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The dimension of nilpotent 2-algebras with two generators

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

An associative algebra over the field F of characteristic p for the prime p is called a p-algebra. Nilpotent p-algebras have been studied in several papers (see for instance [3]). Every nilpotent p-algebra R forms a p-group under the "circle operation" $x \circ y = x + y + xy$ for every two elements $x, y \in R$ (see for instance [4]). This group is called the *adjoint group* R° of R. Examples of nilpotent p-algebras are the subalgebras of the algebra of all upper triangular matrices with dimension n over $GF(p)$.

It is well-known that the adjoint group of a nilpotent p-algebra is a p-group. This raises the question which finite p-groups can occur as the adjoint group of a finite nilpotent p-algebra. The metacyclic groups that can occur as the adjoint groups of a finite nilpotent p-algebra are described by Gorlov in [3]. In [2] we classified for odd primes p all finite p-algebras whose adjoint group has at most two generators. Surprisingly the dimension of these algebras is at most 3 (see [2], Theorem 3.6). This is not the case for $p = 2$, since there exists a finite nilpotent 2-algebra with dimension 5 whose adjoint group has two generators (see [2]). Nevertheless we have the following result.

Theorem 20. *Let R be a nilpotent 2-algebra whose adjoint group has only two generators. Then* $\dim R \leq 5$.

The notation is as follows. An algebra L over the field F is a *one-generator algebra* if there exists an element $a \in L$ such that L is the set of all elements of form $af(a)$ for some polynomial $f \in F[x]$. The n-th power of an algebra R is the subalgebra R^n of R generated by the set of elements of the form $x_1 x_2 \dots x_k$ with $k \geq n$, where $x_1, x_2 \dots, x_k \in R$. In particular $R^1 = R$. The natural number $n = n(R)$ such that $R^n \neq 0$ and $R^{n+1} = 0$ is called the *nilpotency class* of R. The subalgebra L of an algebra R generated by the set of elements $x_1, x_2, \dots, x_k$ will be denoted by $\langle\langle x_1, x_2, \dots, x_k \rangle\rangle$ whereas the subspace of the algebra R generated by these elements is $\langle x_1, x_2, \dots, x_k \rangle$. As usual $Ann(R) = \{x \in R \mid xy = yx = 0$ for all $y \in R\}$ is the *annihilator* of R and $Z(R) = \{x \in R \mid xy = yx$ for all $y \in R\}$ is the *center* of R. Multiplication in the algebra R will be denoted by ".", while the multiplication in its adjoint group R° by "$\circ$". The k-th power of the element $x \in R^\circ$ is $x^{(k)}$ and the k-th power of x in R is x^k. Note that if $k = p^m$ for some positive integer m where p is the characteristic of the ground field F, then we have $x^k = x^{(k)}$. The circle commutator of the elements a and b in R° will be denoted by (a, b). Furthermore $d(R)$ is the minimal number of generators of an algebra R and $d(R^\circ)$ is the minimal number of generators of its adjoint group. The other group-theoretical notation is standard.

The following preliminary result can be found in [2], Theorem 4.3.

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Lemma 1. *Let G be the adjoint group of a finite nilpotent 2-algebra R with* $\dim R = 5$. *If* $d(R) = d(G) = 2$ *then*

$$G = \langle a, b \mid a^4 = b^4 = c^2 = 1, (a, b) = c, (a, c) = 1, (b, c) = a^2 \rangle.$$

In this case $G = (\langle a \rangle \times \langle c \rangle) \rtimes \langle b \rangle$, $Z(G) = \langle a^2 \rangle$ *and G has no abelian subgroup of index* 2.

The next result is proved in [1], Theorem 1.3.

Lemma 2. *Let R be a finitely generated nilpotent algebra of nilpotency class $n(R)$ over an arbitrary field F. Then there exists a one-generator subalgebra L in R such that* $\dim L = n(R)$ *if at least one of the following holds*
(i) $\dim R^i/R^{i+1} = 1$ *for some* $i < n(R)$,
(ii) $n(R) > \dim R - k$ *for some integer* $k \geq 0$ *and* $\dim R \geq 2k + 2$.

The above theorem will now be proved in a series of lemmas.

Lemma 3. *Let* $R = \langle\langle x, y \rangle\rangle$ *be a nilpotent algebra over an arbitrary field F. If* $R^i = R^{i-1}x + R^{i-1}y + R^{i+1}$ *for some positive integer i,* $\dim R^i/R^{i+1} = 2$ *and* $R^i \neq R^{i-1}(\lambda x + \mu y)$ *for all* $\lambda, \mu \in F$, *then there exists an element* $u \in R^{i-1}$ *such that* $R^i = \langle ux, uy \rangle + R^{i+1}$. *Moreover* $R^{i-1} = T \oplus \langle u \rangle$ *with* $Tx \subseteq R^{i+1}, Ty \subseteq R^{i+1}$.

Proof. Clearly $\dim(R^{i-1}x + R^{i+1})/R^{i+1} = 1$ and $\dim(R^{i-1}y + R^{i+1})/R^{i+1} = 1$. Hence $R^{i-1} = T_x \oplus \langle u_1 \rangle$, $R^{i-1} = T_y \oplus \langle u_2 \rangle$ with $T_x x \subseteq R^{i+1}$ and $T_y y \subseteq R^{i+1}$. This implies that $R^{i-1} = T + \langle u_1, u_2 \rangle$ with $T = T_x \cap T_y$. If $\dim T \geq \dim R^{i-1} - 1$, then we are done. Suppose that $\dim T = \dim R^{i-1} - 2$ and $\dim\langle u_1, u_2 \rangle = 2$.

Obviously the elements u_1x, u_2x and u_1y, u_2y are linearly dependent. We may choose our notation such that $u_1x = w_1$ and $u_2y = w_2$ are linearly independent. Now we have $u_2x = \lambda w_1, u_1y = \mu w_2$ with some $\lambda, \mu \in F$. We show that $\lambda\mu = 1$. Indeed, $u_1(\alpha x + \beta y) = \alpha w_1 + \beta\mu w_2$ and $u_2(\alpha x + \beta y) = \alpha\lambda w_1 + \beta w_2$. Since the system $\{u_1(\alpha x + \beta y), u_2(\alpha x + \beta y\}$ is linearly dependent for each pair of elements $\alpha, \beta \in F$ then there exists $\gamma \in F$ such that $\gamma(u_1(\alpha x + \beta y)) = u_2(\alpha x + \beta y)$. Therefore $\gamma\alpha w_1 + \gamma\beta w_2 = \alpha\lambda w_1 + \beta w_2$, which implies that $\gamma\alpha = \lambda\alpha$ and $\gamma\beta\mu = \beta$. If $\alpha = \beta = 1$, then $\gamma = \lambda$ and $\gamma\mu = 1$. Thus $\lambda\mu = 1$. Now it follows that $(u_2 - \lambda u_1)x = \lambda w_1 - \lambda w_1 = 0$, $(u_2 - \lambda u_1)y = w_2 - \lambda\mu w_2 = w_2 - w_2 = 0$. Hence $u_2 - \lambda u_1 \in T$ and $\dim T \geq \dim R^{i-1} - 1$ as asserted. Now $u = u_1$ and $ux = w_1, uy = \mu w_2$ are linearly independent. It follows that $\langle ux, uy \rangle + R^{i+1} = R^i$. The lemma is proved.

Lemma 4. *Let* $R = \langle\langle x, y \rangle\rangle$ *be a nilpotent algebra over an arbitrary field F and* $\dim R^i/R^{i+1} \leq 2$ *for some integer* $i > 1$. *Then* $\dim R^j/R^{j+1} \leq 2$ *for each* $j \geq i$.

Proof. Suppose first that $R^i = R^{i-1}x + R^{i+1}$. Then $R^{i+1} = R^i x + R^{i+2}$. If $\dim R^i/R^{i+1} = 1$ then we are done by [1], Lemma 2.2. Hence we may assume that $R^i = \langle w_1, w_2 \rangle \oplus R^{i+1}$. Then we have

$$R^{i+1} = \langle w_1, w_2 \rangle x + R^{i+2} = \langle w_1x, w_2x \rangle + R^{i+2}.$$

This implies $\dim R^{i+1}/R^{i+2} \leq 2$.

Suppose now that $R^i = R^{i-1}x + R^{i-1}y + R^{i+1}$. By Lemma 3 we may assume that $R^{i-1}x + R^{i+1} = \langle ux \rangle + R^{i+1}$ and $R^{i-1}y + R^{i+1} = \langle uy \rangle + R^{i+1}$ for some $u \in R^{i-1}$. Moreover $R^{i-1} = T \oplus \langle u \rangle$ with $Tx \subseteq R^{i+1}, Ty \subseteq R^{i+1}$. It follows that

$$R^{i+1} = R^i x + R^i y + R^{i+2} = Rux + Ruy + R^{i+2}.$$

We may choose $u = u_1 u_0$ with $u_1 \in R^{i-2}, u_0 \in R$ if $i \geq 3$ and $u_1 = 1, u_0 = u$ for $i = 2$. Then

$$Rux = R(u_1)u_0 x \subseteq R^{i-1}u_0 x = (T \oplus \langle u \rangle)u_0 x \subseteq \langle uu_0 x\rangle + R^{i+2}.$$

Clearly $Tu_0 x \subseteq R^{i+2}$ by the definition of T. Therefore $Rux \subseteq \langle uu_0 x\rangle + R^{i+2}, Ruy \subseteq \langle uu_0 y\rangle + R^{i+2}$, which implies $\dim R^{i+1}/R^{i+2} \leq 2$ as claimed. The lemma is proved.

Lemma 5. *Let R be a nilpotent 2-algebra with $\dim R = 6$ and $d(R) = d(R^{\circ}) = 2$. Then $n(R) = 3, \dim R/R^2 = \dim R^2/R^3 = \dim R^3 = 2$ and R° has no cyclic subgroups of order 8*

Proof. Suppose first that H is a nilpotent 2-algebra with $\dim H = 5, d(H) = d(H^{\circ}) = 2$. If $n(H) = 4$ then by Lemma 2 there exists in H a one-generator subalgebra L with $\dim L = 4$. Hence H° has a cyclic subgroup of order 8 which is not the case.

Suppose now that R is a nilpotent 2-algebra over the field F with $\dim R = 6$ such that $d(R) = d(R^{\circ}) = 2$. By the above and Lemma 4 we have only two possibilities:
$\dim R/R^2 = \dim R^2/R^3 = \dim R^3 = 2, R^4 = 0$ or
$\dim R/R^2 = \dim R^2/R^3 = 2, \dim R^3 = \dim R^4 = 1, R^5 = 0$.

Assume that $n(R) = 4$. By Lemma 2 this implies that there exists a one-generator subalgebra L in R such that $\dim L = 4$. If R° has a cyclic subgroup of order 8, the same assertion holds. Suppose that such a subalgebra L exists in R. Let S be a subalgebra of R having L as a subalgebra with codimension 1. By [2], Theorem 2.2, S is either a commutative or $S = \langle\langle x, y\rangle\rangle$ with $\langle\langle y\rangle\rangle = L, x \in S\backslash L$ and $xy = \lambda y^4, yx = \mu y^4$ for some $\lambda, \mu \in F$. It is clear that $J = \langle\langle y^4\rangle\rangle$ is an ideal in R and S/J is a commutative subalgebra in $H = R/J$ which is impossible by Lemma 1. Therefore $n(R) = 3$ and $\dim R/R^2 = \dim R^2/R^3 = \dim R^3 = 2$. By the above R° has no cyclic subgroups of order 8. The lemma is proved.

The theorem now follows from our final lemma.

Lemma 6. *Let R be a nilpotent 2-algebra with $d(R) = d(R^{\circ}) = 2$. Then $\dim R \leq 5$.*

Proof. Suppose that the algebra R is a counterexample for Lemma 6 with minimal dimension. Then $\dim R = 6$ and by Lemma 5 we have $n(R) = 3$. Let $J = R^3$. Then $J \subseteq Ann(R)$ and $J = \langle d_1, d_2\rangle$ for some $d_1, d_2 \in R$. Obviously there are 3 one-dimensional ideals in R, namely $J_1 = \langle d_1\rangle, J_2 = \langle d_2\rangle, J_3 = \langle d_3\rangle$ with $d_3 = d_1 + d_2$. Consider the natural homomorphisms $\psi_i : R \to R/J_i \simeq H$. By Lemma 1 $H^{\circ} \simeq G$ for each $i \leq 3$. Let M be a subgroup of G generated by elements a and c and N be a subgroup of R° such that $\psi_1(N) = M, d = d_1$. The structure of N can be as follows:

(i) $N = \langle c_1\rangle \rtimes \langle a_1\rangle$, a semidirect product with ${c_1}^4 = {a_1}^4 = 1, {c_1}^2 = d$;
(ii) $N = (\langle a_1\rangle \times \langle d\rangle) \rtimes \langle c_1\rangle$ with ${c_1}^2 = d^2 = {a_1}^4 = 1$.

In each case the commutator subgroup of N is contained in $\langle d\rangle$. Let b_1 be an element of R such that $\psi_1(b_1) = b$. Using defining relations for the elements in $G = (R/J_1)^{\circ}$ (Lemma 1) we have ${b_1}^{(-1)} \circ a_1 \circ b_1 = a_1 \circ c_1 \circ d^{\lambda}$ with $\lambda \in \{0, 1\}$ and ${b_1}^{(-1)} \circ c_1 \circ b_1 = {a_1}^2 \circ c_1 \circ d^{\mu}$ with $\mu \in \{0, 1\}$. It follows that

$${b_1}^{-2} \circ a_1 \circ {b_1}^2 = {b_1}^{-1} \circ a_1 \circ c_1 \circ d^{\lambda} \circ b_1 = a_1 \circ c_1 \circ d^{\lambda} \circ c_1 \circ {a_1}^2 \circ d^{\lambda} \circ d^{\mu} = a_1 \circ {c_1}^2 \circ {a_1}^2 \circ d^{\mu},$$

and

$${b_1}^{-2} \circ c_1 \circ {b_1}^2 = c_1.$$

Clearly $\psi_1(\langle a_1^2\rangle) = \langle a^2\rangle = Z(G)$ which implies that $\dim Ann(R/J_1) = 1$ and ${a_1}^2 \in Ann(R)$. Therefore $\langle {a_1}^2, d\rangle = J$. Take $s = {c_1}^2 \circ {a_1}^2 \circ d^{\mu}$ as a generator of an ideal K of R. Note that

$K = J_i$ for some $i \in \{1, 2, 3\}$. Let ψ be the natural homomorphism R onto R/K. It is easy to see that

$$\psi({b_1}^{-2} \circ a_1 \circ {b_1}^2) = \psi(a_1) = \psi(b_1)^{-2} \circ \psi(a_1) \circ \psi(b_1)^2$$

and

$$\psi({b_1}^{-2} \circ c_1 \circ {b_1}^2) = \psi(c_1) = \psi(b_1)^{-2} \circ \psi(c_1) \circ \psi(b_1)^2.$$

Since $R/K = \langle\langle \psi(b_1), \psi(a_1) \rangle\rangle$ it follows that $\psi(b_1)^2 \in Z(R/K)$. On the other hand $\psi(a_1)^2 \in Z(R/K)$ as well. This contradicts Lemma 1. Thus Lemma 6 and so the theorem are proved.

Резюме. Доказано, что размерность нильпотентной 2-алгебры не превышает 5, если её присоединенная группа имеет не более двух порождающих.

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