Известия Гомельского государственного университета имени
Ф.Скорины, 3(16) Вопросы алгебры (2000), 186–187
2000 Mathematics Subject Classification: 20D10, 20E15

Injectors in finite groups

L.A.Shemetkov

Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

All groups considered are finite. Remember that a subgroup V of G is called $\mathfrak{F}$-maximal if $V \in \mathfrak{F}$ and $V \subseteq U \subseteq G$, $U \in \mathfrak{F}$ always implies $V = U$. An $\mathfrak{F}$-injectors of G is a subgroup V of G with the property that $V \bigcap K$ is an $\mathfrak{F}$-maximal subgroup of K for every subnormal subgroup K of G. We denote by $\pi(\mathfrak{F})$ the set of all primes dividing the orders of groups in $\mathfrak{F}$. If $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ are non-empty classes of groups, then $\times_{i=1}^{n} \mathfrak{F}_i = \mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n$ is the class of groups G which are presented in the form $G = G_1 \times \ldots \times G_n$ where $G_i \in \mathfrak{F}_i$. Other notations see in [1].

The aim of this paper is to find the following application of theorem 1 in [2].

Theorem. *Let $\mathfrak{F} = \mathfrak{F}_1 \times \ldots \times \mathfrak{F}_n$ where $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \varnothing$ for $i \neq j$, $\bigcup_{i=1}^{n} \pi(\mathfrak{F}_i) = \mathbb{P}$, and $\mathfrak{F}_i = \mathfrak{F}_i^2$ is a non-empty saturated Fitting formation for any $i = 1, \dots, n$. Then every group G contains an $\mathfrak{F}$-injectors.*

Proof. First we prove that the theorem is true for every group G such that $C_G(G_{\mathfrak{F}}) \subseteq G_{\mathfrak{F}}$. Suppose that $C_G(G_{\mathfrak{F}}) \subseteq G_{\mathfrak{F}}$. We note that by theorem in [3], every subgroup of G contains $\mathfrak{F}_j$-injectors for any i. Let $\mathfrak{H}_i = \times_{j \neq i} \mathfrak{F}_j$, and V_i be an $\mathfrak{F}_i$-injector of $C_G(G_{\mathfrak{H}_i})$. Then, by theorem 1 in [2], $V_1 \dots V_n$ is an $\mathfrak{F}$-injector of G. So theorem is valid for all groups G with $C_G(G_{\mathfrak{F}}) \subseteq G_{\mathfrak{F}}$.

It is not hard to see that $\mathfrak{F}$ is a saturated Fitting formation. Let H be a group such that $H/Z(H) \in \mathfrak{F}$. It is clear that $H = H_1 \times \ldots H_n$ where H_i is a Hall $\pi(\mathfrak{F}_i)$-subgroup of H and $H_i/Z(H_i) \in \mathfrak{F}_i$ for any $i = 1, \dots, n$. Since $\mathfrak{F}_i$ is saturated, it follows, by Gaschütz-Lubeseder-Schmid theorem, that $\mathfrak{F}_i$ is local. Therefore, $\mathfrak{F}_i = LF(f_i)$ and $f_i(p) \neq \varnothing$ for any $p \in \pi(\mathfrak{F}_i)$. We have that each H_i-chief factor of $Z(H_i)$ is f_i-central in H_i, and so $H_i \in \mathfrak{F}$. Hence $H \in \mathfrak{F}$, and so $\mathfrak{F}$ satisfies the condition of the theorem in [4]. Finally, by the theorem in [4], G contains an $\mathfrak{F}$-injectors, as was to be proved.

We recall that $\mathfrak{E}_\pi$ is the class of π-groups, and $\mathfrak{S}_\pi$ is the class of soluble π-groups.

Corollary 1. *Let $\mathbb{P} = \bigcup_{i=1}^{n} \pi_i$, $\pi_i \cap \pi_j = \varnothing$ for $i \neq j$. Let $\mathfrak{F} = \times_{i=1}^{n} \mathfrak{F}_i$ where $\mathfrak{F}_i \in \{\mathfrak{E}_{\pi_i}, \mathfrak{S}_{\pi_i}\}$ for any $i = 1, \dots, n$. Then every group contains an $\mathfrak{F}$-injector.*

Corollary 2. *Let $\mathbb{P} = \bigcup_{i=1}^{n} \pi_i$, $\pi_i \cap \pi_j = \varnothing$ for $i \neq j$. If $\mathfrak{F} = \times_{i=1}^{n} \mathfrak{E}_{\pi_i}$, then every group contains an $\mathfrak{F}$-injector.*

Corollary 3. *Let $\mathbb{P} = \bigcup_{i=1}^{n} \pi_i$, $\pi_i \cap \pi_j = \varnothing$ for $i \neq j$. If $\mathfrak{F} = \times_{i=1}^{n} \mathfrak{S}_{\pi_i}$, then every group contains an $\mathfrak{F}$-injector.*

Corollary 4 [5, 6]. *Every group has nilpotent injectors.*

Remark. A more detailed analysis shows that the Theorem is valid without the condition $\bigcup \pi(\mathfrak{F}_i) = \mathbb{P}$.

Резюме. Пусть $\mathfrak{F} = \mathfrak{F}_1 \times \ldots \mathfrak{F}_n$, $\bigcup_{i=1}^{n} \pi(\mathfrak{F}_i) = \mathbb{P}$ и $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \varnothing$ при $i \neq j$. Доказано, что если $\mathfrak{F} = \mathfrak{F}^2$ есть непустая насыщенная формация Фиттинга для любого $i = 1, \dots, n$, то каждая конечная группа обладает $\mathfrak{F}$-инъектором.

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Received January 10, 2000

Francisk Scorina Gomel State University
246019 Gomel, Belarus

shemetkov@gsu.unibel.by