Известия Гомельского государственного университета имени
Ф.Скорины, 3(16) Вопросы алгебры (2000), 99–100
2000 Mathematics Subject Classification: 20D10

A new property of local formations of finite groups

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

In this paper we introduce a new definition of a $\mathfrak{F}$-normal subgroup and prove that every local formation $\mathfrak{F}$ consists precisely of finite groups in which each Sylow subgroup is either a normal $\mathfrak{F}$-subgroup or else a $\mathfrak{F}$-normal $\mathfrak{F}$-subgroup.

All groups considered are finite, and all group classes are non-empty. Definitions of a formation and a local formation are due to W.Gaschütz [1]. Let $\mathfrak{F} = LF(f)$ be a local formation where $f : \{\text{primes}\} \to \{\text{formations}, \varnothing\}$. By the definition, $\mathfrak{F}$ consists exactly of groups G having a f-central chief series $G = G_0 \supset G_1 \supset \dots \supset G_t = 1$, $t \geq 0$, in which $G/C_G(G_{i-1}/G_i) \in \mathfrak{F}(p)$ for every prime divisor p of $|G_{i-1} : G_i|$. In particular, if $f(p) = (1)$ for every prime p, then $\mathfrak{F}$ coincides with the class $\mathfrak{N}$ of nilpotent groups.

We use standard notations [2, 3]. Recall that a chief factor H/K of G is called f-central in G if $G/C_G(H/K) \in f(p)$ for every prime divisor p of $|H/K|$. A normal subgroup N of G is called f-hypercentral if all its G-chief factors are f-central in G. Following [2] we denote by $Z^f(G)$ the product of all f-hypercentral subgroup of G. $P^G = \langle P^x : x \in G\rangle$, P_G is the product of all normal subgroup of G contained in P.

Lemma 1. *If K is a normal subgroup of G then $Z^f(G) \cap K$ coincides with the product of all f-hypercentral subgroups of G contained in K.*

Proof follows from the Jordan-Hölder theorem and the fact that the product of two f-hypercentral subgroups is f-hypercentral as well.

Definition. Let A be a subgroup of G. We denote by $Z^f(A, G)$ the largest normal subgroup of G such that $A_G \subseteq Z^f(A, G) \subseteq A^G$ and all G-chief factors between A_G and $Z^f(A, G)$ are f-central in G. Set $N_G^f(A) = Z^f(A, G)N_G(A)$. We call A a f-normal subgroup of G if $N_G(A) \neq G$ but $N_G^f(A) = G$. If $\mathfrak{F} = LF(F)$ and $f = F$ is the largest integrated function then we say $\mathfrak{F}$-normal instead f-normal and write $Z^{\mathfrak{F}}(A, G)$ and $N^{\mathfrak{F}}(G)$.

We note that $Z^f(A, G)/A_G$ is f-hypercentral in G. Moreover, by lemma 1, $Z^f(A, G)/A_G$ coincides with $Z^f(G/A_G) \cap A^G/A_G$.

The following lemma is well-known.

Lemma 2. *Let P be a Sylow subgroup of G, and N a normal subgroup of G. Then $N_G(PN) = N_G(P)N$.*

Proof. Evidently, $N_G(PN) \supseteq N_G(P)N$. Let $x \in N_G(PN)$. Then $PN = P^xN$ and $P^x = P^y$ for some $y \in PN$. Therefore $xy^{-1} \in N_G(P)$ and we have $x \in N_G(P)y \subseteq N_G(P)N$.

Lemma 3. *Let A be a subgroup of G and N a normal subgroup of G. Then the following statements hold:*

1) if $N \subseteq A$ then $N_{G/N}^f(A/N) = N_G^f(A/N)$;

2) if A is f-normal in G then AN is either normal or f-normal in G;

3) if A is f-normal in G, then AN/N is either normal or f-normal in G/N.

Proof. Since $N \subseteq A_G$, the statement 1) follows directly from the definition of $N_G^f(A)$. Prove 2). Let A be f-normal in G, i.e. $BN_G(A) = G$ where $B = Z^f(A, G)$, $N_G(A) \neq G$. Suppose that AN is not normal in G. Let H/K be a G-chief factor such that $B \supseteq H \supset K \supseteq A_G$. By the condition, H/K is f-central. Consider

$$HN/KN \simeq H/H \cap KN = H/K(H \cap N).$$

Clearly, this factor is either identity or G-isomorphic to H/K and f-central. Therefore, we have that BN/A_GN is f-hypercentral in G/A_GN. Since $A_GN \subseteq (AN)_G$ and $BN \subseteq A^GN = (AN)^G$, then BN is contained in $Z^f(AN, G)$. Now from $BN_G(A) = G$ and $N_G(A) \subseteq N_G(AN)$ it follows that $Z^f(AN, G)N_G(AN) = G$, i.e. AN is f-normal in G.

Statement 3) follows from 1) and 2). Lemma is proved.

Theorem. *Let $\mathfrak{F} = LF(f)$ be a local formation. A group G belongs to $\mathfrak{F}$ if and only if every Sylow subgroup of G is either a normal $\mathfrak{F}$-subgroup of G or a f-normal $\mathfrak{F}$-subgroup of G.*

Proof. If $G \in \mathfrak{F}$ then $Z^f(G) = G$ and it is clear that every Sylow subgroup of G is either normal or f-normal in G and belongs to $\mathfrak{F}$.

Conversely, suppose that G is a group such that every Sylow subgroup of G belongs to $\mathfrak{F}$ and is either normal or f-normal in G. Evidently, if G is nilpotent then $G \in \mathfrak{F}$. Suppose that G is non-nilpotent. Let N be a minimal normal subgroup of G. By lemma 3, G/N has the same property, i.e. every Sylow subgroup of G/N belongs to $\mathfrak{F}$ and is either normal or f-normal. By induction, $G/N \in \mathfrak{F}$, and we have that $N = G^{\mathfrak{F}}$ is a unique minimal normal subgroup of G. Let P be an arbitrary non-normal Sylow subgroup of G. If N is not contained in P, then $P_G = 1$ and therefore $Z^f(P, G) = Z^f(G) \cap P^G$. Since P is f-normal, $Z^f(P, G)N_G(P) = G$. From this it follows that $Z^f(P, G) \neq 1$, i.e. G has a f-central minimal normal subgroup. Since N is the unique minimal normal subgroup it follows that N is f-central. From this and $G/N \in \mathfrak{F}$ we have $G \in \mathfrak{F}$. Theorem is proved.

Резюме. Рассматриваются только конечные группы. Вводится новое определение $\mathfrak{F}$-нормальной подгруппы и доказывается, что если $\mathfrak{F}$ — локальная формация, то она состоит в точности из всех групп, в которых каждая силовская подгруппа является либо нормальной, либо $\mathfrak{F}$-нормальной $\mathfrak{F}$-подгруппой.

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Received June 1, 2000