Известия Гомельского государственного университета
имени Ф.Скорины, 1(16), 2003

УДК 512.542

On Weakly *c*-Normal Subgroups of Finite Groups

Lujin Zhu, Long Miao, Wenbin Guo

1. Introduction

In [8], Wang initiated the concept of c-normal subgroups and using the c-normality of maximal subgroups to give some conditions for the solvability and supersolvability of a finite group. In [10], The authors replaced the normal condition with the subnormal condition in the concept of c-normal subgroups, introduced the concept of weakly c-normal subgroup and using the weakly c-normality of some subgroups to determine the structures of some groups. In this paper, we shall continue to study the weakly c-normality of some subgroups of a group G. Some theorems of solvable groups, p-nilpotent groups and p-supersolvable groups are obtained by considering weakly c-normal subgroups.

Throughout this paper, all groups are finite groups. For notations and terminologies not given in this paper, the reader is referred to [2] and [7].

2. Basic Definitions and Preliminary Results

Let π be a set of prime numbers and π' the complement of π in the set of all prime numbers. Let G be a finite group. We denote by $|G|$ the order of G. We write $H \lhd\lhd G$ to indicate that H is a subnormal subgroup of G, and write $M < \cdot G$ to indicate that M is a maximal subgroup of G.

Definition 2.1 [10]. *Let G be a group. A subgroup H of G is called weakly c-normal in G if there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \leq H_G$, where $H_G = \cap_{g\in G} H^g$ is the largest normal subgroup of G contained in H.*

In [10], the authors gave some examples to show that the property of weakly c-normality cannot imply c-normality.

Definition 2.2. *We say a group G weakly c-simple if G has no weakly c-normal subgroup except the identity group 1 and G.*

Let G be a group. We consider the following families of subgroups:

$\mathfrak{F}_c(G) = \{M \mid M < \cdot G$ *with* $|G : M|$ *is composite* $\}$.

$\mathfrak{F}^p(G) = \{M \mid M < \cdot G, N_G(P) \leq M$ *for a* $P \in Syl_p(G)\}$.

$\mathfrak{F}^s(G) = \cup_{p\in\pi(G)} \mathfrak{F}^p(G)$.

$\mathfrak{F}^{sc}(G) = F^s(G) \cap \mathfrak{F}_c(G)$.

and define

$\mathfrak{S}^s(G) = \cap\{M \mid M \in \mathfrak{F}^{sc}(G)\}$ *if* $\mathfrak{F}^{sc}(G)$ *is non-empty; otherwise* $\mathfrak{S}^s(G) = G$.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 [10, Lemma 2.1]. *Let G be a group, then the following statements hold.*

(1) Let H be a subgroup of G. Then H is weakly c-normal in G if and only if there exists a subnormal subgroup N of G such that $G = HN$ and $H \cap N = H_G$.

(2) If H is normal or c-normal in G, then H is weakly c-normal in G.

(3) G is weakly c-simple if and only if G is simple.

(4) If H is weakly c-normal in G and $H \leq M \leq G$, then H is weakly c-normal in M.

(5) Let $K \trianglelefteq G$ and $K \leq H$. Then H is weakly c-normal in G if and only if H/K is weakly c-normal in G/K.

(6) Let H be a π-subgroup of G and N a normal π'-subgroup. If H is weakly c-normal in G, then HN/N is weakly c-normal in G/N. Furthermore, if $N \leq N_G(H)$, then the converse also holds.

Lemma 2.2 [8]. *Let G be a finite group. Then G is supersolvable if and only if $G = \mathfrak{S}^s(G)$.*

Lemma 2.3 [10, Corollary 3.2]. *A group G is solvable if and only if every maximal subgroup of G is weakly c-normal in G.*

Lemma 2.4 [3, Theorem VI. 9.9]. *let G be a solvable group. Suppose there exists a normal series:*

$$\Phi(G) = K_0 \leq K_1 \cdots \leq K_n = F(G)$$

such that $K_i \trianglelefteq G$ and $|K_i/K_{i-1}|$ is a prime number, $1 \leq i \leq n$. Then G is supersolvable.

Lemma 2.5 (1) [5, 10.1.9]. *Let G be a group and p the smallest prime dividing of $|G|$. If G has a cyclic Sylow subgroup, then G is p-nilpotent.*

(2) [9, II.4.6]. *Let G be a group and p the smallest prime divisor of $|G|$. If $H \leq G$ and $|G : H| = p$, then $H \trianglelefteq G$.*

Lemma 2.6. *Let R be a solvable minimal normal subgroup of G and $R_1 < \cdot R$. If R_1 is weakly c-normal in G, then R is a cyclic group of prime order.*

Proof. Since R_1 is weakly c-normal in G, there exists a subnormal subgroup K of G such that $G = R_1K$ and $R_1 \cap K = (R_1)_G = 1$. Hence $R = R \cap (R_1K) = R_1(R \cap K)$ and $R_1 \cap (R \cap K) = 1$. Since R is abelian, $R \cap K \trianglelefteq RK = G$. so $R \cap K = 1$ or $R \cap K = R$ by the minimal normality of R in G. If $R \cap K = 1$, then $R = R_1$, a contradiction. Hence $R \cap K = R$. It follows that $R \leq K$ and $R_1 = R_1 \cap K = 1$. This means that $|R|$ is a prime.

3. Main Results

Lemma 3.1. *Let P be a Sylow p-subgroup of a finite group G. If P is a weakly c-normal subgroup of G, then G is p-soluble group.*

Proof . By hypotheses and Lemma 2.1, there exists $H \lhd\lhd G$ G such that $G = PH$ and $H \cap P = P_G$. If $P_G \neq 1$, then G/P_G satisfies the hypotheses and G/P_G is p-soluble by induction. So G is p-soluble since P_G soluble. If $P_G = 1$, then H is Hall p'-subgroup. Since H is subnormal in G, then $H \lhd G$. It implies that G is p-soluble.

Theorem 3.2. *Let G be a finite group. Then G is solvable if and only if M is weakly c-normal in G for every non-nilpotent maximal subgroup M in $\mathfrak{F}^{sc}$.*

Proof. Since the necessity part is straightforward by Lemma 2.3. We only need to prove the sufficient part. For this purpose, we suppose that the theorem is not true and let G be a minimal counterexample. If $\mathfrak{F}^{sc} = \varnothing$, then $G = S^s(G)$ is supersoluble by Lemma 2.2, a contradiction. Now assume that $\mathfrak{F}^{sc} \neq \varnothing$ and $M \in \mathfrak{F}^{sc}$. If M itself is nilpotent, then, by the well-known Tompson's Theorem [5, Theorem 10.4.2], M must be a group of even order. By [6, Theorem 1], $M_{2'}$ (the $2'$-Hall subgroup of M) is a normal subgroup in G. Trivially, the hypotheses of the theorem is quotient closed. If $M_{2'} \neq 1$, then by the minimality of G and Lemma 2.1, we have that that $G/M_{2'}$ is solvable and consequently G is solvable since $M_{2'}$ is nilpotent. Hence we may assume that $M_{2'} = 1$ and therefore M is a Sylow 2-subgroup of G if $M \in \mathfrak{F}^{sc}$ and M is nilpotent.

Let p be the largest prime of $\pi(G)$ and P is a Sylow p-subgroup of G. The choice of G implies that $N_G(P) < G$. Hence, there exists a maximal subgroup L of G such that

$N_G(P) \leq L$. If $L_G \neq 1$, then G is of course not simple. Now assume that $L_G = 1$. If $[G : L] = q$ is a prime, then, $G = G/L_G$ is isomorphic to a subgroup of S_q, where S_q is the symmetric group of degree q. Thus, $|G| \mid q!$, and q is the largest prime in $\pi(G)$. It follows that $p = q$. Hence $[G : L]$ is not divided by p since $P \leq L$, which is a contradiction. If $[G : L]$ is composite, then since L is not nilpotent by the above proof, we have, by our hypotheses, there exists a subnormal subgroup K of G such that $G = LK$ and $L \cap K = L_G$. This implies that G is not simple. By using induction and in virtue of the fact that if there are two minimal normal subgroups N_1 and N_2 of G , then G can be embedded in $G/N_1 \times G/N_2$. We can easily see that G has a unique minimal normal subgroup N and G/N is solvable. Obviously, N is not solvable and $C_G(N) = 1$.

Let q be the largest prime of $\pi(N)$ and $Q_1 \in Syl_q(N)$. By our choice of G, we have $Q_1 < N$ and Q_1 is not normal in G. Hence there exists a maximal subgroup L of G such that $N_G(Q_1) \leq L$. By using Frattini argument, we have $G = NN_G(Q_1) = NL$. Now, consider $Q \in Syl_q(G)$ with $Q_1 = Q \cap N$. Then, for any $x \in N_G(Q)$, we have $Q_1^x = (Q \cap \cap N)^x = Q \cap N = Q_1$. It follows that $N_G(Q) \leq N_G(Q_1) \leq L$. If $[G : L] = r$ is a prime, then, since $L_G = 1$, we have $G = G/L_G$ is isomorphic to a subgroup of S_r, the symmetric group of degree r. This shows that $|G| \mid r!$ and r is the largest prime of $\pi(G)$, thereby we obtain $r = p$. As $[G : L] = [N : N \cap L]$, it leads to p is a prime factor of $|N|$, and hence $p = q$. By $N_G(Q) \leq N_G(Q_1) \leq L$ together with $Q \in Syl_q(G)$, we infer that q is not a factor of $|G : L|$, in contradiction to that $[G : L] = q$. On the other hand, if $|G : L|$ is composite, then $L \in \mathfrak{F}^{sc}$. If L is nilpotent, then, by the above proof, L is a Sylow 2-subgroup of G. Hence $q = 2$ and thereby N must be a 2-subgroup, contradiction to $Q_1 < N$. This shows that L must be a non-nilpotent group and $L \in \mathfrak{F}^{sc}$. However, by our hypotheses, there exists a subnormal subgroup K of G such that $G = LK$ and $L \cap K = L_G = 1$. It follows that $|G : L| = |K|$. Let A is a minimal subnormal subgroup of G contained in K. It is clear that A is a simple group. If $A \not\subseteq N$, then $A \cap N = 1$. But $N \subseteq N_G(A)$ by [1, Theorem A; 14.3], so $NA = N \times A$. Hence $A \subseteq C_G(N) = 1$. This contradiction shows that $A \subseteq N$. Since $N = A_1 \times A_2 \times \cdots \times A_t$, where $A_1 \simeq A_2 \simeq \cdots \simeq A_t$ is a simple nonabelian group, and since $A \lhd\lhd N$, we have $A \in \{A_1, A_2, \cdots, A_t\}$. We may assume that $A = A_1$ and P_1 be a Sylow q-subgroup of A_1. Then $|Q_1| = |P_1|^t$. Hence q divides $|A_i|$. But then q divides $|K| = |G : L|$. This contrary to that $N_G(Q) \leq L$. Therefore, there is no counterexample and this completes our proof.

Theroem 3.3. *Let G is a group and p is the smallest prime dividing the order of G. If all the maximal subgroups of every Sylow p-subgroup are weakly c-normal in G, then G is p-nilpotent.*

Proof. Let P be a Sylow p-subgroup of G and P_1 a maximal subgroup of P. If $P_1 = 1$, then P is a cyclic group and by Lemma 2.5(1), G is p-nilpotent. Now, we assume that $P_1 \neq 1$. By our hypotheses and by Lemma 2.1, we know that there exists a subnormal subgroup M of G such that $G = P_1M$ and $P_1 \cap M = (P_1)_G$. It follows that $P = P_1(P \cap M)$ and $P \cap M$ is a Sylow p-subgroup of M. It is clear that $|(P \cap M)/(P_1)_G| = p$.

Case 1. If $(P_1)_G = 1$, then $|P \cap M| = p$. Hence M is p-nilpotent by Lemma 2.5(1). Let $M_{p'}$ be the normal Hall p'-subgroup of M, then $M_{p'}$ char M. It is clear that $M_{p'}$ is a Hall p'-subgroup of G as well. But $M_{p'} \lhd G$ since $M_{p'}$ char $M \lhd\lhd G$. So G is p-nilpotent.

Case 2. If $(P_1)_G \neq 1$, then $|P \cap M : (P_1)_G| = p$. So $M/(P_1)_G$ is p-nilpotent by Lemma 2.5(1). Let $H/(P_1)_G$ be the normal Hall p'-subgroup of $M/(P_1)_G$, then we have $H \trianglelefteq M$ and $(P_1)_G$ is Sylow p-subgroup of H. Also by Schure-Zassenhaus theorem there exists a Hall p'-subgroup K of H. It is clear that K is also a Hall p'-subgroup of G. By Frattini argument, we arrive that $M = HN_M(K) = (P_1)_GN_M(K)$ and hence $G = P_1N_G(K) = PN_G(K)$. Therefore,

$N_P(K) = P \cap N_G(K)$ is a Sylow p-subgroup of $N_G(K)$. If $|G : N_G(K)| = |P : N_P(K)| \geq p^2$, then we can let P_2 be a maximal subgroup of P such that $N_P(K) < P_2$. By our hypotheses and Lemma 2.1, we know that there exists a subnormal subgroup M_1 of G such that $G = P_2M_1$ and $M_1 \cap P_2 = (P_2)_G$. It follows that $P = P_2(P \cap M_1)$ and $P \cap M_1$ is a Sylow p-subgroup of M_1. It is clear that $|(P \cap M_1)/(P_2)_G| = p$. We know that $M_1/(P_2)_G$ is p-nilpotent since p is the smallest prime. Let $H_1/(P_2)_G$ be the normal Hall p'-subgroup of $M_1/(P_2)_G$. Then we have $H_1 \trianglelefteq M_1$ and $(P_2)_G$ is a Sylow p-subgroup of H_1, same above proof, we have $M_1 = (P_2)_G N_{M_1}(K_1)$, where K_1 is a Hall p'-subgroup of G. Observe the following group series

$$1 \leq (P_1)_G < H < M < G$$

It is clear that the above series is a subnormal series and every factor in the series is either a p-subgroup or p'-subgroup, hence G is p-solvable. Thus, there exists $g \in P$ such that $K_1^g = K$ and consequently $N_G(K_1)^g = N_G(K)$. Then, $G = P_2N_G(K_1) = P_2N_G(K_1)^g = P_2N_G(K)$ since P_2 is normal in P. It follows that $P = P_2(P \cap N_G(K)) = P_2N_P(K)$. But $N_P(K) < P_2$ and therefore $P = P_2$, a contradiction. Thus, we obtain $|G : N_G(K)| = |P : N_P(K)| \leq p$. Suppose that $|G : N_G(K)| = p$. Then, $N_G(K)$ must be normal in G by Lemma 2.5(2). It follows that $K \trianglelefteq G$ and therefore $[G : N_G(K)] = 1$, a contradiction. This shows that $K \triangleleft G$ and G is p-nilpotent. The proof is complete.

Theorem 3.4. *Assume that G is solvable and every maximal subgroup of Sylow subgroups of $F(G)$ is weakly c-normal in G. Then G is supersolvable.*

Proof. We prove the theorem by induction on $|G|$. We distinguish the following two cases

Case 1. $\Phi(G) \neq 1$. Then there exists a prime p such that $p \mid |\Phi(G)|$. Since $\Phi(G) \leq F(G)$, it follows that $p \mid |F(G)|$. Let P_1 be a Sylow p-subgroup of $\Phi(G)$. Since P_1 char $\Phi(G) \trianglelefteq G$, we have that $P_1 \trianglelefteq G$ and $F(G/P_1) = F(G)/P_1$. Let P_2/P_1 be a maximal subgroup of the Sylow p-subgroup of $F(G)/P_1$. Then P_2 is a maximal subgroup of the Sylow p-subgroup of $F(G)$. By hypotheses P_2 is weakly c-normal in G, we have that P_2/P_1 is weakly c-normal in G/P_1 by Lemma 2.1. Let $(Q_2P_1)/P_1$ be a maximal subgroup of the Sylow q-subgroup of $F(G)/P_1$ $(p \neq q)$. Then Q_2 is a maximal subgroup of the Sylow q-subgroup of $F(G)$. Q_2 is weakly c-normal in G by hypotheses. We have that $(Q_2P_1)/P_1$ is weakly c-normal in G/P_1 by Lemma 2.1. Hence G/P_1 is supersolvable by induction on the order of G. Since $(G/P_1)/(\Phi(G)/P_1) \cong G/\Phi(G)$, we have $G/\Phi(G)$ is supersolvable.

Case 2. $\Phi(G) = 1$. Let P be a Sylow subgroup of $F(G)$. Since P char $F(G) \trianglelefteq G$, we have that $P \trianglelefteq G$ and so $\Phi(P) \leq \Phi(G) = 1$. Hence $\Phi(P) = 1$ for every Sylow subgroup P of $F(G)$.

Since G is solvable and $\Phi(G) = 1$, then $F(G) = R_1 \times R_2 \times \cdots \times R_m$, where R_i are (elementary abelian) minimal normal subgroup of G. Clearly, we may assume that $R_1 \leq P$, where P is Sylow p-subgroup of $F(G)$. We shall now prove that $R_i's(i = 1, 2, \cdots m)$ are all cyclic groups.

If $R_1 = P$, then there exists a maximal subgroup P_1 of $R_1 = P$. By hypotheses, P_1 is weakly c-normal in G. By Lemma 2.6, R_1 is a cyclic group of prime order.

If $R_1 < P$, we can assume that $P = R_1 \times R_2 \cdots \times R_t (t \leq m)$, where $R_i (i = 1, 2, \cdots, t)$ are minimal normal p-subgroup of G. Let R_{11} be a maximal subgroup of R_1 , then $R_{11} \times R_2 \times \times \cdots R_t = P_2$ is a maximal subgroup of P, and so P_2 is weakly c-normal in G by hypotheses. We let T denote the normal subgroup $R_2 \times \cdots \times R_t$ of G, then $P_2 = R_{11}T$. We claim that $(P_2)_G = T$. In fact, it is clear that $T \leq (P_2)_G$. If $(P_2)_G > T$, then $(P_2)_G \cap R_{11} > 1$ by $(P_2)_G = (P_2)_G \cap P_2 = (P_2)_G \cap (R_{11}T) = T((P_2)_G \cap R_{11})$. Hence we have that $1 < (P_2)_G \cap \cap R_{11} \leq (P_2)_G \cap R_1 < R_1$ and $(P_2)_G \cap R_1 \trianglelefteq G$. This contrary to the minimal normality of R_1 in G.

Since P_2 is weakly c-normal in G, by Lemma 2.1 there exists a subnormal subgroup K of G such that $G = P_2K$ and $P_2 \cap K = (P_2)_G = T$, and so $T \leq K$. $R_1 \cap K$ is normal in R_1K since R_1 is abelian, but $G = P_2K = R_{11}TK = R_{11}K = R_1K$, so $R_1 \cap K \lhd G$. If $R_{11} \cap K \neq 1$, then $1 < R_1 \cap K < R_1$, contrary to the minimality of R_1 of G. Hence $R_{11} \cap K = 1$. Since $G = P_2K = R_{11}TK = R_{11}K$ and $R_{11} \cap K = 1 = (R_{11})_G$. Hence R_{11} is weakly c-normal in G. By Lemma 2.6, R_1 is a cyclic group of prime order. With the same discussion as above proof it is clear that $R_i(i = 1, 2, \cdots, t)$ are all cyclic groups of prime order.

Set $K_i = R_1 \times R_2 \times \cdots, \times R_t$, where $i = 1, 2, \cdots, m$. Consider the chain

$$1 = \Phi(G) \leq K_1 \leq K_2 \leq \cdots \leq K_m = F(G).$$

Clearly $K_i \trianglelefteq G$ for each i and $|K_i/K_{i-1}|$ is a prime number. Applying Lemma 2.4, we obtain that G is supersolvable. The proof of theorem is completed.

Abstract

A subgroup H is called weakly c-normal in a group G if there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \leq H_G$, where H_G is the largest normal subgroup of G contained in H. In this paper, we investage the influence of weakly c-normality of some subgroups on structure of finite groups.

References

1. K. Doerk and T. O. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin-New York, 1992.

2. W. Guo, The Theory of Classes of Groups, Science Press/Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

3. B. Huppert, Endliche Gruppen I, Berlin-heidelberg-New York, 1967

4. D. Li and X. Guo, The influence of c-normality of subgroups on the structure of finite groups, *Comm. in Algebra.* **26**(2), 1913-1922(1998).

5. D. J. S. Robinson, A Course in the Theory of Groups, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

6. J. Rose, On finite insolvable groups with nilpotent maximal subgroup, *J.Aglebra*, **48** (1977), 182-196.

7. L. A. Shemetkov, Formations of Finite Groups, Nauka, Moscow, 1978.

8. Y. Wang, c-Normality of groups and its properties, *J.Algebra*, **78**(1996), 101-108.

9. M. Xu, Introduction to finite groups, Science Press, Beijing, 1999 (in chinese)

10. L. Zhu, W. Guo and K.P.Shum, Weak c-normal subgroups of finite groups and its properties, *Comm. in Algebra*, **30**(11), 2002.

Department of Mathematics, Yangzhou University,
Yangzhou 225002, P.R. China
E-mail: ydzhlj@sohu.com

Department of Mathematics, University of
Science and Technology of China,
Hefei 230026, P.R.China

Department of Mathematics, Xuzhou Normal University
Xuzhou 221116, P.R. China
E-mail: wbguo@pub.xz.jsinfo.net

Поступило 21.09.2002