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Minimal formations of universal algebras

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

The definition of the formation of algebraic systems was first introduced by L.A.Shemetkov in [1]. L.A.Shemetkov and A.N.Skiba posed a quesion if any finite formation of algebraic systems possesses minimal subformations (problem 5.15 in [3]). We consider that problem for formations of universal algebras belonging to a Mal'cev variety. A variety is called a Mal'cev variety if it consists of algebras in which all congruences are permutable.

All necessary definitions and notations may be found in [3]. Further we consider universal algebras from some fixed Mal'cev variety. Symbol O_A means the least element of the lattice of congruences on A.

Definition. Let d_i be congruences on an algebra A such that $\alpha_1\alpha_2\ldots\alpha_n = \beta$ and $(\alpha_1\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_n)\bigcap\alpha_i = O_A$ for any $i = 1,\ldots,n$. Then we say that β is the direct product of congruences α_i and write $\beta = \alpha_1\times\ldots\times\alpha_n$.

Remind that a non-one-element algebra A is called simple, if it has no congruences, different from trivial ones.

Lemma. *Let an algebra A be the direct product of simple algebras $A_1, A_2, \ldots, A_n$, α_i be the kernel, corresponding to the projection A on A_i, $\beta_i = \alpha_1\bigcap\ldots\bigcap\alpha_{i-1}\bigcap\alpha_{i+1}\bigcap\ldots\bigcap\alpha_n$. Then:*

1) $\alpha_i = \beta_1\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_n$ is the maximal congruence on A;

2) β_i is the minimal congruence on A and $A^2 = \beta_1\times\ldots\times\beta_n$;

3) for non-trivial congruence π on A the following decomposition holds:

$$A^2 = \pi\times\gamma_1\times\ldots\times\gamma_t$$

where $\gamma_i\in\{\beta_1,\ldots,\beta_n\}$.

Proof. Since $A/\alpha_i \simeq A_i$ is a simple algebra, α_i is a maximal congruence on A. It is obvious that there is a permutation j of numbers $1, 2, \ldots, n$ such that equality (1) holds: $\alpha_{ij} = \beta_{j1}\beta_{j2}\ldots\beta_{jn-1}$. That is why it is enough to point out that $\alpha_n = \beta_1\beta_2\ldots\beta_{n-1}$.

$$\beta_1\beta_2\ldots\beta_{n-1} = (\alpha_2\bigcap\ldots\bigcap\alpha_n)\beta_2\ldots\beta_{n-1} =$$

$$= (\alpha_2\bigcap\alpha_3\bigcap\ldots\bigcap\alpha_n)(\alpha_1\bigcap\alpha_3\bigcap\ldots\bigcap\alpha_n)\beta_3\ldots\beta_{n-1} =$$

$$= (\alpha_2(\alpha_1\bigcap\alpha_3\bigcap\ldots\bigcap\alpha_n)\bigcap\alpha_3\bigcap\ldots\bigcap\alpha_n)\beta_3\ldots\beta_{n-1} =$$

$$(\alpha_3\bigcap\ldots\bigcap\alpha_n)\beta_3\ldots\beta_{n-1} = \ldots = \alpha_{n-1}(\alpha_1\bigcap\ldots\bigcap\alpha_{n-2}\bigcap\alpha_n)\bigcap\alpha_n = \alpha_n.$$

2) The factor $A^2/\alpha_i = \beta_i\alpha_i/\alpha_i$ is perspective to the factor $\beta_i/\beta_i\bigcap\alpha_i = \beta_i$. Hence β_i is the minimal congruence on A. Since

$$\beta_1\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_n\bigcap\beta_i = \alpha_i\bigcap\beta_i = O_A,$$

so we have

$$A^2 = \beta_1\times\ldots\times\beta_n.$$

3) $\alpha_1 \bigcap \ldots \bigcap \alpha_n = O_A$. Consequently there is at least one maximal congruence α_i such that $\pi \not\subseteq \alpha_i = \beta_1 \ldots \beta_{i-1}\beta_{i+1} \ldots \beta_n$. Let us mark $\gamma_j = \beta_{ij}$ where $i_1, \ldots, i_t \in \{1, \ldots, n\}$, $\pi\gamma_1 \ldots \gamma_t = A^2$ and $\pi\gamma_1 \ldots \gamma_{i-1}\gamma_{i+1} \ldots \gamma_t \bigcap \gamma_i = O_A$ for any $i = 1, \ldots, t$. Now we show that $\pi \bigcap \gamma_1 \bigcap \ldots \bigcap \gamma_t = O_A$. It is obvious that $\pi \bigcap \gamma_1 = O_A$, and let us suppose that it was proved that $\pi \bigcap \gamma_1 \ldots \gamma_{t-1} = O_A$. Then

$$\pi \bigcap \gamma_1 \ldots \gamma_t \subseteq \pi\gamma_1 \ldots \gamma_{t-1} \bigcap \gamma_1 \ldots \gamma_t =$$

$$= (\pi\gamma_1 \ldots \gamma_{t-1} \bigcap \gamma_t)\gamma_1 \ldots \gamma_{t-1} = \gamma_1 \ldots \gamma_{t-1}.$$

Consequently $\pi \bigcap \gamma_1 \ldots \gamma_t = O_A$ and $A^2 = \pi \times \gamma_1 \times \ldots \times \gamma_t$. Lemma is proved.

Remind that form A means the formation generated by an algebra A. As it is shown in works [1], form $A = HR_0(A)$. Here $H\mathfrak{X}$ denotes the class of all homomorphic images of $\mathfrak{X}$-systems and $R_0\mathfrak{X}$ denotes the class of all isomorphic copies of finite subdirect products of $\mathfrak{X}$-systems.

Theorem. *Let A be a simple algebra, containing one-element subalgebras. Then form A has no proper non-one-element subformations.*

Proof. Let $\mathfrak{F}$ be a non-one-element formation and $\mathfrak{F} \subset$ form A. Then for any algebra $B \in \mathfrak{F}$, we have $B \simeq H_\alpha$, where H is the subdirect product of isomorphic copies of the algebra A, α is a congruence on A. According to Lemma 3.16 [1]

$$H = A_1 \times \ldots \times A_n,$$

where $A_i \simeq A$ for any $i = 1, \ldots, n$. From the lemma it follows that

$$H^2 = \beta_{j1} \times \ldots \times \beta_{jn} = \alpha \times \beta_{jt} \times \ldots \times \beta_{jn}.$$

Denote $\beta_{jn} = \beta$, $\beta_{j1} \ldots \beta_{jn-1} = \gamma$, $\alpha\beta_{jt} \ldots \beta_{jn-1} = \tau$. Then $H^2 = \gamma \times \beta = \tau \times \beta$ and $H/\tau \in \mathfrak{F}$. Let E be a one-element subalgebra of the algebra H. Then $K = \beta E$ is the subalgebra of the algebra H, coinciding with some equivalence class of β on the algebra H. Since $\gamma\beta = \tau\beta = H^2$, we have $\gamma K = \tau K = H$. And since $\gamma \bigcap \beta = \tau \bigcap \beta = O_A$, so we get $H/\gamma = \gamma K/\gamma \simeq K \simeq A$ and $H/\tau = \tau K/\tau \simeq K \in \mathfrak{F}$. Consequently, $A \in \mathfrak{F}$ and $\mathfrak{F} =$ form A. The theorem is proved.

The formation $\mathfrak{F}$ is called minimal if from $\mathfrak{E} \subset \mathfrak{H} \subseteq \mathfrak{F}$ it always follows that $\mathfrak{H} = \mathfrak{F}$, where $\mathfrak{E}$ is the one-element formation of algebras.

Corollary. Any formation $\mathfrak{F}$ of the universal algebras possessing the condition of maximality for congruences and containing one-element subalgebras has the minimal subformations.

Резюме. Рассматриваются универсальные алгебры из фиксированного мальцевского многообразия. Доказано, что формация, порожденнная простой алгеброй, имеющей одноэлементные подалгебры, не имеет собственных неодноэлементных подформаций.

References

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