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Resolvability of left topological groups

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1. Introduction

All topological spaces under consideration are supposed to be Hausdorff. A topological space X without isolated points is called *maximal* if X has an isolated point in every stronger topology. A topological space X is called *resolvable* (*ω-resolvable*) if X can be partitioned into two (countably many) dense subsets. A topological space X is called *irresolvable* (*ω-irresolvable*) if X is not resolvable (ω-resolvable). Every maximal space is irresolvable.

A topological group G is called *maximal* if G is maximal as a topological space. A maximality is a very exotic phenomenon in the class of topological groups. Every maximal topological group contains a countable open Boolean subgroup (Boolean group is a group of period 2). The examples of maximal topological groups under MA are given in [6], but there exist ZFC-models without maximal topological groups [8].

Comfort and van Mill [2] proved that every nondiscrete topological Abelian group without elements of order 2 is resolvable. This result was strengthened in [10]: every nondiscrete ω-irresolvable topological Abelian group contains a countable open Boolean subgroup. The examples of nonmaximal irresolvable topological groups under MA are given in [16, 17]. There exist ZFC-models in which every nondiscrete topological Abelian group is ω-irresolvable [11]. For another topics concerning the resolvability of topological groups see surveys [1, 9].

A topology on a group G is called *left invariant* if the mapping $x \mapsto gx$ is continuous for every element $g \in G$. A group G endowed with a left invariant topology is called *left topological*. As distinct from the group topologies, every infinite group admits in ZFS a plenty of left invariant topologies that are maximal [13]. Thus resolvability is much more interesting to inverstigate in the class of left topological groups than in the class of topological groups.

By [12, Theorem 1], every nondiscrete irresolvable left topological group is of the first category. Here we improve this result.

Theorem. *Every nondiscrete left group of the second category is ω-irresolvable.*

Note that the examples of nondiscrete irresolvable topological spaces of the second category under some additional set-theoretical assumption are given in [14, 15].

2. Semigroup of ultrafilters

Let G be a discrete group and let βG be the Stone-Čech compactification of G. Identify βG with the set of all ultrafilters on G. Given any subset $A \subseteq G$, put $\bar{A} = \{p \in \beta G : A \in p\}$. Then the family $\{\bar{A} : A \subseteq G\}$ forms an open base of the space βG. Identify G with the subspace of all principal ultrafilters on G and put $G^* = \beta G \setminus G$. Given any filter φ on G, put $\bar{\varphi} = \bigcap_{A \in \varphi} \bar{A}$, $\varphi^* = \bar{\varphi} \bigcap G^*$.

Define the *product* pq of the ultrafilters $p, q \in \beta G$, by the following rule. Given any subset $A \subseteq G$

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

Then pq is an ultrafilter, the multiplication of ultrafilters is associative, and G^* is a subsemigroup of semigroup βG. For more detailed information about semigroup βG and its combinatorial applications see [4, 7].

Let G be a group endowed with a left invariant topology T. Since T is uniquelly determined by the filter τ of all neighbourhoods of the identity of G, then we shall denote the left topological group G by (G,τ). A filter φ on a group G is called *left topological* if φ is a filter of neighbourhoods of identity for some left invariant topology on G. For characterizaions of the left topological filters see [8]. We need only the fact that the subsets $\bar{\tau}$ and τ^* are subsemigroups of semigroup βG for every left topological filter τ on G.

A left topological group (G,τ) is maximal if and only if card $\tau^* = 1$. Moreover, there is a bijection between the set of all maximal left invariant topologies on G and the set of all idempotents of semigroup βG [13].

A filter φ on a left topological group (G,τ) is called an 0-filter if $\tau \subseteq \varphi$ and φ has a base consisting of the open subsets. A filter φ is maximal in the set of all 0-filters on (G,τ) is called an 0-ultrafilter. By Zorn Lemma, every 0-filter is contained in some 0-ultrafilter.

Let (G,τ), be a nondiscrete left topological group and let φ be an 0-ultrafilter on (G,τ). Then (G,τ) is irresolvable if card $\varphi^* = 1$ [13].

Lemma 1. *Let (G,τ) be a left topological group and let φ be 0-ultrafilter on (G,τ). Then (G,τ) is 0-resolvable if and only if the set $\bar{\varphi}$ is infinite.*

Proof. Suppose that (G,τ) is ω-resolvable, but the set $\bar{\varphi}$ is finite, say, $\bar{\varphi} = \{p_1, ..., p_n\}$. Fix any partition $G = \bigcup_{m\in\omega} A_m$ with dense subsets. Since $p_1, ...p_n$ are ultrafilters, we can choose a subset A_k such that $A_k \notin p_1, ..., A_k \notin p_n$. Then $G \setminus A_k \in \varphi$. Since φ is 0-filter, then $G \setminus A_k$ has a nonempty interior, a contradiction.

Assume that $\bar{\varphi}$ is finite. By [11, Lemma 1.1], for every natural number n, (G,τ) can be partitioned into n dense subsets. By [5] (G,τ) is ω-resolvable.

Lemma 2. *Let (G,τ) be a nondiscrete left topological group and let φ be an 0-ultrafilter on (G,τ). If the set $\bar{\varphi}$ is finite then $p\varphi = \varphi$ for every ultrafilter $p \in \bar{\varphi}$.*

Proof. Fix any ultrafilter $p \in \bar{\varphi}$ and denote by ψ the filter on G such that $\bar{\psi} = p\bar{\varphi}$. Since $\bar{\varphi}$ is a subsemigroup of $\bar{\tau}$ then $\varphi \subseteq \psi$. Take any ultrafilter $q \in \bar{\varphi}$. For every subset $P \in p$ and every element $x \in P$ choose any subset $Q_x \in q$. By definition of product of ultrafilters, $\bigcup\{xQ_x : x \in p\} \in pq$ and every element of ultrafilter pq contains a subset of this form. Since $\bar{\varphi}$ is finite then the filter ψ has a base consisting of the subsets of the form $\bigcup\{xH_x : x \in P, H_x \in \varphi\}$. Hence, ψ is an 0-filter, so $\varphi = \psi$.

3. Basic partition

Let G be an uncountable group of cardinality γ with the identity e. Using a minimal well-ordering of G we can construct an increasing chain $\{G_\alpha : \alpha < \gamma\}$ of subgroups of G such that the following conditions are satisfied:

(i) $G_0 = \{e\}$, $G = \bigcup\{G_\alpha : \alpha < \gamma\}$,

(ii) $G_\alpha \subset G_\beta$ for any $\alpha < \beta < \gamma$,

(iii) $G = \bigcup\{G_\alpha : \alpha < \beta\} = G_\beta$ for every limit ordinal $\beta < \gamma$,

(iv) card $G_\alpha < \gamma$ for every $\alpha < \gamma$.

In what follows we fix the filtration $\{G_\alpha : \alpha < \gamma\}$ of G satisfying (i)-(iv). For every ordinal $\alpha < \gamma$, decompose the subset $G_{\alpha+1} \setminus G_\alpha$ on the right cosets by the subgroup G_α and fix some subset X_α of representatives of cosets. Thus $G_{\alpha+1} \setminus G_\alpha = G_{alpha} X_\alpha$.

Take any element $g \in G$, $g \neq e$ and choose the minimal subgroup G_α with $g \in G_\alpha$. By (iii) $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < \gamma$. Then $g \in G_{\alpha+1} \setminus G_{\alpha_1}$ and $g = g_1 x_{\alpha_1}$, $g_1 \in G_{\alpha_1}$, $x_{\alpha_1} \in X_{\alpha_1}$. If $g_1 \neq e$ then choose an ordinal α_2, the elements $g_1 \in G_{\alpha_2}$, $x_{\alpha_2} \in X_{\alpha_2}$ such that $g = g_2 x_{\alpha_2}$. After the finite number $s(g)$ of steps we obtain the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} \dots x_{\alpha_2} x_{\alpha_1},$$

$$\alpha_{s(g)} < \alpha_{s(g)-1 < \dots <} \alpha_2 < \alpha_1, x_{\alpha_1} \in X_{\alpha_1}$$

Note that such representation is unique and put

$$\nu_1(g) = \alpha 1, \nu_2(g) = \alpha 2 \dots, \nu_{s(g)}(g) = \alpha_{s(g)},$$

and $\nu_i(g) = 0$ for every $i > s(g)$. The following Lemma is an elementary observation.

Lemma 3. *Let $g, h \in G$ such that $\nu_1(g) < \nu_k(h)$ for some k. Then $\nu_i(gh) = \nu_i(h)$ for every $i \leq k$.*

Now we define the basic partition $G \setminus \{e\} = \bigcap_{n=1}^{\infty}$ by the rule

$$g \in S_n \Leftrightarrow s(g) = n.$$

Let p be an ultrafilter on G such that $G \setminus \{e\} \in p$. For every natural number n, denote by $\bar{\nu}_n(p)$ the ultrafilter on the segment $[0, \gamma] = \{\alpha : 0 \leq \alpha \leq \gamma$ with the base $\{\nu_n(p) : P \in p, e \notin P$. Since $[0, \gamma]$ is compact in the ordinal topology then every ultrafilter $\bar{\nu}_n(p)$ converges and $lim\nu_{n+1}(p) \leq lim\nu_n(p)$.

An ultrafilter p is called *smooth* if $lim\nu_n(p = \nu)$ for every natural number n.

Lemma 4. *Let (G, τ) be a nondiscrete left topological group and let φ be an 0-ultrafilter on (G, τ). If every ultrafilter $p \in \bar{\varphi}$ is smooth then every subset S_n is nowhere dense in (G, τ).*

Proof. Take any element $g \in G$ and any neighbourhood U of the identity. For every ultrafilter $p \in \bar{\varphi}$, pick a subset $X(p) \in p$ such that $X(p) \subseteq U$ and $\nu_{n+1}(x) > \nu_1(g)$ for every element $x \in X(p)$. By Lemma 3, $gX(p) \bigcup S_n = \emptyset$. Put $V = \bigcap\{X(p) : p \in \bar{\varphi}\}$. Since $V \in \varphi$ then V has s nonempty interior. Clearly, $gV \subseteq gU$ and $gV \bigcup S_n =$. Hence, S_n is nowhere dense.

For every ultrafilter p on G denote $||p|| = min\{cardP : P \in p\}$.

Lemma 5. *Let p be an idempotent of semigroup βG with $||p|| = \gamma$. Then p is smooth.*

Proof. By (iv) $lim\nu_1(p) = \gamma$. Suppose that, for some natural number k, $lim\nu_k(p) = \gamma$ but $lim\nu_{k+1}(p) = \alpha < \gamma$. Pick a subset $P \in p$ such that $\nu_{k+1}(g) \leq \alpha < \nu_k(g)$ for every element $g \in G$. Since $pp = p$ there exist $a \in P$, $Q \in p$ such that $Q \subseteq P$, $aQ \subseteq P$. Pick $b \in Q$ with $\nu_k(b) > \nu_1(a)$. Since $a \in P$ then $a = a_1 a_2$ where $\nu_1(a_1) \leq \alpha < \nu_k(a_2)$. Since $b \in P$ and $\nu_k(b) > \nu_1(a)$ then $b = b_1 b_2$ where $\nu_1(b_1) \leq \alpha$, $\nu_k(a_2) > \nu_1(a_2)$. Since $ab \in P$ then $ab = c_1 c_2$ where $\nu_1(c_1) \leq \alpha < \nu_k(c_2)$. Hence, $a_1 a_2 b_1 b_2 = c_1 b_2$ so $a_1 a_2 b_1 = c_1$ and $a_2 = a_1^{-1} c_1 b_1^{-1}$. Since $\nu_1(a_1) \leq \alpha$, $\nu_1(c_1) \leq \alpha$, $\nu_1(b_1) \leq \alpha$ then $\nu_1(a_2) \leq \alpha$. Hence, $a_2 \in G_{\alpha+1}$, a contradiction with $\nu_1(a_2) > \alpha$.

4. Proof of Theorem

Suppose that (G,τ) is a nondiscrete ω-irresolvable left topological group and show that (G,τ) is of the first category. Let γ be a minimal cardinality of nonempty open subsets of (G,τ). Then (G,τ) has an open subgroup H of cardinality γ. Note that G is a union of left cosets by H and each coset is homeomorphic to H. Hence, if H is of the first category then (G,τ) itself is of the first category. Thus, we may suppose that card $G=\gamma$. For $\gamma=\aleph_0$ the statement is trivial. Therefore, we can use the filtration $\{G_\alpha : \alpha<\gamma\}$ and the basic partition $G\setminus\{e\}=\bigcap_{n=1}^{\infty} S_n$ from section 3.

Prove that each subset S_n is nowhere dense in (G,τ). Let φ be an 0-ultrafilter on (G,τ). By Lemma 1, $\bar{\varphi}$ is finite. To apply Lemma 4 it suffices to show that every ultrafilter $g\in\bar{\varphi}$ is smooth. Note that φ has a base of subset of cardinality γ. Therefore, there exists an ultrafilter $q'\in\bar{\varphi}$ with $\|q'\|$. By Lemma 2, $q'\bar{\varphi}=\bar{\varphi}$. It follows that $\|q\|=\gamma$ for every ultrafilter $q\in\bar{\varphi}$. Since $q\bar{\varphi}=\bar{\varphi}$ for every ultrafilter $q\in\bar{\varphi}$ then the minimal ideal of semigroup $\bar{\varphi}$ coincides with $\bar{\varphi}$. By Rees-Suschkewitsch Theorem [4, Theorem 1.64], $\bar{\varphi}$ is a union of groups. Hence, for every ultrafilter $q\in\bar{\varphi}$ there exists an idempotent $p\in\bar{\varphi}$ such that $qp=q$. By Lemma 5, p is smooth, so q is smooth by Lemma 3.

5. Open problems

Problem 1. Let X be a nondiscrete homogeneous space of the second category. Is X resolvable? ω-resolvable?

Problem 2. Let G be a left topological group of an uncountable pseudocharacter. Is G ω-resolvable? This is so if G is Abelian [12].

Problem 3. Let G be a topological group such that every nonempty open subset of G is uncountable. Is G resolvable?

Abstract. Let G be a group endowed with a nondiscrete Hausdorff topology in which the mapping $x\mapsto gx$ is continuous for every element $g\in G$. Then G can be partitioned into countably many subsets such that either each subset of the partition is dense, or each subset of the partition is nowhere dense.

References

[1] W.W.Comfort, S.Garcia-Ferreira, *Resolvabiliti: a selective survey and some new results,* Topology Appl. 74 (1996), 149–167.

[2] W.W.Comfort, van Mill J., *Groups with only resolvable group topologies,* Proc. Amer. Math. Soc. 120 (1993), 687–696.

[3] A.G.Elkin *Ultrafilters and undecomposable spaces,* Moscow Univ. Math. Bull. 24 (1969), 37–40.

[4] N.Hindmen, D.Strauss, *Algebra in the Stone-Čech compactifications,* De Gruyter Exposition in Mathematics, Vol. 27, 1998.

[5] A.Illanes, *Finite and ω-resolvability,* Proc. Amer. Math. Soc. 124 (1996), 1243–1246.

[6] V.I.Malykhin, *Extremally disconnected and similar groups,* Soviet Math. Dokl. 16 (1975), 21–25.

[7] I.V.Protasov, *Combinatorics of Numbers,* Math. Stud. Monogr. Ser., VNTL Publisher, Vol. 2, 1997.

[8] I.V.Protasov, *Filters and topologies on semigroups,* Matem. Stud., 3 (1994), 15–28 (in Rissian).

[9] I.V.Protasov, *Resolvability of groups,* Matem. Stud., 9 (1998), 130–148 (in Rissian).

[10] I.V.Protasov, *Partitions of direct products of groups,* Ukr. Mat. Zh., 49 (1997), 1386–1395 (in Rissian).

[11] I.V.Protasov, *Irresolvable topologies on groups,* Ukr. Mat. Zh., 50 (1998), 1646–1655 (in Rissian).

[12] I.V.Protasov, *Irresolvable left topological groups,* Ukr. Mat. Zh., 52 (2000), N 6 758–765.

[13] I.V.Protasov, *Maximal topologies on groups,* Sib. Mat. Zh., 39 (1998), 1368–1381 (in Russian).

[14] S.Shelah, *Bear irresolvable spaces and lifting for the layered ideals,* Topology Appl., 33 (1989), 217–221.

[15] S.Shelah, *Iterating forcing and normal ideals on ω_1,* Israel J. Math., 60 (1987), 345–380.

[16] E.G.Zelenyuk, *Topological groups with finite semigroups of ultrafilters,* Matem., Stud., 6 (1996), 41–52 (in Rissian).

[17] E.G.Zelenyuk, I.V.Protasov, *Irresovable and extremally disconnected topologies on groups,* Dokl. NAN Ukr., 1997, N 3, 38–40 (in Rissian).

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