Criterions of Supersolubility for Products of Supersoluble Groups

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Abstract

Let H and T be subgroups of a group G. Then we call H conditionally permutable (or in brevity, c-permutable) with T in G if there exists an element $x \in G$ such that $HT^x = T^xH$. If H is c-permutable with T in $< H,T >$, then we call H completely c-permutable with T in G. By using the above concepts, we will give some new criterions for the supersolubility of a finite group $G = AB$, where A and B are both supersoluble groups. In particular, we prove that a finite group G is supersoluble if and only if $G = AB$, where both A, B are nilpotent subgroups of the group G and B is completely c-permutable in G with every term in some chief series of A. We will also give some applications of our new criterions.

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1 Introduction

Throughout this paper, all groups are finite. A well-known theorem of Fitting says that any group G which is the product of normal nilpotent subgroups of G is nilpotent. However, the above property does not hold for supersoluble groups, as can be seen in Asaad and Shaalan [3], and also Huppert [15]. It is natural to ask under what additional conditions the product of two supersoluble groups is supersoluble ? In the literature, we know, for example, that the product $G = AB$ of two normal supersoluble subgroups A and B is supersoluble if either G' is nilpotent (see [4]) or the subgroups A and B have coprime indices in G ([10]). An interesting approach for solving the supersolubility problem was proposed by Asaad and Shaalan in 1989 ([3]). They have obtained the following nice result: *Assume that $G = AB$ is the product of two supersoluble subgroups A and B. If every subgroup of A is permutable with every subgroup of B, then G is supersoluble.* In addition, they have also generalized the above mentioned result of Baer by replacing the condition of normality of A, B in G and using the following weaker condition: *A permutes with all subgroups of B and B permutes with all subgroups of A.* Their results in [3] were further developed and applied by many authors (see, for example, [1] [5-8], [14], [19]). We also notice that O.H.Kegel has also obtained many elegant results for soluble groups and supersoluble groups by considering the products of their subgroups (see [16-18]).

Our results in this paper are based on c-permutability condition on subgroups of a group. In fact, our concept of c-permutability of subgroups is weaker than the concept of permutability of subgroups. Some new criterions for the supersolubility of products of supersoluble groups are obtained in this paper.

We first recall some definitions. Let H and T be subgroups of a group G. Then, H is said to be permutable with T (or also H and T are permutable) if $HT = TH$.

We note that two subgroups H and T may possibly be not permutable in G but G could have an element x such that $HT^x = T^xH$. For instance, we have the following examples:

a) If $G = AB$ is a finite group, A_p and B_p are Sylow p–subgroups of A and of B respectively, then in general $A_pB_p \neq B_pA_p$ but G has an element x such that $A_pB_p^x = B_p^xA_p$;

b) If P and Q are Sylow subgroups of a finite soluble group G. Then for some $x \in G$, we have $PQ^x = Q^xP$;

c) If M is a maximal subgroup of the group $PSL(2, 7)$, then for every Sylow subgroup P of G there exists an element x such that $MP^x = P^xM$. It is clear also that in general M is not permutable with P .

The above examples motivate the following definition [11].

Definition 1.1. Let H and T be subgroups of the group G. Then

1) H and T are said to be conditionally permutable (or in brevity, *c-permutable*) in G if for some $x \in G$ we have $HT^x = T^x H$ (In this case, we also say that H is c-permutable with T in G).

2) H and T are said to be completely c-permutable in G if H and T are c-permutable in $< H, T >$.

By using the above definition, it is not difficult to note that a group G is soluble if and only if its any two Hall subgroups (associated with different set of primes) are c-permutable in G. We can also prove (see [11, Theorem 3.8]) that a group G is supersoluble if and only if every maximal subgroup of G is c-permutable with all subgroups of G. On the other hand, every group in which any two Hall subgroups or any two maximal subgroups are permutable is always nilpotent. For c-permutability of subgroups, we consider the following elementary example: Let $G = G_p \times S_3 \times G_q$, where $|G_p| = p, |G_q| = q, p \neq q$ and $2, 3 \notin \{p, q\}$. If $A = G_p S_3, B = G_q S_3$, then $S_3 \leq A \cap B$ and so $G = AB$ is a factorization of G in which some subgroups of A are not permutable with some subgroups of B, however, one can easily check that every subgroup of A is completely c-permutable with every subgroup of B. Thus the condition of permutability is generally stronger than the condition of c-permutability. Motivated by the above observation, we are now able to give the following three criterions of supersolubility for products of supersoluble groups.

Theorem A. *Let $G = AB$ be the product of supersoluble groups A and B. If every subgroup of A is completely c-permutable in G with every subgroup of B, then G is supersoluble.*

By the well known Kegel's theorem, we know that a group G is soluble if G is a product of two nilpotent groups. However, such product of nilpotent groups may not be supersoluble in general. The following theorem gives some additional conditions under which the product of two nilpotent groups is supersoluble.

Theorem B. *A finite group G is supersoluble if and only if $G = AB$, where A, B are nilpotent subgroups of G and A has a chief series*

$$1 = A_0 \leq A_1 \leq \ldots \leq A_{t-1} \leq A_t = A \tag{1}$$

such that every A_i is completely c-permutable (permutable) with all subgroups of B, for all $i = 1, \ldots, t$.

Theorem C. *Assume that $G = AB$, where A, B are supersoluble subgroups of a group G. Assume further that either G' is nilpotent or A and B have coprime orders. If A is completely c-permutable with every subgroup of B and B is completely c-permutable with every subgroup of A, then G is supersoluble.*

For notation and terminology not given in this paper, the reader is referred to the monograph of W. Guo [12].

2 Preliminaries

We first cite here some properties of factorizations of groups. Some useful properties of p-supersoluble and p-soluble groups are also included.

The following three lemmas are well known.

Lemma 2.1. *Let A, B be subgroups of a group G. If $G = AB$, then $G = AB^x$ for every $x \in G$.*

Lemma 2.2. *Let H be a proper subgroup of a group G. Then $HH^x \neq G$ for all $x \in G$.*

Lemma 2.3. *Let $G = AB$ and A_p, B_p and G_p be Sylow p-subgroups of A, B and G, respectively. Then there are elements $x, y \in G$ such that $G_p^x = A_p B_p^y$.*

Lemma 2.4[9]. *Let $G = AB$ be the product of the subgroups A and B. If L is a normal subgroup of A and $L \leq B$, then $L \leq B_G$.*

Lemma 2.5[12; 1.7.11]. *If H/K is a chief factor of a group G and p is a prime divisor of $|H/K|$, then $O_p(G/C_G(H/K)) = 1$.*

A group G is said to be dispersive if G has a chain of normal subgroups

$$1 = G_0 \subset G_1 \subset ... \subset G_t = G, \ t \geq 0,$$

where G_i/G_{i-1} is a Sylow p_i–subgroup of G/G_i and $p_1 > \ p_2 > \ ... > \ p_t$.

Lemma 2.6. *Let G be a group. Then the following statements hold:*

(i) if G is supersoluble, then $G' \subseteq F(G)$ and G is dispersive (see [12; 1.9.9]);

(ii) *if $L \trianglelefteq G$ and $G/\Phi(L)$ is supersoluble (dispersive) , then G is supersoluble (respectively, G is dispersive)* (see [12; 1.8.1]) ;

(iii) *G is supersoluble if and only if $|G : M|$ is a prime for every maximal subgroup M of G* (B. Huppert, 1954).

Lemma 2.7[12, 2.4.3]. *Let M_1, M_2 be maximal subgroups of a soluble group G such that $(M_1)_G = (M_2)_G$. Then M_1 and M_2 are conjugate.*

Lemma 2.8. *Let p be a prime number and G a p-soluble group. If $O_{p'}(G) = 1$, then the following statements are equivalent:*

(i) G is p-supersoluble;

(ii) G is supersoluble;

(iii) $G/O_p(G)$ is an abelian group of exponent dividing $p - 1$.

Proof. (i)$\Longrightarrow$ (ii). Since G is p-supersoluble, for every chief p-factor H/K of G, we have $|H/K| = p$ and so by [20; 1,1.4], $G/C_G(H/K)$ is an abelian group of exponent dividing $p-1$. Since $O_{p'}(G) = 1$, the intersection of the centralizers of all such factors is $O_{p',p}(G) = O_p(G)$.

Hence G is supersoluble by [20; 1,1.9]. By using the same arguments, we can also prove that (ii)$\Longrightarrow$ (iii) and (iii)$\Longrightarrow$ (i).

3 The Proof of Theorems A, B and C

A group $G = AB$ is said to be a totally permutable product of the groups A and B if every subgroup of A is permutable with every subgroup of B. By analogy, we call $G = AB$ a *totally* (*completely*) *c-permutable product* of the groups A and B if every subgroup of A is (completely) c-permutable with every subgroup of B. Equipped with the above concepts, we now prove the Theorems stated in section 1.

Proof of Theorem A. Since every subgroup of a supersoluble group is also supersoluble, we only need to show that G is supersoluble if $G = AB$ is a totally completely c-permutable product of supersoluble groups A and B. Assume that the assertion is not true and let G be a counterexample of minimal order. Then A and B are proper subgroups of G. We proceed the proof via the following steps.

(a) If M is a maximal subgroup of G and either $A \subseteq M$ or $B \subseteq M$, then M is supersoluble.

Indeed, by using the Dedekind Law, we have $M = M \cap AB = A(M \cap B)$. Hence M is a totally completely c-permutable product of the groups A and $M \cap B$. This shows that M is supersoluble since $|M| < |G|$.

(b) For every $a \in A$, the group G is a totally completely c-permutable product of the subgroups A and B^a.

By Lemma 2.1 we have $G = AB^a$. Now let $H \leq A$, $T \leq B^a$ and H, $T \leq D \leq G$. Then $H^{a^{-1}} \leq A$, $T^{a^{-1}} \leq B$ and $H^{a^{-1}}$, $T^{a^{-1}} \leq D^{a^{-1}}$. By hypothesis, for some $d \in D$, we have $H^{a^{-1}}(T^{a^{-1}})^{d^{a^{-1}}} = (T^{a^{-1}})^{d^{a^{-1}}}H^{a^{-1}}$. Then $(aHa^{-1})(ad^{-1}a^{-1})(aTa^{-1})(ada^{-1}) = aHd^{-1}Tda^{-1} = ad^{-1}TdHa^{-1}$. This implies that $HT^d = T^dH$.

(c) G has an abelian minimal normal subgroup.

Let L be a minimal normal subgroup of A. Then, by hypothesis, G has an element x such that $LB^x = B^xL$. Assume that $L \subseteq B^x$. Since by Lemma 2.1, $G = AB^x$, we see from Lemma 2.3 that $L^G \subseteq B^x$. But B^x is a supersoluble group, and so any minimal normal subgroup of G contained in $L^G \subseteq B^x$ must be abelian. Hence, we may suppose that L is not contained in B^x. In this case,we may assume that $LB^x \neq G$ and let M be a maximal subgroup of G such that $LB^x \subseteq M$. Let $x = ba$, where $a \in A$, $b \in B$. Then $B^x = B^a$. By Lemma 2.1 again, we have $G = AB^a$. In view of (b), we can see that M is a supersoluble group. However,since $L \subseteq A \cap M$, and so by Lemma 2.3, we have $L^G \subseteq M$. This shows that any minimal normal subgroup of G contained in L^G is still abelian. Finally, we let $G = LB^a$. Since $L \subseteq A$, we see from (b) that G is a totally completely c-permutable product of the groups L and B^a. Let R be a minimal normal subgroup of B^a. Using the same argument as above, we come to the case that $G = LR$. Since L and R are abelian groups, we conclude that G has an abelian minimal normal subgroup.

(d) G/L is a supersoluble group for any non-identity normal subgroup L of G.

Obviously, $G/L = (AL/L)(BL/L)$. Let $H/L \leq AL/L$ and $T/L \leq BL/L$, and let $D =< H, T >$. Then,by our hypothesis,we have $(H \cap A)(H \cap B)^x = (H \cap B)^x(H \cap A)$,for some $x \in D$. Thus, we have

$$(H/L)(T/L)^{xL} = (L(A \cap H)/L)(L(T \cap B)/L)^{xL} = L(A \cap H)(T \cap B)^x/L$$

$$= ((T \cap B)^x L/L)(L(A \cap H)/L) = ((T \cap B)L/L)^{xL}(L(A \cap H)/L) = (T/L)^{xL}(H/L),$$

where $xL \in D/L$. This shows that G/L is the totally completely c-permutable product of the supersoluble groups $AL/L \simeq A/A \cap L$ and $BL/L \simeq B/B \cap L$. Since $|G/L| < |G|$, we conclude that G/L is supersoluble.

(e) G has only one minimal normal subgroup $L = O_p(G) = C_G(L)$, for some prime p, and $G = [L]M$, where M is a maximal subgroup of G with $O_p(M) = 1$ and $|L| \neq p$.

Since the class of all supersoluble groups is closed under subdirect products, in view of (d), L is the only minimal normal subgroup of G. By Lemma 2.6, we also have $L \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G not containing L and $C = C_G(L)$. Then by Dedekind Law, we have $C = C \cap LM = L(C \cap M)$. Since L is abelian, $C \cap M \trianglelefteq G$ and so $C \cap M = 1$. This shows that $L = O_p(G) = C_G(L)$ and $M \simeq G/L$ is a supersoluble group with $O_p(M) = 1$ by Lemma 2.4. Now, by (d) and the choice of G, we have $|L| \neq p$.

(f) p is the largest prime divisor of the order of the group G.

Assume that q is the largest prime divisor of the order of G with $q \neq p$. Let T_1 and T_2 be maximal subgroups of G such that $A \leq T_1$, $B \leq T_2$. Then $T_1T_2 = G$. By Lemma 2.2, T_1 and T_2 are not conjugate in G. Since by Lemma 2.7 all maximal subgroups of G not containing L are conjugate in G, we have either T_1 contains L or T_2 contains L. Let $L \subseteq T_1$ and let G_q be a Sylow q-subgroup of G. Assume that $|G_q| \neq q$. Since by (d), G/L is supersoluble and T_1/L is maximal in G/L, we obtain that $|G/L : T_1/L| = |G : T_1|$ is a prime by Lemma 2.6. Hence, T_1 contains a non-trivial Sylow q-subgroup Q. In view of Lemma 2.6, we have $Q \trianglelefteq T_1$, and consequently, $Q \subseteq C_G(L) = L$. This contradiction shows that $|G_q| = q$. Clearly $q \nmid |A|$. Hence $q \mid |B|$. Assume that $LB \neq G$ and let M_3 be a maximal subgroup of G containing LB. From (a), we know that M_3 is supersoluble. Hence we have $L \neq C_G(L)$ again. This contradicts (e), so $LB = G$. Thus, by applying Dedekind Law again,we have $T_2 = T_2 \cap LB = B(T_2 \cap L) = B$ and clearly $B \cap L = 1$. Let x be an element of G such that $(L \cap A)^x B = B(L \cap A)^x$. Assume that $L \cap A \neq 1$. Then $(L \cap A)^x \neq 1$, and clearly $(L \cap A)^x \nsubseteq B$. This leads to $B(L \cap A)^x = G$. Thus,we have $|G : B| \leq |L \cap A|$. Evidently, $|G : B| = |L|$, and thereby $L \subseteq A$. If L_1 is a maximal subgroup of L,then for some $x \in G$, we have $L_1^x B = BL_1^x$.

Since G is not a supersoluble group, from (d) we see that $L_1 \neq 1$. But then, we can derive that $|L| = |G : B| = |L_1|$, a contradiction. Thus $L \cap A = 1$. Let B_q be a Sylow q-subgroup of B and x an element of G such that $AB_q^x = B_q^x A$. Suppose that $LAB_q^x \neq G$. Then, there exists a maximal subgroup M of G containing LAB_q^x. Thus by (a), M is supersoluble. This leads to $B_q^x \subseteq C_G(L) = L$, a contradiction. Hence, we have shown that $G = LAB_q^x$. Now, we assume that $G = AB_q^x$. In this case, we have $p \nmid |G : A|$, and so any Sylow p-subgroup of A must be a Sylow p-subgroup of G. Thus, $L \leq A \cap L = 1$. However,this contradiction shows that $AB_q^x \neq G$, and consequently, we know that AB_q^x is a maximal subgroup of G. Now in view of Lemma 2.7, we have $AB_q^x = B^y$, for some $y \in G$. This contradiction shows that p is the largest prime divisor of $|G|$.

(g) L is a Sylow p-subgroup of G.

Assume that the assertion is not true. Then, we have $p \mid |G : L|$. This means that $p \mid |M|$, and so by (f) and also by Lemma 2.6, we see that $O_p(M) \neq 1$. This contradicts (e). Hence, L is a Sylow p-subgroup of G.

(h) To complete the proof.

Without loss of generality, we may assume that $p \mid |A|$. Since A is supersoluble, by (f) we know that A has a normal subgroup Z_p of order p. Clearly $Z_p \subseteq L$. Let $B_{p'}$ be a Hall p'-subgroup of B and x an element of G such that $Z_p B_{p'}^x = B_{p'}^x Z_p$. Since evidently $Z_p = L \cap Z_p B_{p'}^x \trianglelefteq Z_p B_{p'}^x$, we see that $B_{p'}^x \subseteq N_G(Z_p)$. In view of (g), the Sylow p-subgroup of B is contained in $N_G(Z_p)$. Hence $Z_p \trianglelefteq G$, and so $Z_p = L$, which contradicts (e). Thus the proof is completed.

Proof of Theorem B. Assume that G is a supersoluble group. Then, by Lemma 2.6, we see that $G' \subseteq F(G)$. Let $A = F(G)$ and B be a subgroup of G such that $AB = G$ and $AB_1 \neq G$, for every proper subgroup B_1 of B. Then, evidently, $A \cap B \subseteq \Phi(B)$. Since $AB/A \simeq B/A \cap B$, $B/A \cap B$ is nilpotent and so B is a nilpotent group. Now considering a chief series of G below $F(G)$, say

$$1 = A_0 \leq A_1 \leq \ldots \leq A_{t-1} \leq A_t = A = F(G)$$

Then we can see immediately that this series is also a chief series of A (since $|A_i/A_{i-1}|$ is a prime for all $i = 1, \ldots, t$) and that A_i is permutable with all subgroups of B for all $i = 1, \ldots, t$.

Now we assume that $G = AB$, where A, B are nilpotent subgroups of G and A has a chief series $1 = A_0 \leq A_1 \leq \ldots \leq A_{t-1} \leq A_t = A$ such that every term of which is completely c-permutable with all subgroups of B. We claim that G is a supersoluble group. Suppose that G is not a supersoluble group and let G be a counterexample of minimal order. Without loss of generality, we may assume that $A_{t-1}B \neq G$ and $G \neq AB_1$ for every proper subgroup B_1 of B. First of all, we note that by the well known Theorem of Kegel in [16], G is a soluble group since it is a product of two nilpotent groups. We now divide our proof into the following steps:

(a) *G/N is supersoluble for every normal subgroup $N \neq 1$ of G.*

Clearly, $G/N = (AN/N)/(BN/N)$, where $AN/N \simeq A/A \cap N$ and $BN/N \simeq B/B \cap N$ are nilpotent groups. Consider the series

$$1 = A_0N/N \leq A_1N/N \leq \ldots \leq A_{t-1}N/N \leq A_tN/N = AN/N \quad (2)$$

of AN/N. Without loss of generality, we may assume that all terms of this series are distinct. Obviously, every term of series (2) is completely c-permutable with all subgroups of the group BN/N (see the proof of Theorem A). Since $A \subseteq N_G(A_iN)$, $A_iN/N \trianglelefteq AN/N$. Since $|A_i/A_{i-1}|$ is a prime, $|A_iN/N : A_{i-1}N/N|$ is also a prime. Hence the series (2) is a chief series of AN/N. Thus our hypothesis is true for G/N. But $|G/N| < |G|$, and so G/N is supersoluble.

(b) *G has only one minimal normal subgroup H such that $H = C_G(H) = O_p(G)$, for some prime p, and $|H| \neq p$.*

Let H be a minimal normal subgroup of G. Because the group G is soluble, we know that H is an elementary abelian p-group for some prime p. Since G/H is supersoluble, $|H| \neq p$. Since the class of all supersoluble groups is closed under subdirect products, we know that H is the only minimal normal subgroup of G. Now, by Lemma 2.6, we have $H \nsubseteq \Phi(G)$. Hence, it follows that $H = C_G(H) = O_p(G)$.

(c) *The orders of A and B are not prime.*

Indeed, if $|A| = q$ for some prime q, then G is a totally completely c-permutable product of two supersoluble groups A and B. By Theorem A, we see that G is supersoluble,however, this contradicts to the choice of G, and hence $|A|$ is not a prime. Next, we assume that $|B| = q$ is a prime. Suppose if possible that $q \neq p$. Then $H \subseteq A$. Since A is nilpotent, by (b), we see that A is a p-group. We now claim that $H = A$. Assume that $p > q$. Then $A/H = G_p/H \lhd G/H$ since G/H is supersoluble. But $H = C_G(H)$, by Lemma 2.5, we have $O_p(G/C_G(H)) = 1$, and so $H = A$. On the other hand, suppose that $q > p$. In this case, let $x \in G$ such that $T = A_{t-1}B^x = B^xA_{t-1}$. Since $A_{t-1} \trianglelefteq A$, by Lemma 2.4, $A_{t-1} \subseteq (A_{t-1}B^x)_G$. Hence $H \subseteq T$. It is clear that the hypothesis still holds for T. This means that the group T is supersoluble, and hence $B^x \trianglelefteq T$. It follows that $B^x \subseteq C_G(H) = H$, a contradiction. Therefore $A = H$ and our claim is established. Consequently,H must be a Sylow p-subgroup of G and so B must be a maximal subgroup of G. Now,by our hypothesis, there exists some $x \in G$ such that $BA_1^x = A_1^xB$. Since B is a maximal subgroup of G and $A_1^x \nsubseteq B$, $G = BA_1^x$. This contradicts our assumption on G. Hence $q = p$. By our hypothesis again, we have $A_1B^x = B^xA_1$, for some $x \in G$. Hence $G = AB^x = A(A_1B^x)$. By using Lemma 2.4, we see that $H \subseteq A_1B^x$ and so $H = A_1B^x$ since the order of H is not a prime. Hence, it follows that $A_1 \trianglelefteq A(A_1B^x) = G$, contrary to (b). Thus (c) is proved.

(d) *For every $x \in G$ and all $i = 1, \ldots, t$, the subgroup A_i is completely c-permutable with* *all subgroups of* B^x (see the proof of Theorem A)

(e) *H is a Sylow p-subgroup of G.*

Assume that the assertion is not true and let q be the largest prime divisor of $|G|$. Then, we see that $p \neq q$, and by (b),we have $O_p(G/C_G(H)) = 1$. Let B_1 be a maximal subgroup of B and let $x, y \in G$ so that $A_{t-1}B^x = B^x A_{t-1}$ and $AB_1^y = B_1^y A$. Then,in view of (d), we see that our hypothesis also holds for the groups $A_{t-1}B^x$ and AB_1^y. By (c), A_{t-1} and B_1^y are non-identity groups. Since A, B are nilpotent, $A_{t-1} \trianglelefteq A$ and $B_1^y \trianglelefteq B^y$. Now, by Lemma 2.4, we have $A_{t-1} \subseteq (A_{t-1}B^x)_G$ and $B_1^y \subseteq (AB_1^y)_G$. It follows that $H \subseteq A_{t-1}B^x \cap AB_1^y$. It is clear that either $q \mid |A_{t-1}B^x|$ or $q \mid |AB_1^y|$. Suppose that the first case holds and let Q be a Sylow q-subgroup of $A_{t-1}B^x$. Then, by Lemma 2.6, we have $Q \trianglelefteq A_{t-1}B^x$, and so $Q \subseteq C_G(H) = H$, a contradiction. The second case can be similarly proved. Thus (e) holds.

(f) *$H \nsubseteq A$ and $H \nsubseteq B$.*

Assume that $H \subseteq A$. Because A is nilpotent, A is a p-group, and so by (e), $A = H$ is a Sylow p-subgroup of G. Clearly, $H \nsubseteq B$ and $H \cap B \trianglelefteq G$. Hence $H \cap B = 1$. Let $x \in G$ such that $A_1^x B = BA_1^x$. It is clear that $1 \neq A_1^x = H \cap A_1^x B \trianglelefteq A_1^x B$. But then we have $A_1^x \trianglelefteq G$ and so $A_1^x = H = A_1$. This contradicts (b). Hence $H \nsubseteq A$. Analogously, we can show that $H \nsubseteq B$.

(h) *The final step.*

Let $B_{p'}$ be a Hall p'-subgroup of B. Then we can easily see that $B_{p'} \neq 1$. Now, let x be an element of G such that $T = AB_{p'}^x = B_{p'}^x A$. Since $B_{p'}^x \trianglelefteq B^x$, $B_{p'}^x \subseteq (B_{p'}^x)^G \subseteq AB_{p'}^x$. Hence $H \subseteq AB_{p'}^x$, and so $H \subseteq A$, this contradicts (f). Thus,the proof is completed.

Proof of Theorem C. We first prove that G is supersoluble whenever $G' \subseteq F(G)$. Assume that the assertion is not true and let G be a counterexample of minimal order. Since $G' \subseteq F(G), G$ is soluble. By using the same arguments as in the proof of Theorem A, one can show that $G = [H]M$, where H is the only minimal normal subgroup of G. Moreover, we can see that $H = O_p(G) = C_G(H)$, for some prime p. Since $G' \subseteq F(G)$, we know that G/H is abelian. But then G/H must be a cyclic group because G/H is an irreducible automorphism group of H. Now, by Lemma 2.5, H is a Sylow p-subgroup of G. It is also clear that $|H| \neq p$.

Let G_q be a Sylow q-subgroup of G, where $q \neq p$. Then G_q is a cyclic group. Now, by Lemma 2.3, we have $G_q = A_q^x B_q^y$, for some Sylow q-subgroups A_q of A, B_q of B and some $x, y \in G$.. Hence we have either $G_q = A_q^x$ or $G_q = B_q^x$. Assume that $H \subseteq A$ and $H \subseteq B$, and let, for example, $G_q \subseteq A$. Since $O_{p'}(A) = 1$, we have $H = O_p(A) = F(A)$. Since A is supersoluble by our hypothesis, we have $exp(A/H)|(p-1)$ by Lemma 2.8. Hence $|G_q| \mid (p-1)$. Thus, if $H \subseteq A \cap B$, we can deduce that $|G/H||(p-1)$. This shows that G/H is an abelian group with exponent dividing $p-1$, and by Lemma 2.8, G is supersoluble, which is a contradiction. Hence we have either $H \nsubseteq A$ or $H \nsubseteq B$. Assume that $H \nsubseteq B$. Then,

$H \cap A \neq 1$. Since A is supersoluble, A has a minimal normal subgroup $L \subseteq H$ with $|L| = p$.

Assume that $p||B|$. Let $A_{p'}$ be a Hall p'-subgroup of A. Then, by hypothesis, for some $x \in G$, we have $T = (A_{p'})^x B = B(A_{p'})^x$. Since we have already known from above that if Q is a Sylow q-subgroup of G with $q \neq p$, then either $Q^y \subseteq A$ or $Q^y \subseteq B$ for some $y \in G$.,we have $|G : T| = p^\alpha$, for some $\alpha \in \mathbb{N}$ and so $G = TH$. Let $B_p = B \cap H$. Then, we have $1 \neq B_p \neq H$ and $B_p = H \cap T \trianglelefteq G$, which is impossible. Consequently, $B \cap H = 1$.

Let $D = LA_{p'}$ and $F = BD^x = DB^x$ for some $x \in G$. In this case,by using the same arguments as above, we can prove that $L = H \cap F \trianglelefteq G$. This contradiction completes the proof of the first case.

Now we will prove that G is supersoluble whenever A and B have coprime indices in G. Assume that the assertion is not true and let G be a counterexample with minimal order. Without loss of generality, we may suppose that $A_1B \neq G \neq AB_1$ for all proper subgroups A_1 of A and B_1 of B. We proceed the proof as follows:

(a) *Every supgroup of A is completely c-permutable in G with all subgroups of B^a for all $a \in A$* (see the proof of Theorem A).

(b) *G has an abelian minimal normal subgroup.*

Let L be a minimal normal subgroup of the supersoluble subgroup A. Then, we have $|L| = p$, for some prime p. By hypothesis, $T = LB^a = B^aL$ for some $a \in A$. In view of (a), every subgroup of B^a is completely c-permutable in T with all subgroups of L. Hence by Theorem A, T is supersoluble, and so by the choice of G we have $T \neq G$. Using Lemma 2.4, we see that $L \subseteq T_G$. Therefore (b) holds.

(c) *G has a unique minimal normal subgroup H such that G/H is supersoluble, moreover, $H = O_p(G) = C_G(H)$, for some prime p and $|H| \neq p$* (see the proof of Theorem A).

(d) *The final step.*

Since $(|A|, |B|) = 1$, we have either $H \subseteq A$ or $H \subseteq B$. Without loss of generality, we may assume that $H \subseteq A$. Let L be a minimal normal subgroup of A contained in H. Let $x \in G$ such that $T = L^xB = BL^x$. Then by hypothesis, $p \nmid |B|, L^x = H \cap T \trianglelefteq T$, and so $L^x \trianglelefteq A^xB = G$. Thus $H = L^x$ is a group of order p, which contradicts (c). This contradiction completes the proof.

4 Some Applications

In this section, we give some applications of our main results.

We first prove the following extension theorem of Theorem A.

Theorem 4.1. *Let p be a prime number and $G = AB$ a totally completely c-permutable* *product of two p-supersoluble groups A and B. Then G is p-supersoluble.*

Proof. Assume that the assertion is false and let G be a counterexample of minimal order. Since the hypothesis of the theorem holds for every factor group of G, we may put $O_{p'}(G) = 1$. Also, we assume that for every proper subgroup A_1 of A and every proper subgroup B_1 of B, we have $A_1B \neq G$ and $G \neq AB_1$. We proceed the proof as follows:

(a) G has a non-trivial normal subgroup which is p-soluble.

By using the same arguments as in the proof of Theorem A, we obtain that both subgroups A and B are simple groups.

It is clear that if both subgroups A, B are either p'-groups or p-groups, then G is p-supersoluble, which contradicts the choice of G. Suppose that A is a p'-group and B a p-group. Assume that G is simple group. Since G has a Hall p'-subgroup A, by Corollary 5.3 in [2], we know that G belongs to one of the following types:

(i) A_p with $p \geq 5$ and $A \simeq A_{p-1}$;

(ii) M_{11} with $p = 11$;

(iii) M_{23} with $p = 23$ and $A = M_{22}$;

(iv) $PSL(2, q)$, where either $p = q$ and $A \simeq A_5$ or A is soluble.

By hypothesis, G has a Hall $\{q, p\}$-subgroup containing B for each prime $q \neq p$. Hence by [13], the case (i) is impossible. It is not difficult to see that the cases (ii)-(iv) are also impossible, for example, we just check case (iii). Recall that the order of the Mathieu group $M_{23} \simeq G$ is $2^7 3^2 5 \cdot 7 \cdot 11 \cdot 23$. Let D be a Hall $\{3, p\}$-subgroup of G containing B. Then $B \trianglelefteq D$. Indeed, it is clear that $|D : N_D(B)| \in \{1, 3\}$. Since $|D : N_G(B)| \equiv 1 (\text{mod } 23)$, $|D : N_D(B)| = 3$ is clearly impossible. Hence,we conclude that $3 \nmid |G : N_G(B)|$. Analogously, one can also see that $5, 7, 11, 23 \nmid |G : N_G(B)|$. So $|G : N_G(B)| = 2^\alpha$, where $1 < \alpha < 7$. But this is also impossible, by Theorem 5.8 in [2].

Thus, we have already shown that G is not a simple group. Let H be a minimal normal subgroup of G. If $p \nmid |H|$, then H is p-soluble. Assume that $p \mid |H|$. Then H is a simple group since if otherwise we will have $H = H_1 \times \ldots \times H_t$, where $t > 1$ and $H_1 \simeq \ldots \simeq H_t$ are isomorphic groups.This is clearly impossible. Since A is a Hall p'-subgroup of H, we have $A \cap H$ is a Hall p'-subgroup of H and so $H = (A \cap H)B$ is the totally completely c-permutable product of the groups $A \cap H$ and B. But as we have shown above, H is not a simple group. This contradiction shows that G has a p-soluble minimal normal subgroup, say L. Thus (a) is proved.

(b) For every non-identity normal subgroup D in G, the quotient G/D is p-supersoluble (see the proof of Theorem A).

(c) $L = O_p(G) = C_G(L)$.

Since the class of all p-supersoluble groups is closed under subdirect products, by (b) we see that L is the unique minimal normal subgroup of G. Moreover, since $O_{p'}(G) = 1$, we have $L \subseteq O_p(G)$. Now using the same argument as in the proof of Theorem A, we see that $L = C_G(L) = O_p(G)$.

(d) If $L \leq A$, then $L \leq B$ and conversely.

Assume that $L \leq A$. Since $L \trianglelefteq A$ and A is p-supersoluble, A has a minimal normal subgroup L_1 such that $L_1 \subseteq L$ and $|L_1| = p$. There exists $x \in G$ such that $L_1^x B = BL_1^x$. Then by Dedekind Law, we have $L = L \cap L_1^x B = L_1^x(L \cap B)$. Since $|L| \neq p$, it follows that $L \cap B \neq 1$. Let L_2 be a minimal normal subgroup of B such that $L_2 \subseteq L \cap B$. Then since $L_2 = L_1^{ab} = L_1^b$, for some $a \in A, b \in B$, we have $L_1 \subseteq B$. Now, by using Lemma 2.4, we obtain that $H = (L_1)^G \subseteq B$.

(e) $O_{p'}(A) = 1 = O_{p'}(B)$.

Assume that $O_{p'}(A) \neq 1$. Then $L \nsubseteq A$ and so by (d) $L \nsubseteq B$. But by Lemma 2.4, $L \subseteq O_{p'}(A)B^x$ for some $x \in G$, and so $L \subseteq B^x$. It follows that $L \subseteq B$, a contradiction. Hence (e) is proved.

(f) A and B are supersoluble. (This part follows directly from (e) and Lemma 2.8.)

Thus, we have proved that G is a totally completely c-permutable product of supersoluble groups A and B. Hence G is supersoluble by Theorem A. The proof is completed.

Corollary 4.2[8]. *Let p be a prime number. Assume that $G = AB$ is a totally permutable product of p-supersoluble groups A and B. Then G is p-supersoluble.*

Now, we apply Theorem B to prove the following characterization theorem for p-supersoluble groups. This theorem can be regarded as a generalized theorem of O. H. Kegel [16].

Theorem 4.3. *Let p be a prime and G a soluble group. Then G is p-supersoluble if and only if $G = AB$, where A is p-nilpotent, B is nilpotent and A has a chief series*

$$1 = A_0 \leq A_1 \leq \ldots \leq A_n = O_{p'}(A) \leq A_{n+1} \leq \ldots \leq A_{t-1} \leq A_t = A$$

such that A_i is completely c-permutable (permutable) with all subgroups of B, for all $i = n, \ldots, t$.

Proof. First, we assume that G is p-supersoluble. Then by Lemma 2.8, $G/O_{p'}(G)$ is supersoluble and $G/O_{p',p}(G)$ is an abelian group. Let $A = O_{p',p}(G)$ and B be a subgroup of G such that $AB = G$ and $B_1A \neq G$ for all proper subgroups B_1 of B. Then B is nilpotent. Since $O_{p'}(A) char A \trianglelefteq G$, we have $O_{p'}(A) \triangleleft G$. Hence the group G below A has a chief series

$$1 = A_0 \leq A_1 \leq \ldots \leq A_n = O_{p'}(A) \leq A_{n+1} \leq \ldots \leq A_{t-1} \leq A_t = A$$

passing through $O_{p'}(A)$. This proves the necessity part of the theorem. The sufficiency part can be proved by using Theorem B and the arguments adopted in the proof of Theorem 4.1. We omit the details.

Using Theorem C and applying the same arguments in the proof of Theorem 4.1, we can prove the following theorem for p-supersoluble groups.

Theorem 4.4. *Let p be a prime and $G = AB$ the product of p-supersoluble groups A and B. Assume that A is completely c-permutable will all subgroups of B and B is completely c-permutable with all subgroups of A. If either $G' \subseteq O_{p',p}(G)$ or A and B have coprime orders, then G is p-supersoluble.*

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