

Formations of finite groups with given properties

S.F.KAMORNIKOV AND L.A.SHEMETKOV

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

Development of the Gaschütz theory of formations made it possible to pose and solve problems of description or enumeration of group classes with given properties. One of such problems, posed by L.A.Shemetkov more than 10 years ago, consists in the following: describe S -closed formations \mathfrak{F} such that every minimal non- \mathfrak{F} -group is either a minimal non-nilpotent group or is of prime order. Local and composition (solubly saturated) formations of this type were studied in [1–4].

It was proved in [5] that a S -closed soluble formation \mathfrak{F} is saturated if every soluble minimal non- \mathfrak{F} -group is either a Schmidt group or a cyclic group of prime order. It was proved in [1] that a S -closed formation is solubly saturated if every minimal non- \mathfrak{F} -group is either a Schmidt group or a cyclic group of prime order. In this article we prove that these results are corollaries of the general theorem.

We consider only finite groups. All group classes considered are non-empty. We use notations from [6, 7]. A group G is called a minimal non- \mathfrak{F} -group if G is not in \mathfrak{F} but all proper subgroups of G are in \mathfrak{F} . We denote by $\mathcal{M}(\mathfrak{F})$ the set of all minimal non- \mathfrak{F} -groups. Groups in $\mathcal{M}(\mathfrak{N})$ are also called Schmidt groups. We say that a Schmidt group A has type (q, p) if $|A| = p^\alpha q^\beta$ and a Sylow q -subgroup of A is normal. \mathfrak{N} is the class of nilpotent groups, \mathfrak{N}_π is the class of nilpotent π -groups (π is a set of primes), \mathfrak{S} is the class of soluble groups. We denote the set of all prime divisors of orders of \mathfrak{F} -groups by $\pi(\mathfrak{F})$. $\mathcal{K}(G)$ denotes the set of all groups which are isomorphic to composition factors of G . A formation \mathfrak{F} is said to be solubly saturated if $G/\Phi(N) \in \mathfrak{F}$ for a soluble normal subgroup N of G always implies $G \in \mathfrak{F}$. We need also the definition of a compositional formation (see [6, 8, 9]). A function $f : \{\text{groups}\} \rightarrow \{\text{formations}, \emptyset\}$ is called a composition satellite if $\mathcal{K}(G_1) = \mathcal{K}(G_2)$ always implies $f(G_1) = f(G_2)$. If $P \neq 1$ is a p -group then we write $f(p)$ instead of $f(P)$. A formation

$$CF(f) = \{G : G/C_G(H/K) \in f(H/K) \text{ for every chief factor } H/K \text{ of } G\}$$

is called a composition formation with a satellite f . Clearly, every local formation is a composition formation. By Baer's theorem ([7], p.373), a formation is a composition formation if and only if it is solubly saturated.

$C^A(G)$ denotes the intersection of centralizers of all chief factors of G which are contained in $D_0(A)$; $C^A(G) = G$ if G has no such chief factors. If $|A| = p$ then $C^A(G) = C^p(G)$. One can notice that $C^p(G)$ coincides with the \mathfrak{E}_{cp} -radical of G where \mathfrak{E}_{cp} is the group class in which every chief p -factor is central.

We formulate in the form of lemmas some famous results that will be used farther on.

Lemma 1 [5]. *Let \mathfrak{F} be an S -closed formation such that $\mathcal{M}(\mathfrak{F}) \cap \mathfrak{S} \subseteq \mathcal{M}(\mathfrak{N}_{\pi(\mathfrak{F})})$. Suppose that \mathfrak{F} contains a Schmidt group of type (q, p) . Then every Schmidt group of type (q, p) belongs to \mathfrak{F} .*

Lemma 2 [10]. *Let H be a simple group in $\mathcal{M}(\mathfrak{S})$. Then $\mathfrak{F} = D_0(\text{form}(H) \cup \mathfrak{S}_{\pi(H)})$ is an S -closed formation.*

Lemma 3 [11]. *Let \mathfrak{X} be a group class. Let $\mathfrak{F} = c\text{form}(\mathfrak{X})$ be the smallest composition formation containing \mathfrak{X} . Then $\mathfrak{F} = CF(f)$ where $f(H) = \emptyset$ for all simple groups H not in*

$\mathcal{K}(\mathfrak{X})$, and $f(H) = \text{form}\{G/C^H(G) : G \in \mathfrak{X}\}$ for every H in $\mathcal{K}(\mathfrak{X})$. Furthermore, f is the minimal composition satellite of \mathfrak{F} .

Theorem 1. Let \mathfrak{F} be an S -closed formation satisfying the following condition: $\mathcal{M}(\mathfrak{F}) \subseteq \mathcal{M}(\mathfrak{N}_{\pi(\mathfrak{F})}) \cup \mathcal{M}(\mathfrak{S})$. Then \mathfrak{F} is a composition formation.

Proof. Let H be a simple non-abelian \mathfrak{F} -group, and all its proper subgroups are soluble. Suppose that $\mathfrak{F} \neq E(H)\mathfrak{F}$, and let G be a group of the smallest order in $E(H)\mathfrak{F} \setminus \mathfrak{F}$. It is clear that $G \in \mathcal{M}(\mathfrak{F})$ and $G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G . We have $G^{\mathfrak{F}} = H_1 \times \dots \times H_n$ where $H_i \simeq H$ for all i . It follows from the assumption of the theorem that $G \in \mathcal{M}(\mathfrak{S})$. If $G^{\mathfrak{F}} = G$, then $G \simeq H \in \mathfrak{F}$. If $G^{\mathfrak{F}} \neq G$, then $G^{\mathfrak{F}} \subseteq \Phi(G)$. The latter contradicts $G^{\mathfrak{F}} \in E(H)$. Thus, the following statement is true:

$$E(H)\mathfrak{F} = \mathfrak{F} \text{ for every non-abelian simple } \mathfrak{F}\text{-group } H \text{ in } \mathcal{M}(\mathfrak{S}). \quad (1)$$

We take again a simple non-abelian \mathfrak{F} -group H which is contained in $\mathcal{M}(\mathfrak{S})$. Assume that there exists a group G in $\mathcal{M}(\mathfrak{S})$ such that $G/\Phi(G) \simeq H$. We want to prove that $G \in \mathfrak{F}$. Let \mathfrak{H} be the set of all S in $\mathcal{M}(\mathfrak{S})$ such that $S/\Phi(S) \simeq H$. Suppose that $\mathfrak{H} \setminus \mathfrak{F} \neq \emptyset$ and G is a group of the smallest order in $\mathfrak{H} \setminus \mathfrak{F}$. It is obvious that $N = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G . Since $H \in \mathfrak{F}$, $N \subseteq \Phi(G)$. Therefore, N is an abelian p -group for some prime $p \in \pi(H)$. Let $|N| = p^n$. Let P be a subgroup of order p in H . Then $N = P_1 \times \dots \times P_n$ where $P_i \simeq P$ for every i . Suppose that $H^* = H_1 \times \dots \times H_n$ where $H_i \simeq H$ for every i . Consider wreath products $W = H^*wr(G/G^{\mathfrak{F}})$ and $W_1 = Nwr(G/G^{\mathfrak{F}})$. Obviously, W contains a subgroup R which is isomorphic to W_1 . By the theorem of Kaloujnine-Krasner, W_1 contains a subgroup that is isomorphic to G . Hence W also contains a subgroup which is isomorphic to G . By statement (1), we have $E(H)\mathfrak{F} = \mathfrak{F}$. Therefore, $W \in \mathfrak{F}$, and $G \in \mathfrak{F}$ because \mathfrak{F} is S -closed.

So, we have proved the following statement:

$$\text{if } G \in \mathcal{M}(\mathfrak{S}) \text{ and } G/\Phi(G) \in \mathfrak{F}, \text{ then } G \in \mathfrak{F}. \quad (2)$$

For each prime p in $\pi(\mathfrak{F})$ we define a set $\pi(p) = \{q \in \pi(G/C^p(G)) : G \in \mathfrak{F}\} \cup \{p\}$. Let f be a composition satellite such that $f(p) = \mathfrak{F}_{\pi(p)}$ for $p \in \pi(\mathfrak{F})$, $f(H) = \mathfrak{F}$ for every non-abelian group H in $\mathcal{K}(\mathfrak{F})$ and $f(H) = \emptyset$ for all simple groups H not in $\mathcal{K}(\mathfrak{F})$. We will show that $\mathfrak{F} = CF(f)$. Clearly, \mathfrak{F} is contained in $CF(f)$. Suppose that $\mathfrak{F} \neq CF(f)$. Let G be a group of minimal order in $CF(f) \setminus \mathfrak{F}$. Since \mathfrak{F} is S -closed, then G is a minimal non- \mathfrak{F} -group having a unique minimal normal subgroup $N = G^{\mathfrak{F}}$. We consider two cases: $N \subseteq \Phi(G)$ and $N \not\subseteq \Phi(G)$.

Let $N \subseteq \Phi(G)$. Then by the condition of the theorem, G is either a Schmidt group or a group in $\mathcal{M}(\mathfrak{S})$. Let G be a Schmidt group. Then $N \subseteq Z(G)$ (see theorem 26.2 in [6]). From this it follows that G/N is a Schmidt group. Now by lemma 1, $G/N \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. So, $G \in \mathcal{M}(\mathfrak{S})$. Then it is clear that $G/N \in \mathcal{M}(\mathfrak{S})$. By (2), $G/N \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. Contradiction.

Let $N \not\subseteq \Phi(G)$. Then by the condition of the theorem, G is either a Schmidt group or is of prime order. If $|G| = p$ is a prime, then from $G \in CF(f)$, $\pi(CF(f)) = \pi(\mathfrak{F})$ and $\mathfrak{F} = S\mathfrak{F}$ it follows that $G \in \mathfrak{F}$. Contradiction.

Let G be a Schmidt group. Since $\Phi(G) = 1$, then $G = [N]Q$ where $|N| = p^\alpha$, $|Q| = q$. Since $G \in CF(f)$, then $G/C_G(N) = G/N \simeq Q \in f(p) = \mathfrak{F}_{\pi(p)}$. Therefore, $q \in \pi(p)$. From the definition of $\pi(p)$ it follows that there exists a group R in \mathfrak{F} such that $R/C^p(R)$ contains a subgroup $Q_1 = M/C^p(R)$ of order q . Let $\{H_i/K_i : i \in I\}$ be the set of all R -chief p -factors

of $C^p(R)$. Then $C = \bigcap_{i \in I} C_R(H_i/K_i)$ contains $C^p(R)$ (we set $C = R$ if $I = \emptyset$). Since $|Q_1| = q$ is prime then either $M \cap C = C^p(R)$ or $M \subseteq C$. Let first $M \subseteq C$. Let $L/C^p(R)$ be an R -chief factor of C . Since L is contained in C , all R -chief p -factors of L are central in L . Clearly, $L = C^p(L)$. Since L is normal in R , L is contained in $C^p(R)$. Contradiction. Now we suppose that $M \cap C = C^p(R)$. It means that Q_1 acts non-trivially on H_j/K_j for some j . Since $|Q_1| = q$ then $C_{Q_1}(H_j/K_j) = 1$. Consider $[H_j/K_j]Q_1$. By lemma 3.32 in [6], $[H_j/K_j]Q_1 \in \mathfrak{F}$. By Maschke's theorem $H_j/K_j = L_1 \times \dots \times L_t$ where L_i is a minimal Q_1 -admissible subgroup of H_j/K_j , $i = 1, \dots, t$. Clearly, there exists $r \in \{1, \dots, t\}$ such that $C_{Q_1}(L_r) = 1$. Let $D = [L_r]Q_1$. Obviously, D is a Schmidt group of type (p, q) . Since D is a subgroup of \mathfrak{F} -group $[H_j/K_j]Q_1$, $D \in \mathfrak{F}$. By lemma 1, every Schmidt group of type (p, q) belongs to \mathfrak{F} . Hence $G \in \mathfrak{F}$. Contradiction.

So, $\mathfrak{F} = CF(f)$. Theorem is proved.

Corollary 1.1 [1]. Let \mathfrak{F} be an S -closed formation such that $\mathcal{M}(\mathfrak{F}) \subseteq \mathcal{M}(\mathfrak{N}_{\pi(\mathfrak{F})})$. Then \mathfrak{F} is a composition formation.

Corollary 1.2 [5]. Every soluble S -closed formation with the condition $\mathcal{M}(\mathfrak{F}) \subseteq \mathcal{M}(\mathfrak{N}_{\pi(\mathfrak{F})})$ is local.

Corollary 1.3. If \mathfrak{F} is an S -closed formation and $\mathcal{M}(\mathfrak{F}) \subseteq \mathcal{M}(\mathfrak{S}_{\pi(\mathfrak{F})})$ then \mathfrak{F} is a composition formation.

Theorem 2. There exists S -closed formation \mathfrak{F} satisfying the following properties:

- 1) $\mathcal{M}(\mathfrak{F}) \cap \mathfrak{S} \subseteq \mathcal{M}(\mathfrak{N}_{\pi(\mathfrak{F})})$;
- 2) \mathfrak{F} is not a composition formation.

Proof. Let $H = PSL(2, 7)$. By lemma 2, $\mathfrak{F} = D_0(\text{form}(H) \cup \mathfrak{S}_{\{2,3,7\}})$ is an S -closed formation. Evidently every soluble minimal non- \mathfrak{F} -group is a group of prime order p not in $\{2, 3, 7\}$. Suppose that \mathfrak{F} is a composition formation, i.e. $\mathfrak{F} = \text{cform}(\mathfrak{F})$. By lemma 3, $\mathfrak{F} = CF(f)$ where $f(H) = \text{form}(G/C^H(G) : G \in \mathfrak{F})$, $f(p) = \text{form}(G/C^p(G) : G \in \mathfrak{F})$ for every $p \in \{2, 3, 7\}$, and $f(R) = \emptyset$ for all R not in $\mathcal{K}(\mathfrak{F})$. Since $C^H(G) = G_{E\mathcal{K}'}$ where \mathcal{K}' is the set of all simple groups not in (H) , then for every group G in \mathfrak{F} we have $G/C^H(G) \in \text{form}(H)$. Since $C^p(G) = G_{\mathfrak{E}_{cp}}$ where \mathfrak{E}_{cp} is the class of groups in which every chief p -factor is central, then for every group G in \mathfrak{F} and every $p \in \{2, 3, 7\}$ we have $G/C^p(G) \in \mathfrak{S}_{\{2,3,7\}}$. It means that $f(p) \subseteq \mathfrak{S}_{\{2,3,7\}}$ for any $p \in \{2, 3, 7\}$.

Suppose that $f(p) \neq \mathfrak{S}_{\{2,3,7\}}$ for some $p \in \{2, 3, 7\}$. Let S be a group of minimal order in $\mathfrak{S}_{\{2,3,7\}} \setminus f(p)$. Then S has a unique minimal subgroup N . Since S is soluble, N is a q -group for some prime q in $\{2, 3, 7\}$. Suppose first that $q \neq p$. Then by the well-known result (see, for example, lemma 18.8 in [12]), there exists faithful irreducible $\mathbb{F}_p S$ -module V . Let $D = [V]S$. Since $D_S = C^p(G) = V$ then from $D \in \mathfrak{F}$ it follows that $D/F_p(G) \simeq S \in f(p)$. Contradiction. Hence $q = p$, i.e. N is a p -group. Let V_1 be a faithful irreducible $\mathbb{F}_r S$ -module for some $r \in \{2, 3, 7\}$, $r \neq p$, and V_2 is a faithful irreducible $\mathbb{F}_p[V_1]S$ -module. Let $F = [[V_2]V_1]S$. Since $F_{\mathfrak{E}_{cp}} = C^p(F) = V_2$, we have $F/V_2 \simeq [V_1]S \in f(p)$. It follows from this that $S \in f(p)$. Contradiction.

So, we proved that $f(p) = \mathfrak{S}_{\{2,3,7\}}$ for every p in $\{2, 3, 7\}$. Thus, $\mathfrak{F} = CF(f)$ where $f(H) = \text{form}(H)$, $f(p) = \mathfrak{S}_{\{2,3,7\}}$ for any $p \in \{2, 3, 7\}$, and $f(R) = \emptyset$ for every simple group R not in $\mathcal{K}(\mathfrak{F})$. Evidently, $SL(2, 7) \in CF(f)$. Therefore, from $\mathfrak{F} = CF(f)$ it follows that $SL(2, 7) \in \mathfrak{F}$. But it is not like this. Theorem is proved.

Резюме. Рассматриваются только конечные группы. Доказано, что непустая S -замкнутая формация разрешимо насыщена, если каждая минимальная не \mathfrak{F} -группа

является либо циклической группой простого порядка, либо группой Шмидта, либо минимальной неразрешимой группой.

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Mathematics Department
Francisk Scorina Gomel State University
246019 Gomel, Belarus

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kamornikov@gsu.unibel.by
shemetkov@gsu.unibel.by