On Π-permutable subgroups of finite groups*

Wenbin Guo
Department of Mathematics, University of Science and Technology of China,
Hefei 230026, P. R. China
E-mail: wbguo@ustc.edu.cn

Alexander N. Skiba
Department of Mathematics, Francisk Skorina Gomel State University,
Gomel 246019, Belarus
E-mail: alexander.skiba49@gmail.com

Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and Π a non-empty subset of the set σ. A set $\mathcal{H}$ of subgroups of a finite group G is said to be a *complete Hall* Π-*set* of G if every member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. A subgroup H of G is called Π-*quasinormal* or Π-*permutable* in G if G possesses a complete Hall Π-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $AH_i^x = H_i^x A$ for any i and all $x \in G$. We study the embedding properties of H under the hypothesis that H is Π-permutable in G. Some known results are generalized.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is always supposed to be a non-empty subset of the set σ and $\Pi' = \sigma \setminus \Pi$.

In practice, we often deal with two limited cases: $\sigma = \{\{2\}, \{3\}, \{5\}, \ldots\}$ and $\sigma = \{\pi, \pi'\}$.

Recall that $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$ [1]. G is called: a Π-*group* if $\sigma(G) \subseteq \Pi$; σ-*primary* [2] if G is a Π-group for some one-element set Π.

*Research is supported by a NNSF grant of China (Grant # 11371335) and Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences.

[0]Keywords: finite group, complete Hall Π-set, σ-subnormal subgroup, Π-permutable subgroup, σ-nilpotent group.

[0]Mathematics Subject Classification (2010): 20D10, 20D15, 20D20, 20D30, 20D35

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* Π-*set* of G if every member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \Pi \cap \sigma(G)$. We say also that G is: Π-*full* if G possesses a *complete Hall* Π-*set*; a Π-*full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \Pi$.

Let $\mathcal{L}$ be some non-empty set of subgroups of G and E a subgroup of G. Then a subgroup A of G is called $\mathcal{L}$-*permutable* if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$-*permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$.

If $\mathcal{S}$ is a complete Sylow π-set of G (that is, every member of $\mathcal{S}$ is a Sylow p-subgroup for some $p \in \pi$ and $\mathcal{S}$ contains exact one Sylow p-subgroup for every $p \in \pi$), then an $\mathcal{L}^G$-permutable subgroup is called π-*permutable* or π-*quasinormal* (Kegel [3]) in G. The $\pi(G)$-permutable subgroups are also called S-*permutable* or S-*quasinormal.*

In this note we study the following generalization of π-permutability.

Definition 1.1. We say that a subgroup H of G is Π-*quasinormal* or Π-*permutable* in G if G possesses a complete Hall Π-set $\mathcal{H}$ such that H is $\mathcal{H}^G$-permutable.

Before continuing, consider some examples.

Example 1.2. (1) G is called σ-*soluble* [2] if every chief factor of G is σ-primary. In view of Theorem A in [1], every σ-soluble group is a Π-full group of Sylow type for each $\Pi \subseteq \sigma$. .

(2) G is called σ-*nilpotent* [4] if G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$. Therefore every subgroup of every σ-nilpotent group G is Π-permutable in G for each $\Pi \subseteq \sigma$.

(3) Now let $p > q > r$ be primes, where q divides $p-1$ and r divides $q-1$. Let $H = Q \rtimes R$ be a non-abelian group of order qr, P a simple $\mathbb{F}_pH$-module which is faithful for H, and $G = P \rtimes H$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{p, r\}$ and $\sigma_2 = \{p, r\}'$. Then G is not σ-nilpotent and $|P| > p$. Since q divides $p-1$, PQ is supersoluble. Hence for some normal subgroup L of PQ we have $1 < L < P$. Then for every Hall σ_1-subgroup V of G we have $L \leq P \leq V$, so $LV = V = VL$. On the other hand, for every Hall σ_2-subgroup Q^x of G we have $Q^x \leq PQ$, so $LQ^x = Q^xL$. Hence L is σ-permutable in G. It is also clear that L is not normal in G, so $LR \neq RL$, which implies that L is not S-permutable in G.

We will also need the following modification of the main concept in [5]: A subgroup A of G is called: σ-*subnormal* in G [2] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ- primary for all $i = 1, \dots t$.

In this definition $(A_{i-1})_{A_i}$ denotes the product of all normal subgroups of A_i contained in A_{i-1}.

We use G^{N_σ} to denote the σ-*nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Our main goal here is to prove the following

Theorem 1.3. *Let H be a Π-subgroup of G and $D = G^{\mathfrak{N}_\sigma}$.*

(i) *If G is Π-full and possesses a complete Hall Π-set $\mathcal{H}$ such that H is $\mathcal{H}^D$-permutable, then H is σ-subnormal in G and the normal closure H^G of H in G is a Π-group.*

(ii) *If H is Π-permutable in G and, in the case when $\Pi \neq \sigma(G)$, G possesses a complete Hall Π'-set $\mathcal{K}$ such that H is $\mathcal{K}$-permutable, then H^G/H_G is σ-nilpotent and the normalizer $N_G(H)$ of H is also Π-permutable. Moreover, $N_G(H)$ is $\mathcal{H}^G$-permutable for each complete Hall Π-set $\mathcal{H}$ of G such that H is $\mathcal{H}^G$-permutable.*

(iii) *If G is a Π'-full group of Sylow type and H is Π'-permutable in G, then H^G possesses a σ-nilpotent Hall Π'-subgroup.*

Consider some corollaries of Theorem 1.3.

Theorem 1.3(i) immediately implies

Corollary 1.4 (Kegel [5]). *If a π-subgroup H of G is S-permutable in G, then H is subnormal in G.*

Now, consider some special cases of Theorem 1.3(ii). First note that in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem 1.3(ii) the following results.

Corollary 1.5. *Let H be a π-subgroup of G. If H is π-permutable in G and, also, H permutes with some Sylow p-subgroup of G for each prime $p \in \pi'$, then the normalizer $N_G(H)$ of H is π-permutable in G.*

In particular, in the case when $\pi = \mathbb{P}$, we have

Corollary 1.6 (Schmid [6]). *If a subgroup H of G is S-permutable in G, then the normalizer $N_G(H)$ of H is also S-permutable.*

Corollary 1.7. *Let H be a π-subgroup of G. If H is π-permutable in G and, also, H permutes with some Sylow p-subgroup of G for each prime $p \in \pi'$, then H/H_G is nilpotent.*

Corollary 1.8 (Deskins [7]). *If a subgroup H of G is S-permutable in G, then H/H_G is nilpotent.*

Recall that G is said to be a π-*decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$, that is, G is the direct product of its Hall π-subgroup and Hall π'-subgroup.

In the case when $\sigma = \{\pi, \pi'\}$ we get from Theorem 1.3(ii) the following

Corollary 1.9. *Suppose that G is π-separable. If a subgroup H of G permutes with all Hall π-subgroups of G and with Hall π'-subgroups of G, then H^G/H_G is π-decomposable.*

In particular, we have

Corollary 1.10. *Suppose that G is p-soluble. If a subgroup H of G permutes with all Sylow p-subgroups of G and with all p-complements of G, then H^G/H_G is p-decomposable.*

Finally, in the case when $\Pi = \sigma$, we get from Theorem 1.3(ii) the following

Corollary 1.11 (Skiba [2]). *Suppose that G is a σ-full group and let H be a subgroup of G. If H is σ-permutable in G, then H^G/H_G is σ-nilpotent.*

From Theorem 1.3(iii) we get

Corollary 1.12. *Let H be a π-subgroup of G. If H permutes with every Sylow p-subgroup of G for $p \in \pi'$, then H^G possesses a nilpotent π-complement.*

A subgroup H of G is called a *S-semipermutable* in G if H permutes with all Sylow subgroups P of G such that $(|H|, |P|) = 1$. If H is S-semipermutable in G and $\pi = \pi(H)$, then H is π'-permutable in G. Hence from Corollary 1.12 we get the following known result.

Corollary 1.13 (Isaacs [8]). *If a π-subgroup H of G is S-semipermutable in G, then H^G possesses a nilpotent π-complement.*

Note that in the group $G = C_7 \rtimes \mathrm{Aut}(C_7)$ a subgroup of order 3 is π'-permutable in G, where $\pi = \{2, 3\}$, but it is not S-semipermutable.

2 Preliminaries

We use: $O^\Pi(G)$ to denote the subgroup of G generated by all its Π'-subgroups; $O_\Pi(G)$ to denote the subgroup of G generated by all its normal Π-subgroups. A subgroup H of G is said to be: a *Hall Π-subgroup* of G [1] if $|H|$ is a *Π-number* (that is, $\pi(H) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$) and $|G : H|$ is a Π'-number.

Lemma 2.1. *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If K is a σ-subnormal subgroup of A, then K is σ-subnormal in G.*

(3) *If K is σ-subnormal in G, then $A \cap K$ and $\langle A, K \rangle$ are σ-subnormal in G.*

(4) *AN/N is σ-subnormal in G/N.*

(5) *If $N \leq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

(6) *If $K \leq A$ and A is σ-nilpotent, then K is σ-subnormal in G.*

(7) *If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.*

(8) *If $|G : A|$ is a Π-number, then $O^\Pi(A) = O^\Pi(G)$.*

(9) *If G is Π-full and A is a Π-group, then $A \leq O_\Pi(G)$.*

Proof. Statements (1)–(8) are known [2, Lemma 2.6]).

(9) Assume that this assertion is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_r = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss

of generality that $M \neq G$. Let $D = A \cap M_G$.

First note that A is not σ-primary. Indeed, assume that A is a σ_i-group. By hypothesis, G has a Hall σ_i-subgroup, say H. Then, by Assertion (7), for any $x \in G$ we have $A \leq H^x$. Hence $A^G \leq H_G \leq O_\Pi(G)$, a contradiction. Hence $|\sigma(A)| > 1$.

Suppose that $D \neq 1$. The subgroup D is σ-subnormal in M_G by Lemma 2.1(1)(3), so the choice of G implies that $D \leq O_\Pi(M_G)$. Hence $O_\Pi(M_G) \neq 1$. But since $O_\Pi(M_G)$ is characteristic in M_G, we have that $O_\Pi(M_G) \leq O_\Pi(G)$. The hypothesis holds for $(G/O_\Pi(G), AO_\Pi(G)/O_\Pi(G))$ by Assertion (4). Therefore $AO_\Pi(G)/O_\Pi(G)) \leq O_\Pi(G/O_\Pi(G)) = 1$. It follows that $A \leq O_\Pi(G)$, a contradiction. Hence $A \cap M_G = 1$, so M is not normal in G. Therefore, G/M_G is a σ_j-group for some $j \in I$. But then $A \simeq AM_G/M_G$ is σ-primary. This contradiction completes the proof.

The first three statements in the next lemma can be proved by the direct calculations and the last statement see [9, A, 1.6(a)].

Lemma 2.2. *Let H, K and N be subgroups of G. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall Π-set of of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G.*

(1) *If $H \leq E \leq G$, then H is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1 \cap E, \ldots, H_t \cap E\}^{K \cap E}$. In particular, if H is Π-permutable in G and either G is a Π-full group of Sylow type or E is normal in G, then H is Π-permutable in E.*

(2) *The subgroup HN/N is $\mathcal{L}^{**}$-permutable, where $\mathcal{L}^{**} = \{H_1N/N, \ldots, H_tN/N\}^{KN/N}$.*

(3) *If G is a Π-full group of Sylow type and E/N is a Π-permutable subgroup of G/N, then E is Π-permutable in G.*

(4) *If K is $\mathcal{L}$-permutable, then $\langle H, K\rangle$ is $\mathcal{L}$-permutable.*

Lemma 2.3 (See Lemma 2.2 in [1]). *Let H be a normal subgroup of G. If $H/H \cap \Phi(G)$ is a Π-group, then H has a Hall Π-subgroup, say E, and E is normal in G.*

We say that a group G is Π-closed if $O_\Pi(G)$ is a Hall Π-subgroup of G. Two integers n and m are called *σ-coprime* if $\sigma(n) \cap \sigma(m) = \emptyset$.

Lemma 2.4. *If a σ-soluble group G has three Π-closed subgroups A, B and C whose indices $|G : A|$, $|G : B|$, $|G : C|$ are pairwise σ-coprime, then G is Π-closed.*

Proof. Suppose that this lemma is false and let G be a counterexample with $|G|$ minimal. Let N be a minimal normal subgroup of G. Then the hypothesis holds for G/N, so G/N is Π-closed by the choice of G. Therefore N is not a Π-group. Moreover, N is the unique minimal normal subgroup of G and, by Lemma 2.3, $N \nleq \Phi(G)$. Hence $C_G(N) \leq N$. Since G is σ-soluble by hypothesis, N is σ-primary, say N is a σ_i-group. Then $\sigma_i \in \Pi'$.

Since $|G : A|$, $|G : B|$, $|G : C|$ are pairwise σ-coprime, there are at least two subgroups, say A and B, such that $N \leq A \cap B$. Then $O_\Pi(A) \leq C_G(N) \leq N$, so $O_\Pi(A) = 1$. But by hypothesis, A is Π-closed, hence A is a Π'-group. Similarly we get that B is a Π'-group and so $G = AB$ is a Π'-group.

But then G is Π-closed. This contradiction completes the proof of the lemma.

Recall that G is called a *Schmidt group* if G is not nilpotent but every proper subgroup of G is nilpotent.

Proposition 2.5. *Let G be a σ-soluble group. Suppose that G is not σ_i'-closed but all proper subgroups of G are σ_i'-closed. Then G is a σ_i-closed Schmidt group.*

Proof. Suppose that this proposition is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G and $\{H_1, \ldots, H_t\}$ a complete Hall σ-set of G. Without loss of generality we can assume that H_1 is a σ_i-group.

(1) $|\sigma(G)| = 2$. *Hence* $G = H_1H_2$.

It is clear that $|\sigma(G)| > 1$. Suppose that $|\sigma(G)| > 2$. Then, since G is σ-soluble, there are maximal subgroups M_1, M_2 and M_3 whose indices $|G : M_1|$, $|G : M_2|$ and $|G : M_3|$ are σ-coprime. Hence $G = M_1M_2 = M_2M_3 = M_1M_3$. But the subgroups M_1, M_2 and M_3 are σ_i'-closed by hypothesis. Hence G is σ_i'-closed by Lemma 2.4, a contradiction. Thus $|\sigma(G)| = 2$.

(2) *If either $R \leq \Phi(G)$ or $R \leq H_2$, then G/R is a σ_i-closed Schmidt group.*

Lemma 2.3 and the choice of G imply that G/R is not σ_i'-closed. On the other hand, every maximal subgroup M/R of G/R is σ_i'-closed since M is σ_i'-closed. Hence the hypothesis holds for G/R. The choice of G implies that G/R is a σ_i-closed Schmidt group.

(3) $\Phi(G) = 1$, *R is the unique minimal normal subgroup of G and $R \leq H_1$.*

Suppose that $R \leq \Phi(G)$. Then R is a r-group for some prime r and, in view of Claim (1), Lemma 2.3 and [10, IV, 5.4], $G = H_1 \rtimes H_2 = P \rtimes Q$, where $H_1 = P$ is a p-group and $H_2 = Q$ is a q-group for some different primes p and q. Assume that $R \leq Q$ and take a subgroup L of order q in $R \cap Z(Q)$. Then it is clear that $R < Q$, so $PR < G$ and hence $PR = P \times Q$ is p-nilpotent. Therefore $L \leq Z(G)$, so $R = L \leq Z(G)$. But for every maximal subgroup M of G we have $R \leq M$ and M/R is nilpotent. Hence every maximal subgroup of G is nilpotent and so G is a σ_i-closed Schmidt group, a contradiction. Similarly, we get that G is a σ_i-closed Schmidt group in the case when $R \leq P$. Therefore $R \nleq \Phi(G)$.

Now assume that G has a minimal normal subgroup $L \neq R$. Then by (3), there are maximal subgroups M and T of G such that $LM = G$ and $RT = G$. By hypothesis, M and T are σ_i'-closed. Hence $G/L \simeq LM/L \simeq M/M \cap L$ is σ_i'-closed. Similarly, G/R is σ_i'-closed and so $G \simeq G/L \cap R$ is σ_i-nilpotent, a contradiction. Hence R is the unique minimal normal subgroup of G, and so $R \leq H_1$.

Final contradiction. In view of Claim (3), $C_G(R) \leq R$. Hence $|H_2|$ is a prime and $RH_2 = G$ since $R \leq H_1$ and every proper subgroup of G is σ_i'-closed. Therefore $R = H_1$, so R is not abelian since G is a not a σ_i-closed Schmidt group. By Claim (1) and Theorem 3.5 in [11], for any prime p dividing $|R|$ there is a Sylow p-subgroup P of G such that $PH_2 = H_2P$. But $H_2P < G$, so $H_2P = H_2 \rtimes P$. This implies that $R \leq N_G(H_2)$ and thereby $G = R \times H_2 = H_1 \times H_2$. This final contradiction completes the proof of the result.

Corollary 2.6. *Let G be a minimal non-σ-nilpotent group, that is, G is not σ-nilpotent, but every proper subgroup of G is σ-nilpotent. If G is a σ-soluble, then G is a Schmidt group.*

Proof. It is clear that G is σ-nilpotent if and only if G is σ_i'-closed for all $\sigma_i \in \sigma$. Hence, for some i, G is not σ_i'-closed. On the other hand, every proper subgroup of G is σ_i'-closed. Hence G is a Schmidt group by Proposition 2.5.

Proposition 2.7. *Let G be a Π-full group of Sylow type. If G possesses a σ-nilpotent Hall Π-subgroup H, then every σ-soluble Π-subgroup of G is contained in a conjugate of H. In particular, any two σ-soluble Hall Π-subgroups of G are conjugate.*

Proof. Suppose that this proposition is false and let G be a counterexample of minimal order. Then some σ-soluble Π-subgroup K of G is not contained in H^x for all $x \in G$. We can assume without loss of generality that every proper subgroup V of K is contained in a conjugate of H, so V is σ-nilpotent. Hence either K is σ-nilpotent or K is a minimal non-σ-nilpotent group. Then in view of Corollary 2.6 and [10, IV, 5.4], K has a normal Hall σ_i-subgroup L for some $\sigma_i \in \sigma(K)$. Now arguing as in the proof of Wielandt's theorem [12, (10.1.9)], one can show that for some $y \in G$ we have $K \leq H^y$. This contradiction completes the proof of the result.

Corollary 2.8. *Let G be a Π-full group of Sylow type. Suppose that every chief factor of G possesses a σ-nilpotent Hall Π-subgroup. Then G possesses a σ-soluble Hall Π-subgroup.*

Proof. Let R be a minimal normal subgroup of G, H a σ-nilpotent Hall Π-subgroup of R and $N = N_G(H)$. By induction, G/R has a σ-soluble Hall Π-subgroup, say U/R. Therefore if R is a Π-group, then U is a σ-soluble Hall Π-subgroup of G. On the other hand, if R is a Π'-group, then $U = R \rtimes V$ by the Schur-Zassenhas theorem, where $V \simeq U/R$ is a σ-soluble Hall Π-subgroup of G. Now suppose that $1 < H < R$. Proposition 2.7 and the Ftattini argument imply that $G = RN$, where $|G : N| = |R/R \cap N|$ is a Π'-number and $N < G$. Then $N/N \cap R \simeq G/R$ possesses a σ-soluble Hall Π-subgroup. Hence in view Proposition 2.7, the hypothesis holds for N, so N possesses a σ-soluble Hall Π-subgroup W by induction. It is clear now that W is a Hall Π-subgroup of G. The corollary is proved.

3 Proof of Theorem 1.3

Suppose that this theorem is false and let (G, H) be a counterexample with $|G| + |G : H|$ as small as possible. Then $H \neq H^G$.

(i), (ii) By hypothesis, G possesses a complete Hall Π-set, say $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Let $E = H_1^G \cdots H_t^G$.

Suppose that Assertion (i) is false. Then in view of Lemma 2.1(9), H is not σ-subnormal in G. Moreover, in this case we have $E = G$. Indeed, since the class of all σ-nilpotent groups is closed under taking subgroups, homomorphic images and the direct products, $E/E \cap D \simeq DE/D$ is σ-nilpotent.

Hence $E^{N_\sigma} \leq D$. It follows that the hypothesis holds for (E, H). Thus in the case when $E < G$ the choice of (G, H) implies that H is σ-subnormal in E and so H is σ-subnormal in G, a contradiction. Therefore $E = G$. Since $H \neq H^G$, it follows that for some $x \in G$ and $H_i \in \mathcal{H}$ we have $H_i^x \nleq N_G(H)$. Now, arguing as in Claim (2) of the proof of Theorem B in [2], one can show that H is σ-subnormal in G. This contradiction completes the proof of (i).

(ii) Suppose that this assertion is false. Then:

(1) *The hypothesis holds for* $(G/H_G, H/H_G)$, *so* $H_G = 1$.

First note that the hypothesis holds for $(G/H_G, H/H_G)$ by Lemma 2.2(2). Assume that $H_G \neq 1$. Then the choice of (G, H) implies that H^G/H_G is σ-nilpotent and $N_{G/H_G}(H/H_G) = N_G(H)/H_G$ is $\mathcal{H}^*$-permutable by Lemma 2.2(2), where

$$\mathcal{H}^* = \{H_1H_G/H_G, \dots, H_tH_G/H_G\}^{G/H_G}.$$

But then, clearly, $N_G(H)$ is $\mathcal{H}^G$-permutable. This shows that Assertion (ii) is true. Therefore the choice of (G, H) implies that $H_G = 1$.

(2) $t > 1$.

Assume that $t = 1$, that is, H is a σ_1-group. Then $HH_1^x = H_1^xH = H_1^x$ for all $x \in G$, so $H^G \leq (H_1)_G \leq O_{\sigma_1}(G)$, which implies that H^G is σ-nilpotent. Hence H is σ-subnormal in G by Lemma 2.1(6). Note also that for any Hall σ_1'-subgroup V of G such that $HV = VH$ we have $H = VH \cap O_{\sigma_1}(G)$, so $V \leq N_G(H)$. Therefore if H is Π-permutable in G and also, in the case when $\Pi \neq \sigma(G)$, H is $\mathcal{K}$-permutable, then $|G : N_G(H)|$ is a σ_1-number, which implies that $N_G(H)H_1^x = G = H_1^xN_G(H)$ for all $x \in G$. This means that $N_G(H)$ is Π-permutable in G. Thus Assertion (ii) is true, a contradiction. Therefore $t > 1$.

Let $L_i = O^{\sigma_i'}(H)$, for all $i = 1, \dots, t$. Then $H = L_1 \cdots L_t$ and $N_G(H) = N_G(L_1) \cap \cdots \cap N_G(L_t)$. Let

$$W_i = H_1^G \cdots H_{i-1}^G H_{i+1}^G \cdots H_t^G,$$

for all $i = 1, \dots, t$, and $W = W_1 \cap \cdots \cap W_t$.

(3) $W_i \leq N_G(L_i)$ *for all* $i = 1, \dots, t$, *so* $W \leq N_G(H)$.

Indeed, since H is σ-subnormal in G by Part (i), Lemma 2.1(8) implies that $H_i^x \leq N_G(O^{\sigma_i}(H))$ for all $x \in G$. This means that $H_i^G \leq N_G(O^{\sigma_i}(H))$. Hence $H_i^G \leq N_G(L_j)$ for all $j \neq i$, so $W_i \leq N_G(L_i)$ for all $i = 1, \dots, t$.

(4) H^G is σ-nilpotent.

Suppose that this is false. Let $K = H_1 \cdots H_tW$. Then:

(a) K *is a subgroup of* G, $H \leq K$ *and* $|K : W|$ *is a* Π-*number*.

First note that $(H_iW/W)^{G/W} = H_i^GW/W$ and

$$WW_i \cap H_i^G W = W(W_i \cap H_i^G W) = W(W_i \cap H_i^G (W_1 \cap \cdots \cap W_t)) =$$

$$= W(W_i \cap W_1 \cap \cdots \cap W_{i-1} \cap W_{i+1} \cap \cdots \cap W_t \cap W_i H_i^G) = W(W \cap E) = W.$$

Therefore

$$E/W = (H_1 W/W)^{G/W} \times \cdots \times (H_t W/W)^{G/W}.$$

This means that $[H_i W/W, H_j W/W] = 1$, for all $i \neq j$. Hence $K = H_1 \cdots H_t W = (H_1 W) \cdots (H_t W)$ is the product of pairwise permutable subgroups, which implies that K is a subgroup of G. It is also clear that K/W is a Hall Π-subgroup of G/W. Hence $|K : W|$ is a Π-number and $WH/W \leq K/W$ by Lemma 2.1(4)(7), so we have (a).

(b) *The hypothesis holds for* (K, H).

Let $\mathcal{K} = \{K_1, \ldots, K_n\}$. Since $|K : W|$ is a Π-number, $K_i \cap K$ is a Hall σ_i-subgroup of K and hence $\mathcal{B} = \{K_1 \cap K, \ldots, K_n \cap K\}$ is a complete Hall Π'-set of K. On the other hand, for any $K_i \in \mathcal{K}$ we have $HK_i \cap K = (K_i \cap K)H$ and so H is $\mathcal{B}$-permutable. Finally, it is clear that H is Π-permutable in K. Hence the hypothesis holds for (K, H).

(c) $K < G$.

Suppose that $K = G$. Then, since $|K : W| = |G : W|$ is a Π-number by Claim (4), for every $K_i \in \mathcal{K}$ and every $x \in G$ we have $K_i^x \leq W \leq N_G(H)$ by Claim (3), so $K_i^x H = HK_i^x$. Therefore H is σ-permutable in G and so $H^G \simeq H^G/H_G$ is σ-nilpotent by Theorem B in [2], contrary to our assumption on H. Hence $K < G$.

(d) $|G : N_G(H)|$ *is a* Π-*number* (Since H is a σ-subnormal Π-subgroup of G, this follows from Lemma 2.1(8)).

(e) Conclusion for (4).

Since $K < G$ by Claim (c), we have that H^K/H_K is σ-nilpotent. Because $|G : N_G(H)|$ is a Π-number by Claim (d), $G = KN_G(H)$. Hence $H^G \simeq H/1 = H^G/H_G = H^K/H_K$ is σ-nilpotent. This contradiction shows that H^G is σ-nilpotent.

Final contradiction for (ii).

Since H^G is σ-nilpotent by (4), H is also σ-nilpotent. Hence H possesses a complete Hall σ-set $\{V_1, \ldots, V_t\}$ such that $H = V_1 \times \cdots \times V_t$. Without loss of generality we can assume that V_i is a σ_i-group for all $i = 1, \ldots, t$. Let $N = N_G(H)$ and $N_i = N_G(V_i)$. Then $N = N_1 \cap \cdots \cap N_t$. Moreover, it is clear that $L_i = V_i$ for all $i = 1, \ldots, t$. Hence $W_i \leq N_G(V_i)$ for all $i = 1, \ldots, t$ by Claim (3). It is also clear that $|G : N_i|$ is a σ_i-number, so $G = N_i H_i$. Hence for any $x \in G$ and $H_i \in \mathcal{H}$ we have

$$NH_i^x = (N_1 \cap \cdots \cap N_t)H_i^x = N_i H_i^x \cap N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t =$$

$$= G \cap N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t = N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_t = H_i^x N$$

and so N is $\mathcal{H}^G$-permutable. Therefore Assertion (ii) is true. This contradiction completes the proof of Assertion (ii).

(iii) Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be a complete Hall Π'-set of G such that H is $\mathcal{L}^G$-permutable. Let $E = H^G$ and R a minimal normal subgroup of G. First note that $m > 1$, Indeed, if $m = 1$, then $L_1 \cap E$ is a σ-nilpotent Hall Π'-subgroup of G, which contradicts the choice of (G, H).

(1) *ER/R possesses a σ-nilpotent Hall Π'-subgroup U/R. Therefore $R \leq E$.*

From Lemma 2.2(2) and the choice of G it follows that $(HR/R)^{G/R} = ER/R$ possesses a σ-nilpotent Hall Π'-subgroup, say U/R. Therefore, if $R \nleq E$, then $E \simeq ER/R$ possesses a σ-nilpotent Hall Π'-subgroup, a contradiction. Hence we have (1).

(2) *$O_{\Pi}(G) = 1$.*

Assume that $R \leq O_{\Pi}(G)$. Then, by the Schur-Zassenhaus theorem, R has a complement V in U, so $V \simeq U/R$ is a σ-nilpotent Hall Π'-subgroup of E, a contradiction. Hence we have (2).

(3) *$L_i^G \nleq C_G(E)$ for all $i = 1, \ldots, t$.*

Assume that $L_i^G \leq C_G(E)$ and let N be a minimal normal subgroup of G contained in L_i^G. Then $N \leq E$ and E/N possesses a σ-nilpotent Hall Π'-subgroup, say U/N, by Claim (1). On the other hand, $N \leq Z(U)$, so U is σ-nilpotent. But a Hall Π'-subgroup of U is a Hall Π'-subgroup of E, a contradiction. Hence we have (3).

(4) *R is the unique minimal normal subgroup of G.*

Suppose that G has a minimal normal subgroup $N \neq R$. Then $N \leq E$ and G/N possesses a σ-nilpotent Hall Π'-subgroup by Claim (1). Therefore $(E/R) \times (E/N)$ possesses a σ-nilpotent Hall Π'-subgroup V. But $E \simeq K \leq (E/R) \times (E/N)$ since $R \cap N = 1$. Hence E possesses a σ-nilpotent Hall Π'-subgroup. Moreover, since $N \simeq RN/R$ possesses a σ-nilpotent Hall Π'-subgroup, E possesses a Hall Π'-subgroup U by Corollary 2.8. But then, by Proposition 2.7, for some $x \in G$ we have $U \leq V^x$ and so U is σ-nilpotent, contrary to the choice of G. Hence we have (4).

Final contradiction for (iii).

Let $x, y \in G$ and $A = H^x$. Then

$$AL_i^y = (HL_i^{yx^{-1}})^x = (L_i^{yx^{-1}}H)^x = L_i^y A$$

by hypothesis. Let $L = A^{L_i} \cap L_i^A$. Then L is a subnormal subgroup of G by [13, 7.2.5]. Suppose that $L \neq 1$ and let L_0 be a minimal subnormal subgroup of G contained in L. Then $V = L_0 \cap L_i$ is a Hall Π'-subgroup of L_0 since $L \leq AL_i$. Moreover, in view of Claim (2), $V \neq 1$ (see, for example, [14, Chapter 1, Lemma 5.35(5)]). We now show that $L_i \cap R$ is a non-identity Hall Π'-subgroup of R. Indeed, if L_0 is abelian, then $L_0 \leq O_{\sigma_i}(G)$, where $\sigma_i = \pi(L_i)$, so R is a σ_i-group by Claim (4). On the other hand, if L_0 is non-abelian, L_0^G is a minimal normal subgroup of G and so, by Claim (4), $L_i \cap R$ is a non-identity Hall Π'-subgroup of R.

Since $m > 1$, Claim (2) implies that there is $j \neq i$ such that for every $x, y \in G$ we have

$(L_j^y)^{H^x} \cap (H^x)^{L_j^y} = 1$ and so

$$[L_j^y, H^x] \le [(L_j^y)^{H^x}, (H^x)^{L_j^y}] = 1.$$

Therefore $L_j^G \le C_G(E)$, contrary Claim (3). Hence Statement (iii) holds.

The theorem is proved.

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