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On the Gaschütz product of ω-local formations of finite groups

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Dedicated to Professor Wolfgang Gaschütz on the occasion of his 80th birthday

All groups considered are finite.

Recall that *a formation* [1] is a class of groups which is closed under taking homomorphic images and subdirect products.

Let ω be a non-empty set of primes. Every function of the form

$$f : \omega \bigcup \{\omega'\} \to \{\text{group formations}\}$$

is called an *ω-local satellite* [2]. Recall that an ωd-group is a group whose order is divisible by at least one number in ω. Following [2] we use $G_{\omega d}$ to denote the greatest normal subgroup in G whose composition factors are ωd-groups ($G_{\omega d} = 1$ if $\pi(R) \bigcap \omega = \varnothing$ for each minimal normal subgroup R in G). If a formation $\mathfrak{F}$ is such that

$$\mathfrak{F} = (G \mid G/G_{\omega d} \in f(\omega') \text{ and } G/F_p(G) \in f(p) \text{ for all } p \notin \omega \bigcap \pi(G))$$

for some ω-local satellite f, then $\mathfrak{F}$ is called an *ω-local formation.* If $\omega = \{p\}$ then the ω-local formations are called *p-local.* In the other extreme case, $\omega = \mathbb{P}$, ω-local formations are just local formations in the usual sense [1, 3]. A formation $\mathfrak{F}$ is called *ω-saturated* if $\mathfrak{F}$ contains each group G with $G/O_\omega(G) \bigcap \Phi(G) \in \mathfrak{F}$. By Theorem 1 of [4], a formation $\mathfrak{F}$ is ω-local if and only if it is ω-saturated.

Let $\mathfrak{M}$ and $\mathfrak{H}$ be non-empty formations. Then the *Gaschütz product* (see [5]) $\mathfrak{F}\mathfrak{M}$ of the formations $\mathfrak{M}$ and $\mathfrak{H}$ is the class of all groups G with $G^{\mathfrak{H}} \in \mathfrak{M}$.

There are ω-local formations $\mathfrak{F}$ which can be represented as a Gaschütz product $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ of two non-ω-local formations $\mathfrak{M}$ and $\mathfrak{H}$. Indeed, let $\mathfrak{F}$ be the class of all groups each composition factor of which is isomorphic to one of the following groups: A_5, Z_2, Z_3, Z_5 (Z_n is a cyclic group of order n). It is clear that the formation $\mathfrak{F}$ is local. Using some ideas of the Kamornikov and Shemetkov's work [6] we consider the formation $\mathfrak{H}$ which consists of all groups whose composition factors are isomorphic to A_5, and the formation $\mathfrak{M}$ which consists of all groups whose chief factors H/K satisfy one of the following conditions:

(a) composition factors of H/K are isomorphic to A_5;

(b) each factor group of $G/C_G(H/K)$ is not isomorphic to A_5.

It is clear that $\mathfrak{F} = \mathfrak{M}\mathfrak{H}$ (see [6] for details) and $A_5 \in \mathfrak{M}$. From [7, 8] it follows that there is a group E with a normal subgroup N such that $E/N \simeq A_5$, $N \subseteq O_2(E) \bigcap \Phi(E)$ and for some chief factor H/K of the group E we have $C_G(H/K) = N$ and $H \subseteq N$. Hence the formations $\mathfrak{M}$ and $\mathfrak{H}$ are not 2-local.

In connection with those examples we note that the Gaschütz product $\mathfrak{M}\mathfrak{H}$ of any two ω-local formations $\mathfrak{M}$ and $\mathfrak{H}$ is an ω-local formation as well (see Theorem 7 of [2]). Using this result and some observations from [9] we prove the following results:

Theorem 1. *Let $\mathfrak{F} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is a p-local formation for every formation $\mathfrak{H}$ if and only if $\mathfrak{F}$ is a p-local formation and $\mathfrak{N}_p \subseteq \mathfrak{F}$.*

In this theorem $\mathfrak{N}_p$ denotes the class of all p-groups. The symbol $\mathfrak{N}_\omega$ denotes the class of all nilpotent ω-groups (an *ω-group* is a group G with $\pi(G) \subseteq \omega$).

Corollary 1. *Let $\mathfrak{F} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is an ω-local formation for every formation $\mathfrak{H}$ if and only if $\mathfrak{F}$ is an ω-local formation and $\mathfrak{N}_\omega \subseteq \mathfrak{F}$.*

Corollary 2 ([10], Theorem I). *Let $\mathfrak{F} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is a saturated formation for every formation $\mathfrak{H}$ if and only if $\mathfrak{F}$ is a saturated formation of full characteristic (i.e. a group of order p belongs to $\mathfrak{F}$ for all $p \in \mathbb{P}$).*

Theorem 2. *Let $\mathfrak{H} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is a p-local formation for every formation $\mathfrak{F}$ if and only if $\mathfrak{H}$ is the class of all groups.*

Corollary 3. *Let $\mathfrak{H} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is a ω-local formation for every formation $\mathfrak{F}$ if and only if $\mathfrak{H}$ is the class of all groups.*

Corollary 4 ([10], Theorem II). *Let $\mathfrak{H} \neq (1)$ be a formation. Then the Gaschütz product $\mathfrak{F}\mathfrak{H}$ is a saturated formation for every formation $\mathfrak{F}$ if and only if $\mathfrak{H}$ is the class of all groups.*

Резюме. Доказано, что произведение формаций $\mathfrak{F}\mathfrak{H}$ p-локально для любой формации $\mathfrak{H}$ тогда и только тогда, когда $\mathfrak{F}$ p-локальна и $\mathfrak{N}_p \subseteq \mathfrak{F}$.

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