq DV/D$ is σ-nilpotent by Lemma 2.2. Let W be a σ_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore we have (d) by Claim (c).

(e) $G_p \neq H_1$.

Assume that $G_p = H_1$. Let Z be a subgroup of order p in $Z(G_p) \cap D$. Then, since $D \leq O^{\sigma_1}(G) = O^p(G)$, Z is normal in G by Lemma 3.1 in [1]. Hence $D = Z < G_p$ and so $D < C_G(D)$. Then $V = C_G(D) \cap M \neq 1$ is a normal subgroup of G and $V \leq H_1 = G_p$ by Claim (d). Let L be a minimal normal subgroup of G contained in V. Then $G_p = D \times L$ is a normal elementary abelian subgroup of G. Therefore every subgroup of G_p is normal in G by Lemma 3.1 in [1]. Hence $|D| = |L| = p$. Let $D = \langle a \rangle$, $L = \langle b \rangle$ and $N = \langle ab \rangle$. Then $N \nleq D$, so in view of the G-isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D$$

we get that $G/C_G(D) = G/C_G(N)$ is a p-group since G/D is σ-nilpotent by Lemma 2.2. But then Claim (d) implies that G is a p-group. This contradiction shows that we have (e).

Final contradiction for (4). In view of Theorem A in [4], G has a σ_1-complement E such that $EG_p = G_pE$. Let $V = (EG_p)^{\mathfrak{N}_\sigma}$. By Claim (e), $EG_p \neq G$. On the other hand, since $D \leq EG_p$ by Claim (a), EG_p is σ-subnormal in G by Lemma 2.6(5) in [1]. Therefore the necessity condition of the theorem holds for EG_p by Claim (2). Hence V is a Hall subgroup of EG_p. Moreover, by Claim (2) we have $V \leq D$, so for a Sylow p-subgroup V_p of V we have $|V_p| \leq |P| < |G_p|$. Hence V is a p'-group and so $V \leq C_G(D) \leq H_1 = G_p$. Thus $V = 1$. Therefore $EG_p = E \times G_p$ is σ-nilpotent and so $E \leq C_G(D) \leq H_1$ by Claim (d). Hence $E = 1$ and so $D = 1$, a contradiction. Thus, D is a Hall subgroup of G.

(5) *$G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(D)$.*

First assume that $O^{\sigma_i}(D) \neq 1$ and let N be a minimal normal subgroup of G contained in $O^{\sigma_i}(D)$. Then G/N is a $P\sigma T$-group by Lemma 2.5(ii), so the choice of G implies that

$$(G/N)/O^{\sigma_i}(D/N) = (G/N)/(O^{\sigma_i}(D)/N) \simeq G/O^{\sigma_i}(D)$$

is a special $P\sigma T$-group. Now assume that $O^{\sigma_i}(D) = 1$, that is, D is a σ_i-group. Since G/D is σ-nilpotent by Lemma 2.2, H_i/D is normal in G/D and hence H_i is normal in G. Therefore all subgroups of D are σ-permutable in G by Lemma 2.3(2)(3) and hypothesis. Since D is a normal Hall subgroup of H_i, it has a complement S in H_i by the Schur-Zassenhaus theorem. Lemma 3.1 in [1] implies that $D \leq O^{\sigma_i}(G) \leq N_G(S)$. Hence $H_i = D \times S$. Therefore

$$G = H_i O^{\sigma_i}(G) = SO^{\sigma_i}(G) \leq N_G(L)$$

for every subgroup L of D. Hence every element of G induces a power automorphism in D. Hence G is a special $P\sigma T$-group.

(6) *Every subgroup H of D is normal in G. Hence every element of G induces a power automorphism in D.*

Since D is nilpotent by Claim (3), it is enough to consider the case when H is a subgroup of the Sylow p-subgroup P of D for some prime p. For some i we have $P \leq O_{\sigma_i}(D) = H_i \cap D$. On the other hand, we have

$$D = O_{\sigma_i}(D) \times O^{\sigma_i}(D)$$

since D is nilpotent. Then

$$HO^{\sigma_i}(D)/O^{\sigma_i}(D) \leq D/O^{\sigma_i}(D) = (G/O^{\sigma_i}(D))^{\mathfrak{N}_\sigma},$$

so $HO^{\sigma_i}(D)/O^{\sigma_i}(D)$ is normal in $G/O^{\sigma_i}(D)$ by Claim (5). Hence $HO^{\sigma_i}(D)$ is normal in G, which implies that

$$H = H(O^{\sigma_i}(D) \cap O_{\sigma_i}(D)) = HO^{\sigma_i}(D) \cap O_{\sigma_i}(D)$$

is normal in G.

(7) *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, $|D|$ is odd.*

Assume that this is false. Then, by Claim (6), D has a maximal subgroup E such that $|D : E| = p$ and E is normal in G. It follows that $C_G(D/E) = G$ since $(p-1, |G|) = 1$. Since D is a Hall subgroup of G, it has a complement M in G. Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ-nilpotent. Therefore G/E is σ-nilpotent by Lemma 2.2. But then $D \leq E$, a contradiction. Hence p does not divide $|D|$. In particular, $|D|$ is odd.

(8) *D is abelian.*

In view of Claim (6), D is a Dedekind group. Hence D is abelian since $|D|$ is odd by Claim (7).

From Claims (4)–(8) we get that the necessity condition of the theorem holds for G.

Sufficiency. This directly follows from Theorem B.

The theorem is proved.

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