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On Hall p'-subgroups and $\mathfrak{F}$-subnormality in finite p-soluble groups

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

1. Introduction

All groups treated are finite.

In [2] a new embedding property for subgroups in every soluble group was introduced and studied. This concept is associated to a saturated formation $\mathfrak{F}$ and it is called $\mathfrak{F}$-Dnormality (see Definition (2.3)). It is a natural extension of the classical normality, which is recuperated when $\mathfrak{F} = \mathfrak{N}$, the class of all nilpotent groups. More precisely, the $\mathfrak{F}$-Dnormal subgroups are defined in terms of Sylow p-subgroups, for the primes p in $char(\mathfrak{F})$, the characteristic of $\mathfrak{F}$. This allows to introduce the local concept of p'-$\mathfrak{F}$-Dnormality, for every $p \in char(\mathfrak{F})$. As in the case of normality, the p'-$\mathfrak{F}$-Dnormality is not a transitive property. In this note we continue with this study and we define p'-$\mathfrak{F}$-subnormal subgroups for every $p \in char(\mathfrak{F})$ (see Definition (3.1)and Proposition (3.3)). They are characterized in terms of reduction of Hall p'-subgroups in p-soluble groups (Theorem (3.11)). Moreover, we see that p'-$\mathfrak{F}$-subnormality, for every prime $p \in char(\mathfrak{F})$, constitutes a local version of the usual concept of $\mathfrak{F}$-subnormality in the soluble universe (see Corollary (3.14)). Thus we can derive an alternative proof of the characterization of the $\mathfrak{F}$-subnormal subgroups in every soluble group, obtained by M. J. Prentice in [9], in terms of reduction of complement $\mathfrak{F}$-basis, for a subgroup-closed saturated formation $\mathfrak{F}$, with arbitrary characteristic. (See Corollary (3.14)).

Some more results about the behaviour of p'-$\mathfrak{F}$-subnormal subgroups are also obtained.

2. Preliminaires

The reader is assumed to be familiar with the theory of saturated formations of finite groups. The relevant definitions, notations and results can be found in [4].

For the sake of completeness we gather some concepts and results which are needed later.

$\mathfrak{S}$ denotes the class of all soluble groups. If $\sigma \subseteq \mathbb{P}$, the set of all prime numbers, $\mathfrak{S}_\sigma$ denotes the class of all soluble σ-groups. $\mathfrak{N}$ denotes the class of all nilpotent groups.

Henceforth $\mathfrak{F}$ denotes a satured formation and $\pi = char(\mathfrak{F}) = \{p \in \mathbb{P} \ : \ Z_p \in \mathfrak{F}\}$ is the characteristic of $\mathfrak{F}$. The canonical local definition of $\mathfrak{F}$ is denoted by F. We write $\underline{f}$ to identify the smallest local definition of $\mathfrak{F}$.

If H is a subgroup of a group G, $\sigma(|G : H|)$ denotes the set of the prime numbers dividing $|G : H|$. $\langle H^G \rangle$ denotes the normal closure of H in G and p denotes a prime number. If $G_p \in Syl_p(G)$, the set of the Sylow p-subgroups of G, we write $G_p \searrow H$ to mean that G_p reduces in H, i.e., $G_p \cap H \in Syl_p(H)$.

A subgroup H of a p-soluble group G is said to be p'-subnormal in G if every Hall p'-subgroup of G reduces in H, i.e., $G_{p'} \cap H$ is a Hall p'-subgroup of H. We write H p'-snG.

Proposition 2.1. *Let H be a p'-subnormal subgroup of a p-soluble group G, let $H \le K \le G$ and let N be a normal subgroup of G. Then:*

(1) HN/N *p'-sn* G/N.

(2) H p'-sn K.

(3) $O_p(G) \leq N_G(O^p(H))$.

Proof. The proofs of (1) and (2) are straightforward.

(3) We argue by induction on $|G|$. Notice that H is also p'-subnormal in $HO_p(G)$. If $HO_p(G) < G$, the inductive hypothesis ensures that

$$[O_p(G), O^p(H)] \leq [O_p(HO_p(G)), O^p(H)] \leq O^p(H)$$

and we are done. Otherwise, $G = HO_p(G)$. Since H is p'-subnormal in G, in this case we have that $O^p(G) = O^p(H)$ and the conclusion is clear.

Remark 2.2. *Given a saturated formation $\mathfrak{F}$, we recall that $F(p) = \mathfrak{S}_p f(p)$, for every $p \in \pi$, whenever f is an integrated local definition of $\mathfrak{F} = LF(f)$ (see* [4, IV. Proposition (3.8)(a)]). *Then it is easily deduced that $H^{F(p)}$ p'-snG if and only if $H^{f(p)}$ p'-snG, for every subgroup H of a p-soluble group G.*

A maximal subgroup M of a group G is said to be $\mathfrak{F}$-normal in G if $G/Core_G(M) \in \mathfrak{F}$; otherwise, it is called $\mathfrak{F}$-abnormal. A subgroup H of a group G is called $\mathfrak{F}$-subnormal in G if either $H = G$ or there exists a chain $H = H_n < H_{n-1} < \ldots < H_0 = G$ such that H_{i+1} is an $\mathfrak{F}$-normal maximal subgroup of H_i, for every $i = 0, \ldots, n-1$ ([4, III. Definition (4.13); IV. Definition (5.12)]).

2.1. Definition. [2, Definition (3.1)] For a prime $p \in \pi = char(\mathfrak{F})$, a subgroup H of a group G is said to be p'-$\mathfrak{F}$-Dnormal in G, if $\sigma(|G : H|) \subseteq \pi$ and $[H_G^p, H^{f(p)}] \leq H$, where $H_G^p = \langle G_p \in Syl_p(G) : G_p \searrow H \rangle$. In this case, we write H p'-$\mathfrak{F}$-DnG.

A subgroup H of a group G is said to be $\mathfrak{F}$-Dnormal in G if H is p'-$\mathfrak{F}$-Dnormal in G, for every $p \in \pi$. We write H $\mathfrak{F}$-DnG.

In particular, the normal subgroups of a group are exactly the $\mathfrak{N}$-Dnormal subgroups. Obviously p'-$\mathfrak{F}$-Dnormality is not a transitive property. The dihedral group of order 8 is an easy example of this fact.

The following results are also true for arbitrary groups without changes in the proof.

Proposition 2.3. [2, Proposition (3.3)] *Let $p \in \pi = char(\mathfrak{F})$ and let H be a p'-$\mathfrak{F}$-Dnormal subgroup of a group G. Let $H \leq K \leq G$ and let $N \trianglelefteq G$. Then:*

(1) *HN/N p'-$\mathfrak{F}$-DnG/N.*

(2) *H p'-$\mathfrak{F}$-DnK.*

(3) *If $L/N \leq G/N$ and L/N p'-$\mathfrak{F}$-DnG/N, then L p'-$\mathfrak{F}$-DnG.*

Remark 2.4. *If M is a maximal subgroup of a soluble group G, then M is $\mathfrak{F}$-normal in G if and only if M is $\mathfrak{F}$-Dnormal in G.*

Notice that for arbitrary finite groups this result is not true. Take for instance, the saturated formation $\mathfrak{F} = LF(F)$ given by $F(p) = \underline{f}(p) = \mathfrak{S}_{\{2,3,5\}}$, for $p \in \{2,3,5\}$, and $F(q) = \mathfrak{S}_q$, for every $q \in \mathbb{P} \setminus \{2,3,5\}$. Then $Alt(4)$ is $\mathfrak{F}$-Dnormal in $Alt(5)$ but $Alt(4)$ is a maximal subgroup of $Alt(5)$ which is not $\mathfrak{F}$-normal in $Alt(5)$. ($Alt(n)$ denotes the alternating group of degree n).

Proposition 2.5. [2, Proposition (3.5)] *Let H be a subgroup of a soluble group G. Then H is $\mathfrak{F}$-subnormal in G if and only if there exists a chain $H = H_1 \leq H_2 \leq \ldots \leq H_n = G$ such that H_i $\mathfrak{F}$-DnH_{i+1}, for every $i = 1, \ldots, n-1$.*

2.2. Definition. (See [11, Definitions (3.1), (3.7)]) Let G be a soluble group. Let $\Sigma = \{G_{p'} : p \in \mathbb{P}\}$ a complement basis of G, i.e., $G_{p'}$ is a Hall p'-subgroup of G, for every prime p. (Obviously, if p does not divide $|G|$, $G_{p'} = G$).

The complement $\mathfrak{F}$-basis associated to Σ is the set

$$\Sigma_{\mathfrak{F}} = \{G_{p'} \cap G^{f(p)}, G_{q'} \ : \ p \in \pi, q \in \mathbb{P} \setminus \pi\},$$

where f is an integrated local definition of $\mathfrak{F}$.

(By [11, Satz (3.2)], $\Sigma_{\mathfrak{F}}$ does not depend on the choice of an integrated f).

Assume moreover that $f(p)$ is subgroup-closed for every $p \in \mathbb{P}$. Let U be a subgroup of G. We say that the complement $\mathfrak{F}$-basis of G reduces in U, if

$$\Sigma_{\mathfrak{F}} \cap U = \{G_{p'} \cap U^{f(p)}, G_{q'} \cap U \ : \ p \in \pi, \ q \in \mathbb{P} \setminus \pi\}$$

is a complement $\mathfrak{F}$-basis of U.

M. J. Prentice proves in [9] that a subgroup U of a soluble group G is $\mathfrak{F}$-subnormal in G if and only if every complement $\mathfrak{F}$-basis of G reduces in U, when $\mathfrak{N} \subseteq \mathfrak{F}$.

3. p'-$\mathfrak{F}$-Subnormality

3.1. Definition. Let $p \in char(\mathfrak{F})$ and let G be a group. A subgroup H of G is said to be p'-$\mathfrak{F}$-subnormal in G if either $H = G$ or there exists a chain $H = H_0 < H_1 < \ldots < H_n = G$ such that H_i is a p'-$\mathfrak{F}$-Dnormal maximal subgroup of H_{i+1}, for every $i = 0, \ldots, n-1$.

Proposition 3.1. *Let $p \in char(\mathfrak{F})$ and let G be a group. Let $H \leq K \leq G$ and let $N \trianglelefteq G$. Then:*

(1) If H p'-$\mathfrak{F}$-sn$K$ and K p'-$\mathfrak{F}$-sn$G$, then H p'-$\mathfrak{F}$-sn$G$.

(2) If H p'-$\mathfrak{F}$-sn$G$, then HN/N p'-$\mathfrak{F}$-snG/N.

(3) If $N \leq H$ and H/N p'-$\mathfrak{F}$-snG/N, then H p'-$\mathfrak{F}$-sn$G$.

Proof. (1) It is obvious from the definition of p'-$\mathfrak{F}$-subnormal subgroups.

(2) We argue by induction on $|G|$. Obviously we can assume $HN \neq G$. Since H is p'-$\mathfrak{F}$-subnormal in G, there exists X a p'-$\mathfrak{F}$-Dnormal maximal subgroup of G containing H. If $N \leq X$, then HN/N is p'-$\mathfrak{F}$-sn in X/N by the inductive hypothesis and clearly HN/N is p'-$\mathfrak{F}$-sn in G/N. Otherwise, $G = XN$ and consequently $G/N \cong X/(X \cap N)$. By the inductive hypothesis we have that $H(X \cap N)/(X \cap N)$ is p'-$\mathfrak{F}$-subnormal in $X/X \cap N$, but $H(X \cap N)/(X \cap N) \cong HN/N$ which concludes the proof.

(3) It is clear from Proposition (2.4(3)).

Proposition 3.2. *Let $p \in char(\mathfrak{F})$ and let G be a p-soluble group. For a subgroup H of G the following are equivalent:*

(i) H is p'-$\mathfrak{F}$-subnormal in G.

(ii) There exists a chain $H = L_0 \leq L_1 \leq \ldots \leq L_t = G$ such H_i is p'-$\mathfrak{F}$-Dnormal in H_{i+1}, for every $i = 0, \ldots, t-1$.

In particular, p'-$\mathfrak{F}$-Dnormal subgroups are p'-$\mathfrak{F}$-subnormal.

Proof. It is obvious that (i) implies (ii).

For the converse, we suppose that H is p'-$\mathfrak{F}$-DnG and we prove that H satisfies (i). Then the result is clear.

We argue by induction on $|G|$. Taking Proposition (2.4) and Proposition (3.2) into account we can assume that $G = HN$, for every normal minimal subgroup N of G.

If $H \neq G$, then there exists a maximal subgroup M of G containing H.

Let N be a minimal normal subgroup of G.

If N is a p'-subgroup, then M p'-$\mathfrak{F}$-DnG. In this case the result follows again by induction.

If N is a p-group, then H is maximal in G, which implies that H is p'-$\mathfrak{F}$-subnormal in G.

Remark 3.3. *The hypothesis about the p-solubility of the group can not be dispensed with in the above proposition. Take for instance the saturated formation $\mathfrak{F} = LF(F)$ with canonical local definition F given by $F(3) = F(5) = \mathfrak{S}_{\{3,5\}}$ and $F(q) = \mathfrak{S}_q$, for every prime $q \neq 3, 5$. Take $G = Alt(5)$ and H a subgroup of G of order 3. Then $H \in F(5) = \underline{f}(5)$ and so H is 5'-$\mathfrak{F}$-Dnormal in G. But it is not difficult to prove that no maximal subgroup of G containing H is 5'-$\mathfrak{F}$-Dnormal in G.*

Lemma 3.4. *Let $p \in char(\mathfrak{F}) = \pi$ and let H be a maximal subgroup of a group G such that $\sigma(|G : H|) \subseteq \pi$. The following are equivalent:*

(i) H is p'-$\mathfrak{F}$-Dnormal in G.

(ii) If $p \in \sigma(|G : H|)$, then $\langle (H^{\underline{f}(p)})^G \rangle \leq H$.

Proof. It is easy to deduce that (ii) implies (i).

For the converse, we argue by induction on the order of G. Assume that $p \in \sigma(|G : H|)$. Let N be a minimal normal subgroup of G. If $N \leq H$, then the results follows by Proposition (2.4) and the inductive hypothesis. Then we can assume that $G = HN$. Take G_p a Sylow p-subgroup of G which reduces in H. Then $G_p = (G_p \cap H)(G_p \cap N)$. Since $G = \langle H, G_p \rangle$, we have that $G = \langle H, G_p \cap N \rangle$. In particular, $[H, G_p \cap N] \trianglelefteq G$. If $[H, G_p \cap N] = 1$, then $G = H(G_p \cap N)$ and obviously $\langle (H^{\underline{f}(p)})^G \rangle \leq H$, because H is p'-$\mathfrak{F}$-Dnormal in G. Otherwise, the case $1 \neq [H, G_p \cap N] \leq N$ implies that $[H, G_p \cap N] = N$. But for every $h \in H$ and $n \in G_p \cap N$, we have that $[n, h] = n^{-1}n^h \in (G_p \cap N)(G_p^h \cap N) \leq H_G^p$. Consequently, $N \leq H_G^p$. Since H is p'-$\mathfrak{F}$-Dnormal in G, it is clear that $\langle (H^{\underline{f}(p)})^G \rangle \leq H$.

Remark 3.5. *In the hypothesis of the above lemma, if $\mathfrak{F}$ is such that $\underline{f}(p) = (1)$ (for instance, if $\mathfrak{F} = \mathfrak{N}$), then H is p'-$\mathfrak{F}$-Dnormal in G if and only if H satisfies the following property:*

If $p \in \sigma(|G : H|)$, then $H \trianglelefteq G$.

Lemma 3.6. *Let $p \in char(\mathfrak{F})$. If H is a p'-$\mathfrak{F}$-Dnormal maximal subgroup of a p-soluble group G, then $H^{F(p)}$ is p'-subnormal in G.*

Proof. If $p \mid |G : H|$, then $\langle (H^{\underline{f}(p)})^G \rangle \leq H$, by Lemma (3.5). Consequently, $H^{\underline{f}(p)}$ is subnormal in G. In particular, $H^{\underline{f}(p)}$, and also $H^{F(p)}$, are p'-subnormal subgroups of G. If $|G : H|$ is a p'-number, then H, and also $H^{F(p)}$, are p'-subnormal subgroups of G.

Proposition 3.7. *Let $p \in char(\mathfrak{F})$ and suppose that $\mathfrak{F}$ is such that $F(p)$ is subgroup-closed. Let H be a subgroup of a p-soluble group G. If H is p'-$\mathfrak{F}$-subnormal in G, then $H^{F(p)}$ is p'-subnormal in G.*

Proof. Assume that $H < G$ and let $H = H_0 < H_1 < \ldots < H_n = G$ a chain of subgroups in which each H_i is a p'-$\mathfrak{F}$-Dnormal maximal subgroup of H_{i+1}, for $i = 0, \ldots, n-1$. We argue by induction of n, the length of the chain. If $n = 1$, the result holds by the previous lemma. If $n > 1$, since H is p'-$\mathfrak{F}$-subnormal in H_{n-1}, the inductive hypothesis ensures that $H^{F(p)}$ is p'-subnormal in H_{n-1}. Moreover, $(H_{n-1})^{F(p)}$ is p'-subnormal in G, by the previous lemma. But $H^{F(p)} \leq (H_{n-1})^{F(p)}$, because $F(p)$ is subgroup-closed. Consequently, $H^{F(p)}$ is p'-subnormal in G.

Remark 3.8. *The hypothesis on $F(p)$ being subgroup-closed is necessary to obtain Proposition (3.8). Take for instance the saturated formation $\mathfrak{F} = LF(f)$ locally defined by the formation function f given by: $f(p) = (1)$, for every prime $p \neq 2, 3$, and $f(2) = f(3) = (G :$ G is a soluble group whose Carter subgroups are 2-groups). Notice that $F(2)$ is not subgroup closed.*

Let $G = Sym(4)$ be the symmetric group of degree 4 and take $H = \langle(1,2,3)\rangle \leq G$. Notice that the subgroup $K = \langle(1,2,3),(2,3)\rangle$ belongs to $\underline{f}(2)$, which implies that K is $2'$-$\mathfrak{F}$-Dnormal in G. Consequently, the chain $H < K < G$ ensures that H is $2'$-$\mathfrak{F}$-subnormal in G. But $H = H^{F(2)}$ is not $2'$-subnormal in G.

Proposition 3.9. *Let $p \in char(\mathfrak{F}) = \pi$. Let H be a subgroup of a p-soluble group G. If $H^{F(p)}$ is p'-subnormal in G and $|G : H|$ is a π-number, then H is p'-$\mathfrak{F}$-subnormal in G.*

Proof. Arguing by induction on $|G|$, and taking Proposition (2.1) and Proposition (3.2) into account, we can assume that $G = NH$, for every minimal normal subgroup N of G.

If one of these subgroups, N say, is a p'-group, then it is easy to deduce that H is p'-$\mathfrak{F}$-subnormal in G.

If N is a p-group, then H is a maximal subgroup of G with $Core_G(H) = 1$ and G is a primitive group of type 1.

Since $H^{F(p)}$ is p'-subnormal in G, it follows from Proposition (2.1)(3) that $[O_p(G), H^{F(p)}] \leq H^{F(p)}$. This implies that $H^{F(p)} \trianglelefteq G$ and consequently $H \in F(p)$. Moreover, $H \in \underline{f}(p)$, because $O_p(H) = 1$. Therefore H is p'-$\mathfrak{F}$-subnormal in G.

As a consequence of Proposition (3.8) and Proposition (3.10), we can state the following result:

Theorem 3.10. *Let $p \in char(\mathfrak{F}) = \pi$ and suppose that $\mathfrak{F}$ is such that $F(p)$ is subgroup-closed. Let H be a subgroup of a p-soluble group G. Then the following statements are equivalent:*

(i) H is p'-$\mathfrak{F}$-subnormal in G.

(ii) $H^{F(p)}$ is p'-subnormal in G and $\sigma(|G : H|) \subseteq \pi$.

As a consequence of this result, it follows that the p'-$\mathfrak{N}$-subnormal subgroups are exactly the p'-subnormal subgroups. Thus Satz (4.6) of [1], appears now as a particular case. Notice that the p'-$\mathfrak{N}$-subnormal subgroups coincide with the p-subnormal subgroups such as defined in [1, Definition (4.4)] for soluble groups.

Corollary 3.11. *Let $p \in char(\mathfrak{F})$ and suppose that $\mathfrak{F}$ is such that $F(p)$ is subgroup-closed. Let G be a p-soluble group. If $H \leq K \leq G$ and H p'-$\mathfrak{F}$-sn G, then H p'-$\mathfrak{F}$-sn$K$.*

Proof. It follows easily from Theorem (3.11) and Proposition (2.1).

Remark 3.12. (1) *The example in Remark (3.9) also proves that the above corollary does not hold without the hypothesis on $F(p)$. Notice that in this example H is not $2'$-$\mathfrak{F}$-subnormal in $Alt(4)$.*

(2) From Remark (2.2) it follows that $F(p)$ can be replaced by $f(p)$, for any integrated formation f of $\mathfrak{F}$, in Lemma (3.7), Proposition (3.8), Proposition (3.10) and Theorem (3.11).

Corollary 3.13. *Assume that $\mathfrak{F}$ is a subgroup-closed and let H be a subgroup of a soluble group G. The following are equivalent:*

(i) H is $\mathfrak{F}$-subnormal in G.

(ii) H is p'-$\mathfrak{F}$-subnormal in G, for every $p \in char(\mathfrak{F})$.

(iii) Every complement $\mathfrak{F}$-basis of G reduces in H and $\sigma(|G:H|) \subseteq char(\mathfrak{F})$.

Proof. If (i) is assumed, then (ii) is easily deduced taking into account that an $\mathfrak{F}$-normal maximal subgroup of a group is p'-$\mathfrak{F}$-Dnormal in the group, for every $p \in char(\mathfrak{F})$.

The equivalence between (ii) and (iii) follows by Theorem (3.11).

Assumed that (ii) holds. We are proving (i) by induction on $|G|$. Let N be a minimal normal subgroup of G. We can deduce from Proposition (3.2) and the inductive hypothesis that HN is $\mathfrak{F}$-subnormal in G.

If $HN < G$, again the inductive hypothesis and Corollary (3.12) provides that H is $\mathfrak{F}$-subnormal in HN and so H is $\mathfrak{F}$-subnormal in G.

If $G = HN$, then H is maximal in G. Consequently H is $\mathfrak{F}$-subnormal in G.

Remark 3.14. (1) *The above result provide an alternative proof and a slight improvement of the characterization of $\mathfrak{F}$-subnormality proposed by Prentice in [9].*

(2) From Corollary (3.12) and Corollary (3.14) it is easily deduced the following result: If G is a soluble group and $H \leq K \leq G$ such that H $\mathfrak{F}$-sn$G$, then H $\mathfrak{F}$-sn K. (In fact, this result was proved by Förster in [5] for finite groups).

(3) The equivalence between (i) and (ii) in Corollary (3.14) is not true for finite groups. In the example of Remark (2.5), $Alt(4)$ satisfies (ii) in $Alt(5)$ but it is not $\mathfrak{F}$-subnormal in $Alt(5)$.

Theorem 3.15. *Let $p \in char(\mathfrak{F})$ and suppose that $\mathfrak{F}$ is such that $F(p)$ is subgroup-closed. Assume that one the following conditions holds:*

(a) $G = AB$ is a p-soluble group with A and B subgroups of G. Let H be a subgroup of $A \cap B$ such that H is p'-$\mathfrak{F}$-subnormal in A and H is p'-$\mathfrak{F}$-subnormal in B.

(b) G is a p-soluble group and for every prime q dividing $|G|$, there exists a Sylow q-subgroup G_q of G such that the subgroup H is p'-$\mathfrak{F}$-subnormal in $\langle H, G_q\rangle$.

Then H is p'-$\mathfrak{F}$-subnormal in G.

Proof. (a) Assume that the result is not true and let G be a counterexample of minimal order. Consider the pairs (X, Y) of subgroups of G such that $G = XY$, for which there exists a subgroup Z satisfying Z p'-$\mathfrak{F}$- sn$X$, Z p'-$\mathfrak{F}$-sn$Y$ but Z is not p'-$\mathfrak{F}$-subnormal in G. Among all these pairs we choose a pair (A, B) with $|A| + |B|$ maximun. Let H be a subgroup such that H p'-$\mathfrak{F}$-sn$A$, H p'-$\mathfrak{F}$-sn$B$ but H is not p'-$\mathfrak{F}$-sn$G$.

We claim that A and B are maximal subgroups of G. We assume that A is properly contained in a maximal subgroup M of G. Then $M = A(M \cap B)$. By Corollary (3.12) and the choice of G, we conclude that H is p'-$\mathfrak{F}$-subnormal in M. But $G = MB$. Therefore, the choice of the pair (A, B) implies that H is p'-$\mathfrak{F}$-subnormal in G, which is a contradiction.

Let N be a minimal normal subgroup of G. Assume that $N \leq A$. By Proposition (3.2) and the choice of G, we have that HN p'-$\mathfrak{F}$-sn$G$. Since $HN \leq A$, then H is p'-$\mathfrak{F}$-subnormal in HN and consequently H is p'-$\mathfrak{F}$-subnormal in G, a contradiction.

Consequently, we can assume that $Core_G(A) = Core_G(B) = 1$. In particular, $G = AN$.

If N is a p'-group, then A is p'-$\mathfrak{F}$-subnormal in G. Then H is p'-$\mathfrak{F}$-subnormal in G, a contradiction.

Assume that N is a p-group. In this case, G is a primitive group of type 1. Since G is p-soluble, it follows that A and B are conjugated in G, but this contradicts that $G = AB$ and proves (a).

(b) We argue by induction on $|G|$. Let N be a minimal normal subgroup of G. Taking into account Proposition (3.2) and the inductive hypothesis, we can obtain that HN p'-$\mathfrak{F}$-sn$G$. If N is a p-group, then $HN \leq \langle H, G_p \rangle$. By the hypothesis and Corollary (3.12), it follows that H p'-$\mathfrak{F}$-snHN and then H p'-$\mathfrak{F}$-sn$G$. If N is a p'-group, then H p'-$\mathfrak{F}$-snHN and again H p'-$\mathfrak{F}$-sn$G$.

Remark 3.16. (a) *For a subgroup-closed saturated formation $\mathfrak{F}$ and a soluble group, it is easily deduced from Corollary* (3.14) *that Theorem* (3.16) *is also true if the word "p'-$\mathfrak{F}$-subnormal" is replaced by "$\mathfrak{F}$-subnormal".*

For subnormal subgroups in finite groups, these results were considered by Wielandt (see [2, 7.7.1]*) and by Casolo in* [3], *respectively.*

(b) *Theorem* (3.15) *also provides information about p'-subnormal subgroups by taking* $\mathfrak{F} = \mathfrak{N}$.

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Резюме. Рассматриваются только конечные группы. Исследуется обобщение нормальности, связанное с насыщенной формацией. Изучаемые обобщенно нормальные подгруппы характеризуются в терминах редукции холловых p'-подгрупп в p-разрешимых группах. Устанавливается связь с $\mathfrak{F}$-субнормальностью.

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