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X-semipermutable subgroups of finite groups

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Abstract

Let X be a non-empty subset of a group G. Then we call a subgroup A of G a X-*semipermutable* subgroup of G if A has a supplement T in G such that for every subgroup T_1 of T there exists an element $x \in X$ such that $AT_1^x = T_1^x A$. In this paper, we study the properties of X-semipermutable subgroups. In particular, a new version of the famous Schur–Zassenhaus Theorem in terms of X-semipermutable subgroups is given.

Keywords: Finite group; X-semipermutable group; 2-maximal subgroup; Supersoluble group; Nilpotent group; Hall subgroup

1. Introduction

Throughout this paper, all groups are finite.

A subgroup A of a group G is said to be permutable with a subgroup B if $AB = BA$. A subgroup A is said to be a permutable or a quasinormal subgroup of G if A is permutable with all subgroups of G. But we often meet the situation $AB \neq BA$, nevertheless there exists an element $x \in G$ such that $AB^x = B^x A$, for instance, we have the following cases:

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(1) Let $G = AB$ be a group. If A_p and B_p are Sylow p-subgroups of A and of B respectively, then $A_p B_p \neq B_p A_p$ in general, but G has an element x such that $A_p B_p^x = B_p^x A_p$.
(2) If A and B are Hall subgroups of a soluble group G, then there exists an element $x \in G$ such that $AB^x = B^x A$ (cf. [1, I, (4.11)]).
(3) If A and B are normally embedded subgroups (see Definition (7.1) in [1, I]) of a soluble group, then A is permutable with some conjugate of B (cf. [1, I, (4.17)]).
(4) If $|G : A|$ is a prime power, then in every conjugacy class of Sylow subgroups of G, there is a subgroup P such that $AP = PA$.

Let A, B be subgroups of a group G and X a non-empty subset of G. Then by the above examples and some other examples of such kind, the following definitions are inspired:

Definition 1.1.

(1) A is said to be X-*permutable* with B if there exists some $x \in X$ such that $AB^x = B^x A$;
(2) A is said to be X-*permutable in* G if A is X-permutable with all subgroups of G;
(3) A is said to be X-*semipermutable in* G if A is X-permutable with all subgroups of some supplement T of A in G.

It is clear that our definition of X-*semipermutable subgroups* is a generalization of the usual definition of *permutable subgroups*.

Throughout this paper, we will use $X(A)$ to denote the set of all supplements T of A in the group G such that A is X-permutable with all subgroups of T. Thus A is X-semipermutable in G if and only if $X(A) \neq \emptyset$.

The properties of X-permutable subgroups and some of its applications have already been considered in our previous papers (see [2–6]). In this paper, we consider the applications of X-semipermutable subgroups in the structure of a given group G. First of all, we give the following mew version of Schur–Zassenhaus Theorem in finite groups in terms of X-semipermutable subgroups.

Theorem 1.2. *Let A be a Hall subgroup of a group G and $X = F(G)$ the Fitting subgroup of G. Suppose that A is X-semipermutable in G. Then A is complemented in G. Any two complements of A in G are conjugate under the condition that $X(A)$ contains a soluble group.*

We say that a subgroup M of a group G has non-primary index if $|G : M|$ has at least two different prime divisors. A subgroup H of a group G is said to be a 2-maximal subgroup (see [10, p. 24]) or a second maximal subgroup of G if H is a maximal subgroup of some maximal subgroup M of G. The following theorems give further applications of X-semipermutable subgroups.

Theorem 1.3. *Let G be a group and $X = F(G) \cap G'$. Then the following statements are equivalent.*

(1) *For every 2-maximal subgroup E of G of non-primary index in G with the property that G/E_G is not a supersoluble group satisfying the condition $|F(G/E_G)| = |O_p(G/E_G)| > p$, where p is a prime, the set $X(E)$ contains a supersoluble group.*
(2) *G is supersoluble.*

(3) *For every* 2*-maximal subgroup E of G, G/E_G is supersoluble, and if E satisfies that $|F(G/E_G)|$ is a prime or $|F(G/E_G)|$ has at least two distinct prime divisors, then $T \in X(E)$, for any minimal supplement T of E in G.*

It is not difficult to show that, in any supersoluble group G, any its 2-maximal subgroup E of non-primary index with $|F(G/E_G)| = |O_p(G/E_G)| > p$, where p is a prime, is not $F(G)$-semipermutable in G.

Theorem 1.4. *Let G be a group and $X = F(G)$. Then G is nilpotent if and only if $X(E)$ contains a nilpotent group for every* 2*-maximal subgroup E of G having non-primary index.*

2. The basic lemmas

In this section, we give some general properties of X-semipermutable subgroups.

The statements of the following two lemmas are evident.

Lemma 2.1. *Let A, B, X be subgroups of G and $K \trianglelefteq G$. Then the following statements hold*:

(1) *If A is X-permutable with B, then B is X-permutable with A.*
(2) *If A is X-permutable with B, then AK/K is XK/K-permutable with BK/K in G/K.*
(3) *If $K \leqslant A$, then A/K is XK/K-permutable with BK/K in G/K if and only if A is X-permutable with B in G.*
(4) *If A is X-permutable with B and $X \leqslant M \leqslant G$ then A is M-permutable with B.*
(5) *If A is X-permutable with B and $X \leqslant N_G(A)$ then A is permutable with B.*
(6) *If F is a permutable subgroup of G and A is X-permutable with B then AF is X-permutable with B.*

Lemma 2.2. *Let $G = AT$ and T_1 be a subgroup of T. Assume that A is G-permutable with T_1. Then A is T-permutable with T_1.*

The following lemma is also well known.

Lemma 2.3. *Let A, B be proper subgroups of a group G with $G = AB$. Then $G = AB^x$ and $G \neq AA^x$, for all $x \in G$.*

For the X-semipermutable subgroups, we have the following lemma:

Lemma 2.4. *Let A and X be subgroups of G. Then the following statements hold*:

(1) *If N is a permutable subgroup of G and A is X-semipermutable in G, then NA is a X-semipermutable subgroup of G.*
(2) *If $N \trianglelefteq G$, A is X-semipermutable in G and $T \in X(A)$, then AN/N is XN/N-semipermutable in G/N and $TN/N \in (XN/N)(AN/N)$.*
(3) *If A/N is XN/N-semipermutable in G/N and $T/N \in (XN/N)(A/N)$, then A is X-semipermutable in G and $T \in X(A)$.*
(4) *If A is X-semipermutable in G and $A \leqslant D \leqslant G$, $X \leqslant D$, then A is X-semipermutable in D.*
(5) *If A is a maximal subgroup of G, T is a minimal supplement of A in G and $T \in G(A)$, then $T = \langle a \rangle$ is a cyclic p-group, for some prime p and $a^p \in A$.*

(6) *If* $T \in X(A)$ *and* $A \leqslant N_G(X)$*, then* $T^x \in X(A)$*, for all* $x \in G$.
(7) *If* A *is* X*-semipermutable in* G *and* $X \leqslant D$, *then* A *is* D*-semipermutable in* G.

Proof. (1) The proof of this part follows directly from Lemma 2.1(6).

(2) It is obvious that TN/N is a supplement of AN/N in G/N. If T_1/N is a subgroup of TN/N, then $T_1/N = (T_1 \cap NT)/N = N(T_1 \cap T)/N$ and so AN/N is XN/N-permutable with T_1/N in G/N by Lemma 2.1(2). Hence, $TN/N \in (XN/N)(AN/N)$.

(3) The proof of this part is the same as the proof in (2).

(4) This part is evident.

(5) Let M be a maximal subgroup of T. Then, by Lemma 2.2, there exists $t \in T$ such that $AM^t = M^tA$. Since T is a minimal supplement of A in G, $AM \neq G$ and so $AM^t \neq G$ by Lemma 2.3. Since A is a maximal subgroup of G, $M^t \leqslant A$. Suppose that T has a maximal subgroup M_1 which is not a conjugate of M. Then, by using the same arguments as above, we can easily show that $M_1^{t_1} \leqslant A$, for some $t_1 \in T$. It is clear that $M^t \neq M_1^{t_1}$, and hence $T = \langle M^t, M_1^{t_1} \rangle \leqslant A$. Thus, it follows that $G = AT = A$, a contradiction. This shows that T must be a cyclic group of prime power order and $M \leqslant A$.

(6) By Lemma 2.3, T^x is a supplement of A in G. Let T_1 be a subgroup of T^x. We now proceed to show that A is X-permutable with T_1. Since $G = AT$, $x = at$ for some $a \in A, t \in T$, and hence $T^x = T^a$. Note that $T_1^{a^{-1}} \leqslant T$ and $A = A^{a^{-1}}$. Now, for some $d \in X$, by our hypothesis, we have $A(T_1^{a^{-1}})^d = (T_1^{a^{-1}})^d A = A^a(d^{-1})^a(T_1^{a^{-1}})^a d^a = AT_1^{d^a} = T_1^{d^a} A$, where $d^a \in X$ since $A \in N_G(X)$. This shows that $T^x \in X(A)$.

(7) This part is evident. □

3. The proof of Theorem 1.2

To start with, we first cite the following result of H. Wielandt (see Theorem 3.8 in [11]).

Lemma 3.1. *Let* P *be a Sylow* p*-subgroup of a group* G. *Assume that* p^2 *divides* $|G|$ *and every subgroup of* P *is the intersection of* P *with some normal subgroup of* G. *Then* G *is* p*-soluble.*

By using Lemma 3.1, we have proved the following result in [2].

Lemma 3.2. *(See [2, Theorem 3.7].) Let* $G = AT$, *where* A *is a Hall* π*-subgroup of a group* G *and* T *is a nilpotent Hall* π'*-subgroup of* G. *Assume that* A *is* $F(G)$*-permutable with all subgroups of* T. *Then* G *is* p*-supersoluble, for every prime* p *such that* p^2 *divides* $|T|$.

Proof of Theorem 1.2. Let π be the set of all different primes dividing $|A|$. We first prove that if T is a minimal supplement of A in G such that $T \in X(A)$, then T is a complement of A in G. Suppose that this assertion is false and let G be a counterexample of minimal order. Let $D = A \cap T$. Since A is a Hall π-subgroup of G and $|G : A| = |T : A \cap T|$, we can easily observe that D is a Hall π-subgroup of T. Assume that p divides $|X|$ for some prime $p \in \pi'$ and let P be a Sylow p-subgroup of X. Since $P \operatorname{char} X \trianglelefteq G$, $P \trianglelefteq G$ and so $P \leqslant T$. It is clear that T/P is a minimal supplement of AP/P in G/P and $T/P \in (XP/P)(AP/P)$ by Lemma 2.4(2). Since $XP/P \leqslant F(G/P)$, we see that our hypothesis is still valid for G/P. Hence, by the choice of G, T/P is a complement of AP/P in G/P and so T/P is a Hall π'-subgroup of G/P. Since P is a π'-subgroup of G, T is a Hall π'-subgroup of G and so T is a complement of A in G. This contradiction shows that X is a π-group. It follows that $X \leqslant A$ and so by Lemma 2.1(5), A is permutable with all subgroups of T. Now let M be a maximal subgroup of T. Assume that

$D \nsubseteq M$. Then, we have $T \cap AM = M(T \cap A) = MD = DM = T$ and so $G = AT = ADM = AM$, which contradicts the minimality of T. This shows that $D \leqslant \Phi(T)$. However, since D is a Hall π-subgroup of T, we deduce that $D = 1$ and hence T is a complement of A in G.

Now suppose that there exists a soluble group $T \in X(A)$. Then, without loss of generality, we may suppose that T is a minimal supplement of A in G and so T is a Hall π'-subgroup of G. We now prove that any two complements T_1 and T_2 of A in G are conjugate in G. Assume that this statement is false and G is a counterexample of minimal order. We proceed the proof by the following steps.

(1) $D = O_\pi(G) = 1$.

Assume that $D \neq 1$. Then, it is clear that $D \leqslant A$ and A/D is a Hall π-subgroup of G/D. By Lemma 2.4, we see that the hypothesis still holds for A/D in G/D. Hence, by $|G/D| < |G|$ and by the choice of G, we see that T_1D/D and TD/D are conjugate in G/D, that is, $T_1D = T^xD$, for some $x \in G$. However, since the group T^xD is evidently π'-soluble, T_1 and T^x are conjugate in the group T_1D (cf. [7, VI, 1.7]). It follows that T_1 and T are conjugate in G. Analogously, we can prove that T_2 and T are also conjugate in G. Therefore, T_1 and T_2 are conjugate in G. This contradiction shows that (1) holds.

(2) $O_{\pi'}(G) = 1$. (This equality can be proved by using the same arguments as in (1).)
(3) $X = 1$.

Indeed, if $X \neq 1$ and let L be a minimal normal subgroup of G contained in X, then either $L \leqslant O_\pi(G)$ or $L \leqslant O_{\pi'}(G)$. But these two cases are impossible in view of (1) and (2).

(4) *T has at least one non-cyclic Sylow subgroup.*

Assume that all Sylow subgroups of T are cyclic. Then, T_1, T_2 and T are supersoluble (cf. [7, VI, 10.3]) and so T and T_1 have normal Sylow p-subgroups P and P_1 respectively, where p is the largest prime divisor of $|T| = |T_1|$. Let $N = N_G(P)$ and $N_1 = N_G(P_1)$. Since P_1, P are Sylow p-subgroups of G, $P = P_1^x$ for some $x \in G$. It follows that $N = N_1^x$ and so $T_1^x \leqslant N$. Since $N = N \cap AT = T(A \cap N)$, T and T_1^x are complements of $A \cap N$ in N. Now let T_0 be a subgroup of T. Then by (3) and by our hypothesis, $AT_0 = T_0A$ and so $AT_0 \cap N = T_0(A \cap N) = (A \cap N)T_0$. Thus, the hypothesis still holds for $N \cap A$ in N. But in view (1) and (2), we can see that $N_1 \neq G \neq N$. Now, by the choice of G, T and T_1^x are conjugate in N. It follows that T and T_1 are conjugate in G. Analogously, we can prove that T_2 and T are also conjugate in G. Therefore, T_1 and T_2 are conjugate in G. This contradiction shows that (4) holds.

(5) *For some prime divisor p of $|T|$, T has a p-subgroup P such that $1 \nsubseteq O_p(T) \leqslant P$ and $|P| > p$.*

Let $F = F(T)$. Since T is soluble, $F \neq 1$. If F has a Sylow q-subgroup Q such that $Q \neq T_q$, where T_q is a Sylow q-subgroup of T, we can take $P = T_q$. Suppose that the order of any Sylow subgroup of F is a prime. Then, we just write $|F| = p$, a prime. In this case, $T/C_T(F)$ is a cyclic group of order dividing $p - 1$ because it is isomorphic to some subgroup of $Aut(F)$. But since T is soluble, $C_T(F) \leqslant F$. It follows that all Sylow subgroups of T are cyclic which contradicts (4). Hence, F has at least two distinct Sylow subgroups, say P_1 and P_2. Let N_i be the normal closure of P_i in G and let $D = N_1 \cap N_2$. Since $AP_i = P_iA$, by our hypothesis and (3), we have $N_i = P_i^G = P_i^{AT} = P_i^A \leqslant AP_i$, $i = 1, 2$, and consequently, D is a π-group.

This leads to $D \leqslant O_\pi(G)$. Thus, by (1), $D = 1$, and thereby $A \nsubseteq N_i$, for some N_i. Let, for example, $A \nsubseteq N_1$. Then, by using the same arguments as in the proof of (1), we can show that $E = N_1T = N_1T_1^x$, for some $x \in G$. Let T_0 be a subgroup of T. Then $AT_0 = T_0A$ and it is easy to see that $E \cap AT_0 = T_0(N_1T \cap A) = T_0(N_1 \cap A) = (N_1 \cap A)T_0$. On the other hand, since $G = AT$, we have $E = E \cap AT = T(N_1T \cap A) = T(N_1 \cap A)$. This shows that the hypothesis still holds on E. Since $|E| < |G|$, T and T_1^x are conjugate in G by our choice of G. Analogously, T_2 and T are conjugate in G. Therefore, T_1 and T_2 are conjugate in G. This contradiction shows that (5) holds.

(6) *Final contradiction.*

Let D be the normal closure of the subgroup $O = O_p(T)$ in G. Then $D = O^G = O^{AT} = O^A$. Now, by our hypothesis and (3), PA is a subgroup of G and A is permutable with every subgroup of P. Then, by (5) and Lemma 3.2, AP is a p-supersoluble group. Since $D \leqslant AP$, D is p-supersoluble. Hence, either $O_{p'}(D) \neq 1$ or $O_p(D) \neq 1$. Assume that the former case holds. Then, $O_{p'}(D)$ *char* $D \trianglelefteq G$ and hence $O_{p'}(D) \trianglelefteq G$. However, since $O_{p'}(D) \leqslant A$, we have $O_\pi(G) \neq 1$, this contradicts (1). In the second case, we can similarly derive a contradiction by using (2). This completes the proof. □

4. The proofs of Theorems 1.3 and 1.4

In proving Theorem 1.3, we need the following known results.

Lemma 4.1. *(See [8, Theorem 3].) Let A and B be subgroups of a group G such that $G \neq AB$ and $AB^x = B^xA$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leqslant N$ or $B \leqslant N$.*

Lemma 4.2. *(See [9, Theorem 3.4].) A group G is soluble if $G = AB$, where A is a supersoluble subgroup, B is a cyclic subgroup of G of odd order.*

We need also the following two lemmas.

Lemma 4.3. *Let G be a group and X a normal soluble subgroup of G. Then G is soluble if any its 2-maximal subgroup E of non-primary index in G with the property that G/E_G is not a supersoluble group satisfying $|F(G/E_G)| = |O_p(G/E_G)| > p$, where p is a prime, is X-semipermutable in G.*

Proof. Assume that the lemma is false and let G be a counterexample of minimal order. Then

(1) *G is not a simple group.*

Assume that G is a simple non-abelian group. Then $X = 1$ and G has a non-supersoluble maximal subgroup, say M, by [7, VI, 9.6]. Assume that M has non-primary index in G and T is a maximal subgroup of M. Then, it is obvious that T is a 2-maximal subgroup of G satisfying the conditions in the lemma. Hence, by our hypothesis, T is X-semipermutable in G. This implies that T is also X-semipermutable in M by Lemma 2.4(4), and so $|M : T|$ is a prime by Lemma 2.4(5). This shows that every maximal subgroup of M has prime index, and consequently M is supersoluble by the well-known Huppert's Theorem [7, VI, 9.5]. This contradiction shows that $|G : M| = p^a$, for some prime p. Evidently, M has a maximal subgroup T such that

$(p, |M : T|) = 1$. Since G is a simple non-abelian group, $G/T_G \simeq G$ is not supersoluble. Thus T is X-semipermutable in G by our hypothesis. Let $A \in X(T)$, where A is a minimal supplement of T in G, and let A_1 be a proper subgroup of A. Then $G = TA$ and TA_1 is a proper subgroup of G. Let $x \in G$ and $x = at$, where $t \in T$ and $a \in A$. Since $X = 1$, $T(A_1)^a = (A_1)^a T$ and so $((A_1)^a T)^t = (A_1)^x T$ is a subgroup of G. Therefore, G is not simple by Lemma 4.1. This contradiction completes the proof of (1).

(2) *For every minimal normal subgroup N of G, the quotient group G/N is soluble.*

Indeed, let M/N be a 2-maximal subgroup of G/N of non-primary index in G/N with the property that $(G/N)/(M/N)_{G/N}$ is not a supersoluble group satisfying the condition $|F((G/N)/(M/N)_{G/N})| = |O_p((G/N)/(M/N)_{G/N})| > p$, where p is a prime. Then, since $G/M_G \simeq (G/N)/(M_G/N) = (G/N)/(M/N)_{G/N}$, we have that M is X-semipermutable in G by using our hypothesis. Hence, by Lemma 2.4(2), M/N is XN/N-semipermutable in G/N, where XN/N is a normal soluble subgroup of G/N. This shows that our hypothesis still holds on G/N. Thus, by the choice of G, G/N is soluble.

(3) *G has a unique minimal normal subgroup L.* (This part follows directly from (2).)
(4) *Final contradiction.*

By the above claim (2), we only need to prove that L is soluble. Assume that this assertion is not true. Then by our claim (3), we have $X = 1$. By (1), G is not simple and hence $L \neq G$. By (2), G has a normal maximal subgroup M such that $L \leqslant M$ and consequently $|G : M|$ is a prime. Let $|G : M| = p$. Now, we claim that there exists a maximal subgroup T of M such that $M = LT$ and $(|M : T|, p) = 1$. Indeed, if p divides $|L|$, L_p is a Sylow p-subgroup of L and P is a Sylow subgroup of G containing L_p, then $P \leqslant N = N_G(L_p)$. By using the usual Frattini argument, we have $G = LN$ and $M = M \cap LN = L(M \cap N)$. Since L is not soluble, $N \neq G$. Therefore, $M \cap N \neq M$. Let T be a maximal subgroup of M containing $M \cap N$. Then, $M = LT$ and $(|M : T|, p) = 1$. Next, we assume that $(|L|, p) = 1$. Then, it is clear that $L \nleqslant \Phi(G)$ and so $L \nleqslant \Phi(M)$. Hence, there exists a maximal subgroup T of M such that $M = LT$. It follows that p does not divide $|M : T| = |L|/|L \cap T|$. Hence, our claim is established. This shows that T is a 2-maximal subgroup of G having non-primary index. Since $L \nsubseteq T$, $T_G = 1$ and so $G/T_G \simeq G$ is not supersoluble. Thus, by our hypothesis, T is X-semipermutable in G. Since $X = 1$ and T is maximal subgroup of M, we may take $a \in M \setminus T$ such that $\langle a \rangle$ is a minimal supplement of T in M. Then, it is easy to see that $|M : T| = q$ and so $|G : T| = pq$, for some prime $q \neq p$. Let $A \in X(T)$ and A be a minimal supplement of T in G. Then $G = TA$ and $A \cap T \leqslant \Phi(A)$. Thus, A is a $\{p, q\}$-group. Let A_p be a Sylow p-subgroup of A. Since $X = 1$, $D = TA_p = A_pT$. Clearly, $|G : D| = q$. Moreover, since $|G : M| = p$ and $M = LT$, we see that $L \nleqslant D$. Thus $D_G = 1$, and by considering the permutation representation of G on the right cosets of D, we see that G is isomorphic with some subgroup of the symmetric group S_q of degree q. It follows that D is a Hall q'-subgroup of G and $G = DZ_q$, where Z_q is a subgroup of order q. Now we have seen that every maximal subgroup of D is a 2-maximal subgroup of G satisfying the condition of the lemma, hence it is X-semipermutable in G, and consequently, by Lemma 2.4 (4), every maximal subgroup of D is X-semipermutable in D. It follows that the index of every maximal subgroup of D is a prime. Hence, D is supersoluble and thereby G is a soluble group by Lemma 4.2. This contradiction completes the proof. □

Lemma 4.4. *Consider a soluble group $G = [L]M$, where L is a non-cyclic minimal normal subgroup of G. Let L be a p-group and M has a maximal subgroup E such that $|M : E| = q \neq p$ is a prime and let T be a minimal supplement of E in G having a normal maximal subgroup K of T. If E is L-permutable with all maximal subgroups of T, then $|T : K| \neq p$.*

Proof. Suppose that $|T : K| = p$. Let $D = E \cap T$ and D_p be a Sylow p-subgroup of D. If $D \nleqslant K$, then $KD = T$ and so $G = ET = EDK = EK$, this clearly contradicts the minimality of T. Hence $D \leqslant K$. Let $x \in L$ and form $V = EK^x = K^x E$. Since $G = ET$, we have $x = et$, for some $e \in E$ and $t \in T$. Hence, we obtain $V^{e^{-1}} = EK^t = EK$. Let $Y = V^{e^{-1}}$ and Y_p be a Sylow p-subgroup of Y. Now, let G_p be a Sylow p-subgroup of G containing Y_p. Then, $|G_p : Y_p| = (|E_p||T_p|/|D_p|) : (|E_p||K_p|/|D_p|) = p$, where E_p is a Sylow p-subgroup of E. On the other hand, since $|G : LE| = q$, we have $|G_p| = |L||E_p|$. Thus, $L \nsubseteq Y$ and $L \cap Y \neq 1$. This result leads to $G = LY$ and so $Y \cap L$ is normal in G which contradicts the minimality of L. Hence, the lemma is proved. □

Proof of Theorem 1.3. (1) ⇒ (2). Suppose that the assertion is false and let G be a counterexample of minimal order. Let L be a minimal normal subgroup of G. By making use of Lemma 2.4(3), it is not difficult to see that our hypothesis still holds on the quotient group G/L, and so by the choice of G, G/L is supersoluble. Since the class of all supersoluble groups is a saturated formation, L is a unique minimal normal subgroup of G and $L \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G such that $L \nsubseteq M$. Then, by Lemma 4.3, G is a soluble group and hence $G = [L]M$. It is easy to see that $L = C = C_G(L) = F(G) = O_p(G)$, for some prime p, and $|L| \neq p$. Hence $L = X$. Since $M \simeq G/L$ is supersoluble, M contains a maximal subgroup E such that $|M : E| = q \neq p$. Since $L \nsubseteq E$, $E_G = 1$ and so $G/E_G \simeq G$ is not supersoluble. Therefore, by our hypothesis, E is L-semipermutable in G and the set $L(E)$ contains a supersoluble group, say T. Without loss of generality, we may assume that T is a minimal supplement of E in G. Obviously, $E \neq 1$. Let $D = E \cap T$ and D_p be a Sylow p-subgroup of D. Then, since $|G : E| = |L|q$, we have $|T : D| = p^a q$, where $|L| = p^a$ and $a > 1$. Let r be the largest prime divisor of $|G|$. Assume that $r = p$. Because G/L is a supersoluble group and $O_p(G/L) = 1$, we see that L is a Sylow subgroup of G. It is now clear that $L \leqslant T$. Let L_1 be a maximal subgroup of L. Then by our hypothesis again, we have $A = L_1 E = EL_1$. Evidently, $|G : A| = pq$ and $A_G = 1$. Hence, A is L-semipermutable in G. If $T_1 \in L(A)$ and Q is a Sylow q-subgroup of T_1, then $B = Q^x A = AQ^x$, for some $x \in L$. However, since $|G : B| = p$, we have $LB = G$. This shows that $|L| \neq |L \cap B| \neq 1$ and $L \cap B \trianglelefteq G$, which contradicts the minimality of L. Thus, without loss of generality, we may assume that $r = q$. Let K be a maximal subgroup of T such that $|T : K| = p$. Since G is soluble, it is clear that T is a $\{p, q\}$-group. Since $p < q$ and T is supersoluble, K is normal in T, which contradicts Lemma 4.4. Thus, the contradiction shows that G is supersoluble.

(2) ⇒ (3). Let E be a 2-maximal subgroup of G such that either $|F(G/E_G)|$ is a prime or $|F(G/E_G)|$ has at least two distinct prime divisors. Let T be a minimal supplement of E in G. We now going to prove, by using induction on $|G|$, that E is G'-permutable with all subgroups of T. For this purpose, we let T_1 be a subgroup of T. We first suppose that $E_G \neq 1$. Then the assertion is obviously true for G/E_G and therefore E/E_G is $(G/E_G)'$-permutable with $T_1 E_G/E_G$. But, since $(G/E_G)' = G' E_G/E_G$, by Lemma 2.1(3), E is G'-permutable with T_1.

Now we assume that $E_G = 1$. Let $F = F(G)$ and $\pi = \pi(F)$ be the set of all prime divisors of $|F|$. We first suppose that $|G : E| = p^2$ for some prime p. Since $E_G = 1$, it is obvious that F is a Sylow p-subgroup of G because G is a supersoluble group. Hence, by our hypothesis,

$|F|$ is a prime. This shows that $|G:E| = p^2$ is impossible. Now suppose that $|G:E| = pq$ with $p > q$. If $|\pi| > 2$ and R is a Sylow d-subgroup of F, where $q \neq d \neq p$, then, it is clear that $R \leqslant E_G$, which is impossible because $E_G = 1$. Hence, $\pi \subseteq \{p, q\}$. Since G is supersoluble, G has a normal Sylow r-subgroup, where r is a largest prime divisor of $|G|$. It follows that p is the largest prime divisor of $|G|$.

Assume that F is a cyclic group of prime power order. Then F is a p-group. Since $E_G = 1$ and $|G:E| = pq$, we see that $F \nsubseteq E$ and so $|F| = p$. Since G is soluble, $\Phi(G) < F(G)$. This leads to $\Phi(G) = 1$ and so $G = [F]M$, for some maximal subgroup M of G and $C_G(F) = F$. Hence M is a cyclic group. Without loss of generality, we may assume that $E \leqslant M$. We now prove that E is G'-permutable with T_1. In fact, if A is a Hall p'-subgroup of T_1, then $T_1 = PA$, where $P = T_1 \cap F$ is a Sylow p-subgroup of T_1. Since any two Hall p'-subgroups of a soluble group are conjugate, by $G = F(G)M$, we see that $A^x \subseteq M$, for some $x \in G'$. Therefore, $ET_1^x = E(T_1 \cap F)A^x = (T_1 \cap F)A^x E = T_1^x E$.

Next, we assume that $|\pi| = 2$, and let F_p and F_q be the Sylow p-subgroup and the Sylow q-subgroup of F, respectively. Then, it is clear that $G = FE$. Let R be a Sylow r-subgroup of F. If $|R| > r$, then $D = R \cap E \neq 1$. Since R char $F \trianglelefteq G$, $R \trianglelefteq G$. Obviously, $|R:D| = r$ and so $D \trianglelefteq R$. Let $F = R \times Q$, where Q is the another Sylow subgroup of F. Then $Q \subseteq N_G(D)$. It follows that D is a normal subgroup of G. Because $E_G = 1$, we have $D = 1$. This shows that $|F| = pq$. Assume that q divides $|E|$ and q, p divide $|T_1|$. Let $\{E_2, \ldots, E_t\}$ be a Sylow system of E and $\{D_1, D_2\}$ a Sylow system of T_1, where D_1 is a p-group. Then, by [7, VI, 2.3, 2.4], G has Sylow systems $\Sigma = \{P_1, \ldots, P_t\}$ and $\Sigma_1 = \{Q_1, \ldots, Q_t\}$ such that $E_i \leqslant P_i$, for all $i = 2, \ldots, t$ and $D_i \leqslant Q_i$ for $i = 1, 2$. Moreover, the systems Σ and Σ_1 are conjugate, i.e. G has an element x such that $Q_i^x = P_i$, for all $i = 1, \ldots, t$. It is clear that $P_1 = D_1$ is a Sylow p-subgroup of G and $E_3 = P_3, \ldots, E_t = P_t$. If $D_2^x \leqslant M_2$, then $T_1{}^x E = P_1 E = ET_1^x$. On the other hand, if $D_2^x \nleqslant E_2$, then by $|G:E| = pq$, we have $|P_2 : E_2| = q$ and hence $P_2 = D_2^x E_2$. It follows that $T_1^x E = G = ET_1^x$. Since by [7, VI, 11.10], we know that $N_G(\Sigma_1)$ covers all central chief factors of G, we have $G = G' N_G(\Sigma_1)$, and consequently, $x = fn$, where $f \in G'$ and $n \in N_G(\Sigma_1)$. Therefore, we have proved that $ET_1^f = T_1^f E$. Analogously, we can also consider the cases either $(|E|, q) = 1$ or $(|T_1|, p) = 1$.

Finally, since G is a supersoluble group, we have $G' \leqslant F(G)$ and so $X = G'$. Therefore, E is indeed X-permutable with all subgroups of T. Hence every minimal supplement of E in G is contained in $X(E)$.

The implication (3) ⇒ (2) is evident. The implication (2) ⇒ (1) is, indeed, a special case of the implication (2) ⇒ (3). Thus the proof of the theorem is completed. □

Corollary 4.5. *Let G be a group and $X = F(G) \cap G'$. Then G is a nilpotent group if and only if for every 2-maximal subgroup M of G having non-primary index, the set $X(M)$ contains a supersoluble group and every minimal subgroup of G is contained in the hypercenter of its normalizer.*

A group G is called p-decomposable if $G = O_p(G) \times O_{p'}(G)$.

Theorem 1.4 is a direct corollary of the following theorem.

Theorem 4.6. *Let G be a group, $X = F(G)$ and p a prime. Suppose that for every 2-maximal subgroup E of G of non-primary index, the set $X(E)$ contains a p-decomposable group. Then the group G is p-decomposable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then p divides $|G|$. Let L be a minimal normal subgroup of G. It is not difficult to show that the hypothesis of the theorem still holds on G/L and so by our choice of G, G/L is p-decomposable. It is well known that the class of all p-decomposable groups is a saturated formation. Hence L is the only minimal normal subgroup of G and $L \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G such that $L \nsubseteq M$. By Lemma 4.3, G is soluble group. Hence, $G = [L]M$ and $L = C_G(L) = F(G) = X = O_q(G)$, for some prime q. It is clear that $M_G = 1$ and for some maximal subgroup E of M, we have $(|M : E|, q) = 1$. Hence by our hypothesis, E is L-semipermutable in G and the set $L(E)$ contains a p-decomposable subgroup T, which is a minimal supplement of E in G. Let T_p and $T_{p'}$ be a Sylow p-subgroup and a Hall p'-subgroup of T, respectively. It is clear that E has a non-primary index in G. Hence $T_{p'} \neq 1$ and $T = T_p \times T_{p'}$. Assume that $p = q$. Then since $L = O_p(G)$ and G/L is p-decomposable, L is the Sylow p-subgroup of G. It follows that $L \leqslant T$ and so $T_{p'} \leqslant C_G(L) = L$. If $|L| = q$, then since $G/L \simeq Aut(L)$, L is a Sylow q-subgroup of G. Hence, we can also see that $L \leqslant T$ and so $T_{p'} \leqslant C_G(L) = L$. This contradiction shows that $p \neq q$ and $|L| \neq q$. Hence, without loss of generality, we may assume that $|M : E| = p$. Since G is soluble, any minimal supplement of E in G is a $\{p, q\}$-group. Hence, $T_{p'}$ is a Sylow q-subgroup of T. This shows that T has a normal maximal subgroup K such that $|T : K| = q$, which is impossible by Lemma 4.4. This completes the proof. □

5. Remarks and questions

We make the following remarks and questions:

(1) The example of the group A_5 shows that in Theorem 1.2, the subgroup A may be non-normal in G and G is not necessary either π-soluble or π'-soluble, where π is the set of all prime divisors of $|A|$.

(2) In connection with Theorem 1.2, the following question naturally arises.

Question 5.1. *Let A be a Hall soluble subgroup of a group G and $X = F(G)$. Assume that A is X-semipermutable in G. Is it true that any two complements of A in G are conjugate?*

(3) In connection with Theorem 1.3, it is naturally to ask the following question:

Question 5.2. *Is a group G supersoluble if all its 2-maximal subgroups of non-primary index are $F(G)$-semipermutable in G?*

(4) By using the same arguments as in the proof of Theorem 1.3, the following result may be obtained

Theorem 5.3. *A group G is supersoluble if and only if every maximal subgroup of G is $F(G)$-semipermutable in G.*

(5) In the supersoluble group $G = S_3 \times Z_3$, where S_3 is the symmetric group of degree 3 and $|Z_3| = 3$, there exists a 2-maximal subgroup E of order 3 which is not G-permutable with any Sylow 2-subgroups of G. Hence E is not G-semipermutable in G.

(6) Finally, we give the following application of Theorem 1.2.

Theorem 5.4. *Let $|G| = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, where $p_1 > p_2 > \cdots > p_t$. Let $\pi_i = \{p_1, \ldots, p_i\}$ for all $i = 1, 2, \ldots, t$ and $X = F(G)$. Then G is a Sylow tower group if and only if G has a Hall π_i-subgroup which is X-semipermutable in G, for all $i = 1, 2, \ldots, t - 1$.*

Proof. In fact, we only need to prove that if A is a Hall X-semipermutable subgroup of G and $p > q$, for all primes p and q such that p divides $|A|$ and q divides $|G : A|$, then A is normal in G. We now prove this assertion by using induction on $|G|$. We first let π be the set of all prime divisors of $|A|$.

We first claim that $AL \trianglelefteq G$, for any non-identity normal subgroup L of G. Indeed, the hypothesis of the theorem still holds for G/L by Lemma 2.4(2), and so AL/L is normal in G/L by induction, which implies that $AL \trianglelefteq G$. If L is a π-group, then $AL = A \trianglelefteq G$. Hence, we may assume that $O_\pi(G) = 1$. Since $O_\pi(X)$ char $X \trianglelefteq G$, X is a π'-group. Let $T \in X(A)$, where T is a minimal supplement of A in G. Then T is a complement of A in G (see the proof of the first statement in Theorem 1.2). Thus, T is a Hall π'-group of G. Suppose that $X \neq 1$. Then, it is clear that $X \leqslant T$ and so the hypothesis of the theorem still holds on AX, by Lemma 2.4(4). If $AX \neq G$, then A is normal in AX by induction, and so that A is normal in G because A char $AX \trianglelefteq G$. Now, let $AX = G$. Then, $X = T$. Let $Z = Z(X)$. Assume that $Z \neq X$. Then AZ is a proper normal subgroup of G. Since our hypothesis holds on AZ, by induction, $A \trianglelefteq AZ$. It follows that $A \trianglelefteq G$. Now let $Z = X$. Then, in this case, our hypothesis still holds on AD, where D is any proper subgroup of X. Thus $D \leqslant N_G(A)$ by induction. Now, without loss of generality, we may assume that X has prime power order. If X is a non-cyclic group, then, it is obvious that $A \trianglelefteq AZ$. Hence we may assume that $T = X = F(G) = O_p(G) = C_G(T)$ is a cyclic p-group, for some prime p. In this case, G/T is an abelian group. It follows that G is supersoluble and so $A \trianglelefteq G$. Finally, suppose that $X = 1$ and let M be a maximal subgroup of T. Then, same as above, one can also see that A char AM. But AM is normal in G because $|G : AM| = p_t$ is the smallest prime divisor of $|G|$. Hence, we also obtain that $A \trianglelefteq G$. This completes the proof. □

The following corollary is immediate.

Corollary 5.5. *Let p the largest prime divisor of a group G and $X = F(G)$. Then G is p-closed if and only if a Sylow p-subgroup of G is X-semipermutable in G.*

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